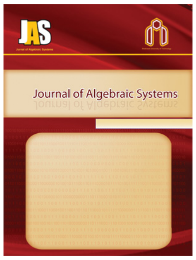
Journal of Algebraic Systems



ISSN:(2345-5128)(2345-511X)
Journal Homepage: www.jas.shahroodut.ac.ir

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E. Tamimi* and A. Ghaffari

To cite this article: E. Tamimi* and A. Ghaffari (9 June 2024): A survey on χ-module Connes amenability of semigroup algebras, Journal of Algebraic Systems, DOI: 10.22044/JAS.2023.12900.1702

To link to this article: https://doi.org/10.22044/JAS.2023.12900.1702

Published online: 9 June 2024

Journal of Algebraic Systems, vol. xx, no. xx, (202x), pp xx-xx
https://doi.org/10.22044/JAS.2023.12900.1702

A SURVEY ON χ-MODULE CONNES AMENABILITY OF SEMIGROUP ALGEBRAS

E. TAMIMI* AND A. GHAFFARI

Abstract. We shall study the χ-module Connes amenability of a semigroup algebra $l^1(S)$, where χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w^*-continuous and S is an inverse weakly cancellative semigroup with subsemigroup E of idempotents. We are mainly concerned with the study of χ-module normal, virtual diagonals. We characterize the χ-module Connes amenability of a semigroup algebra $l^1(S)$. Also, we show that if $l^1(S)$ as a Banach module over $l^1(E)$ has an *id*-module normal, virtual diagonal then it is *id*-module Connes amenable. Other characterizations of χ-module Connes amenability of $l^1(S)$ is presented.

1. Introduction

Connes amenability of dual Banach algebras were introduced by Runde in [19]. In [20], Runde showed that if a Banach algebra is Connes amenable, it has a normal, virtual diagonal. In [7], φ-Connes amenability of dual Banach algebras, where φ is a character from a Banach algebra onto $\mathbb{C}$ is investigated. In [1], Amini introduced the notion of module amenability for semigroup algebras. Amini proved that for an inverse semigroup S with subsemigroup E of idempotents, $l^1(S)$ as a Banach module over $l^1(E)$ is module amenable if S is amenable and vice versa. Also, in [8] Ghaffari et al. studied φ-module Connes amenability of dual Banach algebras, where φ is a w^*-continuous bounded module homomorphism from a Banach algebra to itself. For more study on module amenability, we can refer to [1, 2, 17, 21]. In [14], Minapoor introduced ideal Connes amenability of l^1-Munn algebras and its application to semigroup algebras. Also, ideal Connes amenability of Lau product of Banach algebras is studied by Minapoor et al. [15]. Module operator virtual diagonals on the Fourier algebra of an inverse semigroup is studied by Amini and Rezavand in [3]. In [13], Lau and Zhang studied the fixed point properties of semigroups of non-expansive mappings on weakly compact convex subsets of a Banach space. Also, they provide a characterization for the existence

Published online: 1 April 2024
MSC(2010): Primary: 43A20, 43A10; Secondary: 22D15.
Keywords: χ-module Connes amenable; Module χ-derivation; χ-module normal virtual diagonal; Weakly cancellative semigroup.
Received: 1 April 2023, Accepted: 29 May 2023.
*Corresponding author.

of a left invariant mean on the space of weakly almost periodic functions on separable semitopological semigroups in terms of fixed point property for non-expansive mappings. In [10, Lemma 2.1], Lau showed that if S is a semitopological semigroup and X is a left translation invariant subspace of the closed subalgebra of $l^\infty(S)$ consisting of continuous functions, where X has a left invariant mean, then S is left reversible.

In [11], Lau and Zhang studied the algebraic and analytic properties of semigroups related to fixed point properties of non-expansive mappings. Also, they investigate the extremely left amenability of a semitopological semigroup in terms of existence a left invariant mean on the left uniformly continuous functions. In [12], Lau and Zhang investigated the fixed point properties for semigroups of non-expansive mappings on convex sets in dual Banach spaces. Also, they showed that any left invariant mean on a left invariant subspace of $l^\infty(S)$ that S is a left reversible semitopological semigroup extends to a strictly increasing left subinvariant submean on $l^\infty(S)$.

All of these concepts generalized the earlier concept of amenability for the semigroup algebras introduced by Johnson [9].
In this paper, we introduce the concept of χ-module Connes amenability for semigroup algebra $l^1(S)$ and we characterize the χ-module Connes amenability in terms of χ-modul normal virtual diagonals. In particular, we show that if $\chi : l^1(S) \longrightarrow l^1(S)$ is a bounded module homomorphism that is w^*-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-module Connes amenable, then it has a χ-module normal virtual diagonal. Other results and hereditary properties in this direction are also obtained.

2. χ-Module Connes amenability

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $x \in S$ is called an idempotent if $x = x^* = x^2$. The set of idempotent elements of S is denoted by E. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st, R_s(t) = ts, (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ (see [6]).

Let S be an inverse semigroup, $s, t \in S$ and E be a subsemigroup of idempotents. We consider an equivalence relation on S where $s \sim t$ if and only if there is $q \in E$ such that $sq = tq$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group (see [16]). Also, E is a symmetric subsemigroup of S. Thus, $l^1(S)$

is a Banach $l^1(E)$-module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ via

$$\delta_q.\delta_s = \delta_s, \quad \delta_s.\delta_q = \delta_{sq} = \delta_s * \delta_q \qquad (s \in S, q \in E). \tag{2.1}$$

It is known that, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$-module (see [1]).

Throughout this paper, it is assumed that $l^1(S) = (c_0(S))^*$ is a dual Banach algebra, and $\mathcal{U} = l^1(E)$ is a Banach algebra such that $l^1(S)$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(\delta_s\delta_t) = (\alpha.\delta_s).\delta_t, \;\; (\alpha\beta).\delta_s = \alpha.(\beta.\delta_s) \quad (\delta_s, \delta_t \in l^1(S), \alpha, \beta \in \mathcal{U}). \tag{2.2}$$

Definition 2.1. Let S be a discrete semigroup and $l^1(S)$ be a dual Banach algebra. A module homomorphism from $l^1(S)$ to $l^1(S)$ is a map $\chi : l^1(S) \to l^1(S)$ with

$$\chi(\alpha.\delta_s + \delta_t.\beta) = \alpha.\chi(\delta_s) + \chi(\delta_t).\beta, \qquad \chi(\delta_s\delta_t) = \chi(\delta_s)\chi(\delta_t)$$

for every $\delta_s, \delta_t \in l^1(S)$ and $\alpha, \beta \in \mathcal{U}$.

Throughout this paper $\mathcal{H}_{w^*}(l^1(S))$ will denote the space of all bounded module homomorphisms from $l^1(S)$ to $l^1(S)$ that are w^*-continuous.

Definition 2.2. Let S be a discrete semigroup, $l^1(S) = (c_0(S))^*$ be a dual Banach algebra, X be a dual Banach $l^1(S)$-bimodule and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. A bounded map $D : l^1(S) \to X$ is called a module χ-derivation if for every $\delta_s, \delta_t \in l^1(S)$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D(\alpha.\delta_s \pm \delta_t.\beta) = \alpha.D(\delta_s) \pm D(\delta_t).\beta \tag{2.3}$$

and

$$D(\delta_s\delta_t) = D(\delta_s).\chi(\delta_t) + \chi(\delta_s).D(\delta_t). \tag{2.4}$$

Also, X is called symmetric, if

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in X).$$

When X is symmetric, each $x \in X$ defines a module χ-derivation

$$(D)_x(\delta_s) = \chi(\delta_s).x - x.\chi(\delta_s) \qquad (\delta_s \in l^1(S)). \tag{2.5}$$

Derivations of this form is called inner module χ-derivations.

Let X be a dual Banach $l^1(S)$-bimodule. Then X is called normal if for each $x \in X$, the maps

$$l^1(S) \to X; \qquad \delta_s \to \delta_s.x, \quad \delta_s \to x.\delta_s \tag{2.6}$$

are w^*-continuous. Moreover, if X is an $\mathcal{U}$-bimodule such that for $x \in X$, $\alpha \in \mathcal{U}$ and $\delta_s \in l^1(S)$ we have

$$\alpha.(\delta_s.x) = (\alpha.\delta_s).x, \quad (\delta_s.\alpha).x = \delta_s.(\alpha.x), \quad (\alpha.x).\delta_s = \alpha.(x.\delta_s), \tag{2.7}$$

then X is called a normal Banach left $l^1(S)$-$\mathcal{U}$-module. Similarly for the right and two sided actions.

Definition 2.3. Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra, $\chi \in \mathcal{H}_{w^*}(l^1(S))$ and $\mathcal{U}$ be a semigroup algebra such that $l^1(S)$ is a Banach $\mathcal{U}$-module. Then $l^1(S)$ is called χ-module Connes amenable if for any symmetric normal Banach $l^1(S)$-$\mathcal{U}$-module X, each w^*-continuous module χ-derivation $D : l^1(S) \to X$ is inner.

In the sequel, we characterize the χ-module Connes amenability of semigroup algebras.

Proposition 2.4. *Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$ such that $\chi(l^1(S)) = l^1(S)$. If $l^1(S)$ is χ-module Connes amenable, then $l^1(S)$ is module Connes amenable.*

Proof. Let E be a symmetric normal Banach $l^1(S)$-$l^1(S)$-module and $D : l^1(S) \to E$ be a w^*-continuous module derivation. Set $\rho = D \circ \chi$. The mapping $\rho : l^1(S) \to E$ is a module χ-derivation. Since $\chi \in \mathcal{H}_{w^*}(l^1(S))$, ρ is w^*-continuous. Thus there exists $r \in E$ such that $\rho(\delta_s) = r.\chi(\delta_s) - \chi(\delta_s).r$ for all $\delta_s \in l^1(S)$. By hypothesis, there exists $\delta_t \in l^1(S)$, such that $\chi(\delta_t) = \delta_s$. Hence

$$\begin{aligned} D(\delta_s) &= D(\chi(\delta_t)) = \rho(\delta_t) \\ &= r.\delta_s - \delta_s.r. \end{aligned}$$

It follows that $l^1(S)$ is module Connes amenable. □

In [5, Theorem 1.3], Dales and Strauss showed that if S is an infinite, weakly cancellative semigroup, then S is strongly Arens irregular. Also, in [5, Theorem 1.6] by similar conditions they showed that $l^1(S)$ is strongly Arens irregular. Using this notions we prove the following theorem.

Theorem 2.5. *Let S be a finite discrete semigroup, $l^1(S)$ be a dual Arens regular semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. Then $l^1(S)$ is χ-module Connes amenable if and only if $l^1(S)^{**}$ is χ^{**}-module Connes amenable.*

Proof. Consider the following commutative diagram,

Let $l^1(S)$ be χ-module Connes amenable, E be a symmetric normal Banach $l^1(S)^{**}$-$l^1(E)$-module and $D : l^1(S)^{**} \to E$ be a w^*-continuous module

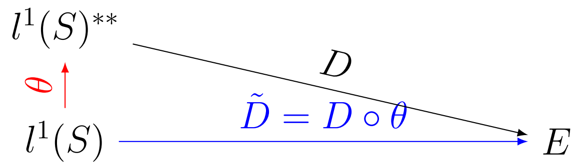

FIGURE 1. Commutative diagram.

χ^{**}-derivation. Let $\theta : l^1(S) \to l^1(S)^{**}$ be the canonical map. It is clear that θ is w^*-continuous. Define a module action of $l^1(S)$ on E by

$$x \circledast \delta_s = x.\theta(\delta_s), \quad \delta_s \circledast x = \theta(\delta_s).x \qquad (\delta_s \in l^1(S), x \in E). \tag{2.8}$$

This module action is well-defined and turns E into a normal Banach $l^1(S)$-$l^1(E)$-module. Define a derivation $\tilde{D} : l^1(S) \to E$ by $\tilde{D} = D \circ \theta$. Then we obtain

$$\begin{aligned}\tilde{D}(\delta_s\delta_t) = D \circ \theta(\delta_s\delta_t) &= D \circ \theta(\delta_s).\chi^{**}(\theta(\delta_t)) + \chi^{**}(\theta(\delta_s)).D \circ \theta(\delta_t)\\ &= D \circ \theta(\delta_s).\theta(\chi(\delta_t)) + \theta(\chi(\delta_s)).D \circ \theta(\delta_t)\\ &= D \circ \theta(\delta_s) \circledast \chi(\delta_t) + \chi(\delta_s) \circledast D \circ \theta(\delta_t)\\ &= \tilde{D}(\delta_s) \circledast \chi(\delta_t) + \chi(\delta_s) \circledast \tilde{D}(\delta_t).\end{aligned}$$

Thus, using the relations (2.3) and (2.4), $\tilde{D}$ is a module χ-derivation that is w^*-continuous. Since $l^1(S)$ is χ-module Connes amenable, there exists $x \in E$ such that

$$\begin{aligned}\tilde{D}(\delta_s) &= D \circ \theta(\delta_s)\\ &= x \circledast \chi(\delta_s) - \chi(\delta_s) \circledast x\\ &= x.\theta(\chi(\delta_s)) - \theta(\chi(\delta_s)).x.\end{aligned} \tag{2.9}$$

Let $\Gamma \in l^1(S)^{**}$. Since $\overline{\theta(l^1(S))}^{w^*} = l^1(S)^{**}$ and using Goldstine Theorem, there exists a net $\{\delta_\alpha\} \subseteq l^1(S)$ such that $\theta(\delta_\alpha) \to \Gamma$ in the w^*-topology. It can be shown that χ^{**} is w^*-continuous. So, $\chi^{**}(\theta(\delta_\alpha)) \to \chi^{**}(\Gamma)$. Thus, by relation (2.9)

$$\begin{aligned}D(\Gamma) &= \lim_\alpha D \circ \theta(\delta_\alpha)\\ &= \lim_\alpha x.\theta \circ \chi(\delta_\alpha) - \theta \circ \chi(\delta_\alpha).x\\ &= \lim_\alpha x.\chi^{**} \circ \theta(\delta_\alpha) - \chi^{**} \circ \theta(\delta_\alpha).x\\ &= x.\chi^{**}(\Gamma) - \chi^{**}(\Gamma).x.\end{aligned}$$

Therefore, D is inner.

Conversely, consider the following commutative diagram,

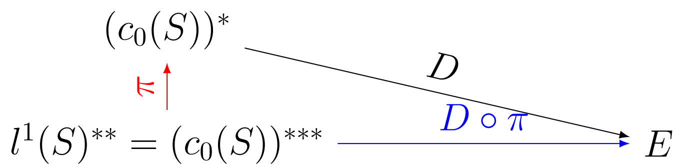

FIGURE 2. Commutative diagram.

Suppose that E is a symmetric normal Banach $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to E$ is a w^*-continuous module χ-derivation. Let

$$\pi : (c_0(S))^{***} \to (c_0(S))^*, \qquad \pi(\Lambda) = \Lambda \mid_{\theta(c_0(S))}$$

be the Dixmier projection. Since, π from $l^1(S)^{**}$ onto $l^1(S)$ is a module homomorphism, E is a Banach $l^1(S)^{**}$-$l^1(E)$-module with the bimodule actions are defined as follows:

$$\Lambda \circleddash x = \pi(\Lambda).x, \;\; x \circleddash \Lambda = x.\pi(\Lambda) \qquad (x \in E, \Lambda \in l^1(S)^{**}).$$

It is easy to see that E is a symmetric normal Banach $l^1(S)^{**}$-$l^1(E)$-module. Consider the mapping $D \circ \pi : l^1(S)^{**} \to E$, in Figure 2. We have,

$$\begin{aligned} D \circ \pi(\Lambda\Gamma) &= D(\pi(\Lambda)\pi(\Gamma)) \\ &= D \circ \pi(\Lambda).\chi \circ \pi(\Gamma) + \chi \circ \pi(\Lambda).D \circ \pi(\Gamma) \\ &= D \circ \pi(\Lambda).\pi(\chi^{**}(\Gamma)) + \pi(\chi^{**}(\Lambda)).D \circ \pi(\Gamma) \\ &= D \circ \pi(\Lambda) \circleddash \chi^{**}(\Gamma) + \chi^{**}(\Lambda) \circleddash D \circ \pi(\Gamma). \end{aligned}$$

It follows that $D \circ \pi$ is a module χ^{**}-derivation. Since $l^1(S)^{**}$ is χ^{**}- module Connes amenable, by Definition 2.3, there exists $x \in E$ such that for all $\Lambda \in l^1(S)^{**}$,

$$\begin{aligned} D \circ \pi(\Lambda) &= \chi^{**}(\Lambda) \circleddash x - x \circleddash \chi^{**}(\Lambda) \\ &= \pi(\chi^{**}(\Lambda)).x - x.\pi(\chi^{**}(\Lambda)) \\ &= \chi(\pi(\Lambda)).x - x.\chi(\pi(\Lambda)). \end{aligned}$$

Therefore $D(\delta_s) = \chi(\delta_s).x - x.\chi(\delta_s)$ for all $\delta_s \in l^1(S)$, and hence by (2.5), D is inner. □

Theorem 2.6. *Let S be a discrete semigroup, $l^1(S)$ be a non-abelian symmetric dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. If $l^1(S)$ is χ-module Connes amenable, then $l^1(S)$ has a bounded approximate identity for $\chi(l^1(S))$.*

Proof. By Theorem 2.5, we assume that $l^1(S)$ is a symmetric Banach $l^1(S)$-$l^1(E)$-module whose space of underlying is $l^1(S)$, and acts on $l^1(S)$ via

$$\delta_s \vartriangle f := \delta_s f, \qquad f \vartriangle \delta_s := 0, \qquad (\delta_s \in l^1(S), f \in l^1(S)).$$

Consider the identity map $I : l^1(S) \to l^1(S)$. By relations (2.3) and (2.4), it is easy to see that $I \circ \chi$ is a module χ-derivation. By the hypothesis, since $l^1(S)$ is χ-module Connes amenable, there exists $e_{l^1(S)} \in l^1(S)$ such that

$$I \circ \chi(\delta_s) = \chi(\delta_s) \vartriangle e_{l^1(S)} - e_{l^1(S)} \vartriangle \chi(\delta_s).$$

Therefore, $\chi(\delta_s) = \chi(\delta_s) \vartriangle e_{l^1(S)}$ and the element $e_{l^1(S)}$ is a bounded approximate identity for $\chi(l^1(S))$. □

An analogue of the following theorem for commutative dual Banach algebra is given in [8, Theorem 2.5].

Theorem 2.7. *Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. If $l^1(S)$ is χ-module Connes amenable, then $l^1(S)$ is $\eta \circ \chi$-module Connes amenable for any $\eta \in \mathcal{H}_{w^*}(l^1(S))$.*

Proof. Let E be a symmetric normal Banach $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to E$ be a module $\eta \circ \chi$-derivation that is w^*-continuous. It is sufficient to show that D is inner. For this purpose we equip E with the module actions defined by

$$\delta_s \bullet x = \eta(\delta_s).x, \;\; x \bullet \delta_s = x.\eta(\delta_s), \qquad (\delta_s \in l^1(S), x \in E).$$

Now, by using relations (2.6) and (2.7), E becomes a symmetric normal Banach $l^1(S)$-$l^1(E)$-module. We have

$$\begin{aligned} D(\delta_s\delta_t) &= D(\delta_s).\eta \circ \chi(\delta_t) + \eta \circ \chi(\delta_s).D(\delta_t) \\ &= D(\delta_s) \bullet \chi(\delta_t) + \chi(\delta_s) \bullet D(\delta_t). \end{aligned}$$

Thus, there exists $y \in E$ such that

$$D(\delta_s) = y \bullet \chi(\delta_s) - \chi(\delta_s) \bullet y = y.\eta \circ \chi(\delta_s) - \eta \circ \chi(\delta_s).y, \qquad (\delta_s \in l^1(S)).$$

The above argument completes the proof. □

3. χ-Module normal virtual diagonals

Let $\mathcal{U}$ be a Banach algebra. Let S be a discrete semigroup and $l^1(S)$ be a dual Banach algebra that is hold in compatible module actions of (2.2).

Let $\mathcal{I}$ be the closed ideal of the projective tensor product of $l^1(S)$ with itself, $l^1(S)\widehat{\otimes}l^1(S)$. We put

$$\mathcal{I} =< \alpha.(\delta_a \otimes \delta_b) - (\delta_a \otimes \delta_b).\alpha >, \qquad (\alpha \in \mathcal{U}, \delta_a, \delta_b \in l^1(S)).$$

We denote the module projective tensor product of $l^1(S)$ and itself by $l^1(S)\widehat{\otimes}_{\mathcal{U}}l^1(S)$, and we consider $l^1(S)\widehat{\otimes}_{\mathcal{U}}l^1(S) \approx \dfrac{l^1(S)\widehat{\otimes}l^1(S)}{\mathcal{I}}$.

Suppose that $\mathcal{J} =< (\alpha.\delta_a).\delta_b - \delta_a.(\delta_b.\alpha) >$, is the closed ideal of $l^1(S)$. It is clear that $\overline{\mathcal{J}}^{w^*} = \mathcal{J}$. Then by use of [18], the quotient algebra $\frac{l^1(S)}{\mathcal{J}}$ is dual with predual

$$^{\perp}\mathcal{J} = \{\Phi \in c_0(S) : \langle \Phi, j\rangle = 0 \text{ for all } j \in \mathcal{J}\}.$$

Also, we have

$$\mathcal{J}^{\perp} = \{\Psi \in l^{\infty}(S) : \langle \Psi, j\rangle = 0 \text{ for all } j \in \mathcal{J}\}.$$

Throughout this section, it is assumed that

$$\mathcal{L}^2_{\omega^*}(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C}) = \Big\{\mathcal{F}|\ \mathcal{F} : \frac{l^1(S)}{\mathcal{J}} \times \frac{l^1(S)}{\mathcal{J}} \to \mathbb{C}\Big\}, \tag{3.1}$$

that $\mathcal{F}$ is separately w^*-continuous two-linear map, and suppose that

$$\Theta : l^1(S)\widehat{\otimes}_{\mathcal{U}}l^1(S) \to \frac{l^1(S)}{\mathcal{J}}$$

is the multiplication operator with $\Theta(\delta_s \otimes \delta_t + \mathcal{I}) = \delta_s\delta_t + \mathcal{J}$. Using the Open Mapping Theorem the quotient map Θ^*, maps $^{\perp}\mathcal{J}$ onto $\mathcal{L}^2_{\omega^*}(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})$. It follows that Θ^{**} drops to an $l^1(S)$-$\mathcal{U}$-module homomorphism

$$\Theta^{**} : \mathcal{L}^2_{\omega^*}(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})^* \to \frac{l^1(S)}{\mathcal{J}}.$$

Also, consider the map that is defined by

$$\tilde{\chi} : \frac{l^1(S)}{\mathcal{J}} \to \frac{l^1(S)}{\mathcal{J}}, \qquad \tilde{\chi}(\delta_s + \mathcal{J}) = \chi(\delta_s) + \mathcal{J}, \qquad \delta_s \in l^1(S).$$

Definition 3.1. Let S be a discrete semigroup, $l^1(S)$ be a dual Banach algebra and $\chi : l^1(S) \to l^1(S)$ be a bounded w^*-continuous module homomorphism. An element $\mathfrak{D} \in \mathcal{L}^2_{\omega^*}(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})^*$ is called a χ-module normal virtual diagonal for $l^1(S)$ if

$$\Theta^{**}(\mathfrak{D}).(\chi(\delta_q) + \mathcal{J}) = (\chi(\delta_q) + \mathcal{J}) \qquad (\delta_q \in l^1(S)), \tag{3.2}$$

and

$$\mathfrak{D}.(\chi(\delta_q)+\mathcal{J})=(\chi(\delta_q)+\mathcal{J}).\mathfrak{D} \qquad (\delta_q\in l^1(S)). \tag{3.3}$$

In the sequel, we prove the main theorem of this section. Indeed, we show that if $l^1(S)$ is an unital dual Banach semigroup algebra with an *id*-module normal virtual diagonal, then $l^1(S)$ is module Connes amenable.

Theorem 3.2. *Let S be a discrete semigroup and $\mathcal{U}$ be a dual Banach algebra. Let $l^1(S)$ be an unital dual Banach $\mathcal{U}$-module that contain an id-module normal virtual diagonal. Then $l^1(S)$ is id-module Connes amenable.*

Proof. Let X be a symmetric normal Banach $l^1(S)$-$\mathcal{U}$-module. First, we note that $l^1(S)$ has an identity. It is sufficient to show that $l^1(S)$ is *id*-module Connes amenable. We suppose that X is unital. Let $D: l^1(S)\to X$ be a w^*-continuous module derivation. It is clear that X is a normal Banach $\frac{l^1(S)}{\mathcal{J}}$-$\mathcal{U}$-module. Let $X=(X_*)^*$. Since X is symmetric, D vanishes on $\mathcal{J}$. We define

$$\tilde{D}:\frac{l^1(S)}{\mathcal{J}}\to X, \qquad \tilde{D}(\delta_s+\mathcal{J}):=D(\delta_s) \qquad (\delta_s\in l^1(S)).$$

For every $x\in X_*$, we correspond $\Omega_x:\frac{l^1(S)}{\mathcal{J}}\times\frac{l^1(S)}{\mathcal{J}}\to\mathbb{C}$ via

$$\Omega_x(\delta_s+\mathcal{J},\delta_t+\mathcal{J})=\langle x,(\delta_s+\mathcal{J})\tilde{D}(\delta_t+\mathcal{J})\rangle \quad (\delta_s,\delta_t\in l^1(S)). \tag{3.4}$$

It is clear that $\Omega_x\in\mathcal{L}^2_{w^*}(\frac{l^1(S)}{\mathcal{J}},\mathbb{C})$. By using (3.1) and Definition 3.1, suppose that $\mathcal{F}\in\mathcal{L}^2_{w^*}(\frac{l^1(S)}{\mathcal{J}},\mathbb{C})$ and $\mathfrak{D}\in\mathcal{L}^2_{w^*}(\frac{l^1(S)}{\mathcal{J}},\mathbb{C})^*$. We define

$$\langle\mathfrak{D},\mathcal{F}\rangle=\int_{l^1(S)\widehat{\otimes}_{\mathcal{U}}l^1(S)}\mathcal{F}(\delta_s+\mathcal{J},\delta_t+\mathcal{J})d\mathfrak{D}(\delta_s+\mathcal{J},\delta_t+\mathcal{J}). \tag{3.5}$$

More generally, suppose that X is a dual Banach space. From Definition 3.1 and above notation it follows that for $\mathcal{F}\in\mathcal{L}^2_{w^*}(\frac{l^1(S)}{\mathcal{J}},\mathbb{C})$,

$$\mathfrak{D}.(\delta_q+\mathcal{J})=(\delta_q+\mathcal{J}).\mathfrak{D}$$

is equivalent to

$$\int\mathcal{F}(\delta_q\delta_s+\mathcal{J},\delta_t+\mathcal{J})d\mathfrak{D}=\int\mathcal{F}(\delta_s+\mathcal{J},\delta_t\delta_q+\mathcal{J})d\mathfrak{D}.$$

Suppose that $\mathcal{G}:\frac{l^1(S)}{\mathcal{J}}\times\frac{l^1(S)}{\mathcal{J}}\to X$ is a bilinear map such that

$$\delta_s+\mathcal{J}\longrightarrow\mathcal{G}(\delta_s+\mathcal{J},\delta_t+\mathcal{J}), \qquad \delta_t+\mathcal{J}\longrightarrow\mathcal{G}(\delta_s+\mathcal{J},\delta_t+\mathcal{J})$$

are w^*-continuous. Now for $x \in X_*$ and $\int \mathcal{G} d\mathfrak{D} \in X$ define

$$\Big\langle \int \mathcal{G} d\mathfrak{D}, x \Big\rangle = \int \Big\langle \mathcal{G}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}), x \Big\rangle d\mathfrak{D}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}).$$

By using (3.2) and (3.3), for each $\delta_s, \delta_t \in l^1(S)$ and $\Phi \in c_0(S)$ we have

$$\Big\langle \int \delta_s\delta_t + \mathcal{J} d\mathfrak{D}, \Phi + \mathcal{J}^{\perp} \Big\rangle = \Big\langle \mathfrak{D}, \Theta^*(\Phi + \mathcal{J}^{\perp}) \Big\rangle = \Big\langle \Theta^{**}(\mathfrak{D}), \Phi + \mathcal{J}^{\perp} \Big\rangle.$$

Now, put

$$H(x) = \langle \mathfrak{D}, \Omega_x \rangle \qquad (x \in X_*). \tag{3.6}$$

Let $\delta_q \in l^1(S)$, by relations (3.5) and (3.6), we obtain

$$\begin{aligned}
\Big\langle (\delta_q + \mathcal{J}).H - H.(\delta_q + \mathcal{J}), x \Big\rangle &= \Big\langle (H, x.(\delta_q + \mathcal{J}) - x.(\delta_q + \mathcal{J}) \Big\rangle \\
&= \int \Big\langle (\delta_s\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x \Big\rangle d\mathfrak{D} \\
&= \Big\langle D, \Omega_{x.(\delta_q+\mathcal{J})-(\delta_q+\mathcal{J}).x} \Big\rangle \\
&= \int \Omega_{x.(\delta_q+\mathcal{J})-(\delta_q+\mathcal{J}).x} d\mathfrak{D} \\
&= \int \Big\langle x.(\delta_q + \mathcal{J}) - (\delta_q + \mathcal{J}).x, (\delta_s \\
&+ \mathcal{J})\tilde{D}_{\mathcal{U}}(\delta_t + \mathcal{J}) \Big\rangle d\mathfrak{D} \\
&= \int \Big\langle x, (\delta_q + \mathcal{J})(\delta_s + \mathcal{J})\tilde{D} \Big\rangle (\delta_t + \mathcal{J}) \\
&- (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \Big\rangle d\mathfrak{D} \\
&= \int \Big\langle x, (\delta_q\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J}) \\
&- (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \Big\rangle d\mathfrak{D} \\
&= \int \Big\langle (x, (\delta_s + \mathcal{J})\tilde{D}(\delta_t\delta_q + \mathcal{J}) \\
&- (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \Big\rangle d\mathfrak{D} \\
&= \int \Big\langle (x, (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J})
\end{aligned}$$

$$- (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J})$$
$$+ (\delta_s + \mathcal{J})(\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J})\Big\rangle d\mathfrak{D}.$$

Thus we have

$$\begin{aligned}\Big\langle(\delta_q + \mathcal{J}).H - H.(\delta_q + \mathcal{J}), x\Big\rangle &= \int\Big\langle(\delta_s + \mathcal{J})(\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x\Big\rangle d\mathfrak{D}\\ &= \int\Big\langle(\delta_s\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x\Big\rangle d\mathfrak{D}\\ &= \int\Big\langle(\delta_s\delta_t + \mathcal{J}), x\Big\rangle d\mathfrak{D}.\tilde{D}(\delta_q + \mathcal{J})\\ &= \int\langle\Theta^{**}(\mathfrak{D}).\tilde{D}(\delta_q + \mathcal{J}), x\Big\rangle\end{aligned}$$

Therefore, $D(\delta_q) = \delta_q.H - H.\delta_q$ holds. □

In Theorem 3.2, it is shown that if an unital semigroup algebra $l^1(S)$ has a module normal virtual diagonal, then $l^1(S)$ is module Connes amenable.

In the next section we prove an important inherited property for above theorem.

4. Some inherited properties

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual semigroup algebra with predual $c_0(S)$, i.e $l^1(S) = (c_0(S))^*$ [4, Theorem 4.6].

Lemma 4.1. *Let S be a weakly cancellative semigroup with idempotents E and $l^1(S)$ be a dual Arens regular semigroup algebra. Moreover, let $l^1(S)$ be a dual Banach $l^1(E)$-module. If $\chi \in \mathcal{H}_{w^*}(l^1(S))$, then the following conditions are equivalent:*

(1) *$l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is $\chi\widehat{\otimes}_{l^1(E)}\chi$-module Connes amenable,*
(2) *$l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$ is $\chi^{**}\widehat{\otimes}_{l^1(E)}\chi^{**}$-module Connes amenable.*

Proof. Let $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ be $\chi\widehat{\otimes}_{l^1(E)}\chi$-module Connes amenable and E be a symmetric normal Banach $l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$-module. Moreover, let $D : l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**} \to E$ be a w^*-continuous module $\chi^{**}\widehat{\otimes}_{l^1(E)}\chi^{**}$-derivation. Let

$$\mu : l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S) \to l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$$

be the canonical map. It is clear that μ is w^*-continuous. Without loss of generality, by [5, Theorem 1.3] suppose that S is a finite semigroup. Since $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is a dual Banach $l^1(E)$-module, by the hypothesis and by using Theorem 2.5 the proof of the first part is complete. For the converse, we apply the second part of Theorem 2.5. □

Remark 4.2. In fact, the proof of Lemma 4.1 say that for any symmetric normal Banach $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$-$l^1(E)$-module E, each w^*-continuous module $\chi\widehat{\otimes}_{l^1(E)}\chi$-derivation from $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ to E is inner if and only if for any symmetric normal Banach $l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$-$l^1(E)$-module E, each w^*-continuous module $\chi^{**}\widehat{\otimes}_{l^1(E)}\chi^{**}$-derivation from module projective tensor product $l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$ to E is inner where, S is a weakly cancellative semigroup with idempotents E.

Corollary 4.3. *Under the conditions of Lemma 4.1, the module projective tensor product* $\overbrace{l^1(S)\widehat{\otimes}_{l^1(E)}\cdots\widehat{\otimes}_{l^1(E)}l^1(S)}^{n}$ *is* $\overbrace{\chi\widehat{\otimes}_{l^1(E)}\cdots\widehat{\otimes}_{l^1(E)}\chi}^{n}$*-module Connes amenable if and only if* $\overbrace{l^1(S)^{**}\widehat{\otimes}_{l^1(E)}\cdots\widehat{\otimes}_{l^1(E)}l^1(S)^{**}}^{n}$ *is* $\overbrace{\chi^{**}\widehat{\otimes}_{l^1(E)}\cdots\widehat{\otimes}_{l^1(E)}\chi^{**}}^{n}$*-module Connes amenable.*

Theorem 4.4. *Let S be a weakly cancellative semigroup with idempotents E, $l^1(S)$ be an unital dual Banach $l^1(E)$-module and $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ be a dual Banach $l^1(E)$-module. If $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$ and $l^1(S)$ is χ-module Connes amenable, then $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is $\chi\otimes_{l^1(E)}\chi$-module Connes amenable.*

Proof. Consider the following commutative diagram,

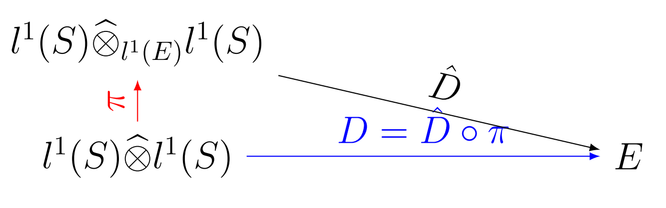

FIGURE 3. Commutative diagram.

Let $l^1(S)$ be χ-module Connes amenable. Let E be a symmetric normal Banach $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$-$l^1(E)$ module and

$$\widehat{D} : l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S) \to E$$

be a module $\chi\otimes_{l^1(E)}\chi$-derivation that is w^*-continuous. Consider the quotient map $\pi : l^1(S)\widehat{\otimes}l^1(S) \to l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$. Define

$$\delta_s \otimes \delta_t).x = \pi(\delta_s \otimes \delta_t) \circledcirc x, \quad x.(\delta_s \otimes \delta_t) = x \circledcirc \pi(\delta_s \otimes \delta_t)$$

for every $\delta_s, \delta_t \in l^1(S)$ and $x \in E$. Since π is w^*-continuous, E is a normal Banach $l^1(S)\widehat{\otimes}l^1(S)$-$l^1(E)$ module. Put

$$\widehat{D} \circ \pi : l^1(S)\widehat{\otimes}l^1(S) \to E.$$

It is clear that $D = \widehat{D} \circ \pi$ is w^*-continuous module $\chi \otimes \chi$-derivation. It is known that if $\widehat{D} \circ \pi$ is inner, then $\widehat{D}$ is inner. For this purpose, suppose that $e_{l^1(S)}$ is an identity element for $l^1(S)$. We define

$$\delta_s \blacktriangleleft x = (\delta_s \otimes e_{l^1(S)})x, \;\; x \blacktriangleleft \delta_s = x(\delta_s \otimes e_{l^1(S)}) \quad (\delta_s \in l^1(S), x \in E).$$

For $\delta_s \in l^1(S), x \in E$ and $\delta_u \in l^1(E)$, we obtain

$$\begin{aligned}
\delta_s \blacktriangleleft (\delta_u.x) - (\delta_s.\delta_u) \blacktriangleleft x &= (\delta_s \otimes e_{l^1(S)}).(\delta_u.x) - (\delta_s.\delta_u \otimes e_{l^1(S)}).x \\
&= (\delta_s \otimes e_{l^1(S)}).(\delta_u.x) - (\delta_u.\delta_s \otimes e_{l^1(S)}).x \\
&= (\delta_s \otimes e_{l^1(S)}).(\delta_u.x) - (\delta_u.(\delta_s \otimes e_{l^1(S)})).x \\
&= (\delta_s \otimes e_{l^1(S)}).(\delta_u.x) - ((\delta_s \otimes e_{l^1(S)}).\delta_u).x \\
&= (\delta_s \otimes e_{l^1(S)}).(\delta_u.x) - ((\delta_s \otimes e_{l^1(S)}).(\delta_u.x) \\
&= 0.
\end{aligned}$$

Also, the same argument is hold for the right actions. Therefore, E is a symmetric normal Banach $l^1(S)$-$l^1(E)$-bimodule. Put

$$D_{l^1(S)} : l^1(S) \to E, \qquad D_{l^1(S)}(\delta_s) = D(\delta_s \otimes e_{l^1(S)}) \qquad (\delta_s \in l^1(S)).$$

We obtain

$$\begin{aligned}
D_{l^1(S)}(\delta_s\delta_t) &= D(\delta_s \otimes e_{l^1(S)}).\chi \otimes \chi(\delta_t \otimes e_{l^1(S)}) \\
&\quad + \chi \otimes \chi(\delta_s \otimes e_{l^1(S)}).D(\delta_t \otimes e_{l^1(S)}) \\
&= D_{l^1(S)}(\delta_s) \blacktriangleleft \chi(\delta_t) + \chi(\delta_s) \blacktriangleleft D_{l^1(S)}(\delta_t).
\end{aligned}$$

Since $l^1(S)$ is χ-module Connes amenable, there is $g \in E$ such that $D_{l^1(S)} = ad_g$. Thus, $\tilde{D} = D - ad_g$ vanishes on $l^1(S) \otimes e_{l^1(S)}$. Setting

$$\delta_s \blacktriangleright x = (e_{l^1(S)} \otimes \delta_s).x, \quad x \blacktriangleright \delta_s = x.(e_{l^1(S)} \otimes \delta_s) \tag{4.1}$$

for every $\delta_s \in l^1(S)$ and $x \in E$. The module actions defined in (4.1), makes E into an $l^1(S)$-$l^1(E)$-bimodule for $\delta_s \in l^1(S)$ and $x \in E$. Let us now,

$D'_{l^1(S)}(\delta_s) := \tilde{D}(e_{l^1(S)} \otimes \delta_s)$. Set

$$\mathcal{Y} = \{y \in E_* : \langle \tilde{D}(e_{l^1(S)} \otimes \delta_s), y\rangle = 0\}.$$

Since $\tilde{D}$ is w^*-continuous, by use of [21, Theorem 4.9] and applying the similar argument we conclude $(\frac{E_*}{\mathcal{Y}})^* = \overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$. Furthermore, $\overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$ is a w^*-closed submodule of E. Thus, $\overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$ is a symmetric normal Banach $l^1(S)$-$l^1(E)$-module. By hypothesis, there exists $\delta_f \in \overline{\tilde{D}(e_{l^1(S)} \otimes_{l^1(S)} \delta_s)}^{w^*}$ such that

$$\begin{aligned}\tilde{D}(e_{l^1(S)} \otimes \delta_s) &= D'_{l^1(S)}(\delta_s) = \chi(\delta_s) \blacktriangleright \delta_f - \delta_f \blacktriangleright \chi(\delta_s)\\ &= \chi \otimes_{l^1(S)} \chi(e_{l^1(S)} \otimes \delta_s).\delta_f - \delta_f.\chi \otimes_{l^1(S)} \chi(e_{l^1(S)} \otimes \delta_s)\end{aligned}$$

and $\tilde{D} - ad_{\delta_f} |_{(e_{l^1(S)} \otimes l^1(S))}$ is vanishes on $e_{l^1(S)} \otimes l^1(S)$. All in all,

$$\tilde{D} - ad_{\delta_f} = D - ad_{\delta_g} - ad_{\delta_f} |_{l^1(S)\widehat{\otimes}_{l^1(E)} l^1(S)} = 0.$$

□

Example 4.5. Suppose that $\mathbb{N}$ is the set of positive integers,

(1) It is well-known that the semigroup algebra $l^1(\mathbb{N}, \max)$ has an identity. Since $\mathbb{N}$ is weakly cancellative, $l^1(\mathbb{N}, \max)$ is a dual semigroup algebra such that $l^1(\mathbb{N}, \max) = (c_0(\mathbb{N}))^*$. By [6, Theorem 5.13], $l^1(\mathbb{N}, \max)$ is not Connes amenable. Moreover, $l^1(\mathbb{N}, \max)$ is module amenable on $l^1(E_{(\mathbb{N}, \max)})$, so it is *id*-module Connes amenable. Using Theorem 2.6, it has a bounded approximate identity for $\chi(l^1(\mathbb{N}, \max))$.

(2) If $S = (\mathbb{N}, \min)$, then the semigroup algebra $l^1(\mathbb{N}, \min)$ is a non-unital Banach algebra with a bounded approximate identity. Furthermore, $l^1(\mathbb{N}, \min)$ is module amenable on $l^1(E_{(\mathbb{N}, \min)})$, so it is *id*-module Connes amenable.

Acknowledgments

We thank the referees for their time and valuable comments.

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Ebrahim Tamimi
Department of Mathematics, Velayat University, P.O. Box 9917638733, Iranshahr, Iran.
Email: e.tamimi@velayat.ac.ir

Ali Ghaffari
Department of Mathematics, University of Semnan, P.O. Box 35195-363, Semnan, Iran.
Email: aghaffari@semnan.ac.ir