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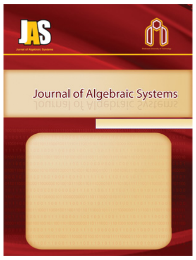

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FUZZY NEUTROSOPHIC PRIME IDEALS OF BCK-ALGEBRAS

B. SATYANARAYANA, SH. BAJI* AND D. DEVANANDAM

Abstract. In this research paper, we introduce and analyse the notion of fuzzy neutrosophic prime ideals (FNPIs) in a commutative BCK-algebra $\mathcal{K}$. It represents a further extension of prime ideals in the context of fuzzy neutrosophic sets. We provide an example that shows that not every fuzzy neutrosophic ideal of a commutative BCK-algebra $\mathcal{K}$ is a FNPI of $\mathcal{K}$. We also prove that a fuzzy neutrosophic set of $\mathcal{K}$ is a FNPI of $\mathcal{K}$ if, for all $a, b, c \in [0, 1]$, the upper (a,b)-level cut and lower c-level cut are prime ideals of $\mathcal{K}$.

1. Introduction

L. A. Zadeh [14], a professor of computer science at the University of California introduced the concept of fuzzy sets (FSs) in 1965. Fuzzy sets analyzed the degree of membership of members of the set, and Xi [13] applied this concept to the ideals of BCK/BCI algebras. In 1986 Atanassove [3] generalized a fuzzy set to an intuitionistic fuzzy set (IFS) by including another function called a non-membership function, and Jun and Kim [8] introduced intuitionistic fuzzy ideals of BCK-algebras. In 1995, Smarandache ([11], [12]) introduced the neutrosophic set (NS), which discusses the degree of uncertainty. In [2], Arockiarani et al. introduced the concept of fuzzy neutrosophic sets (FNSs). In [9] Y. B. Jun et al. introduced fuzzy prime ideals in commutative BCK-algebras. Later in 2014, Saleem Abdullah [1] had given the notion of intuitionistic fuzzy prime ideals of commutative BCK-algebras. In this paper, we give a notion of FNPIs of commutative BCK-algebras and investigate some of their properties.

2. Preliminaries

Definition 2.1. ([5] , [6]) Let $\mathcal{K}$ be a non-empty set with a binary operation "$\diamond$" and a constant "0". Then, $(\mathcal{K}, \diamond, 0)$ is called a BCK-algebra if it follows the following axioms for all $p_0, r_0, u_0 \in \mathcal{K}$:

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$$((p_0 \diamond r_0) \diamond (p_0 \diamond u_0)) \diamond (u_0 \diamond r_0) = 0 \tag{2.1}$$

$$(p_0 \diamond (p_0 \diamond r_0)) \diamond r_0 = 0 \tag{2.2}$$

$$p_0 \diamond p_0 = 0 \tag{2.3}$$

$$0 \diamond p_0 = 0 \tag{2.4}$$

$$p_0 \diamond r_0 = 0 \;\; and \;\; r_0 \diamond p_0 = 0 \Rightarrow p_0 = r_0 \tag{2.5}$$

Definition 2.2. [6] A non-empty subset $\mathcal{I}$ of a BCK/BCI-algebra $\mathcal{K}$ is called an ideal, if it satisfies

(I-1) $0 \in \mathcal{I}$.

(I-2) $p_0 \diamond r_0 \in \mathcal{I}$ and $r_0 \in \mathcal{I} \Rightarrow p_0 \in \mathcal{I}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.3. [6] An ideal $\mathcal{I}$ of a BCI-algebra $\mathcal{K}$ is called a closed ideal, if it satisfies $0 \diamond p_0 \in \mathcal{I}$, for all $p_0 \in \mathcal{I}$.

Definition 2.4. [4] An ideal of a commutative BCK-algebra $\mathcal{K}$ is said to be prime, if it satisfies $p_0 \wedge r_0 \in \mathcal{I}$ implies $p_0 \in \mathcal{I}$ or $r_0 \in \mathcal{I}$.

Proposition 2.5. [4] *Every prime ideal of a BCK-algebra $\mathcal{K}$ is an ideal of $\mathcal{K}$.*

Definition 2.6. [14] Let $\mathcal{K}$ be a non-empty set. A fuzzy set in the set $\mathcal{K}$ is a mapping $\mathcal{N}_T : \mathcal{K} \to [0, 1]$.

Definition 2.7. [14] The complement of a fuzzy set $\mathcal{N}_T$ is denoted by $(\mathcal{N}_T)^c$ and is also a fuzzy set defined as $(\mathcal{N}_T)^c = 1 - \mathcal{N}_T$. Also, $(\mathcal{N}_T{}^c)^c = \mathcal{N}_T$.

Definition 2.8. [9] A fuzzy ideal $\mathcal{N}_T$ of a commutative BCK-algebra is said to be a fuzzy prime ideal if $\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.9. [7] A fuzzy ideal $\mathcal{N}_T$ of a commutative BCK-algebra $\mathcal{K}$ is called an anti-fuzzy prime ideal of $\mathcal{K}$ if $\mathcal{N}_T(p_0 \wedge r_0) \geq min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.10. [3] An intuitionistic fuzzy set $\mathcal{N}$ in a non-empty set $\mathcal{K}$ is an object having the form $\mathcal{N} = \{(p_0, \mathcal{N}_T(p_0), \mathcal{N}_F(p_0))/p_0 \in \mathcal{K}\}$ where the functions $\mathcal{N}_T : \mathcal{K} \to [0, 1]$ and $\mathcal{N}_F : \mathcal{K} \to [0, 1]$ denote the grade of membership and non-membership of each element $p_0 \in \mathcal{K}$ to the set $\mathcal{N}$ respectively, and $0 \leq \mathcal{N}_T(p_0) + \mathcal{N}_F(p_0) \leq 1$ for all $p_0 \in \mathcal{K}$.

Definition 2.11. [8] An IFS $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ of a BCK-algebra $\mathcal{K}$ is an IFI of $\mathcal{K}$ if it follows the conditions:
(IFI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$ *and* $\mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$.
(IFI-2) $\mathcal{N}_T(p_0) \geq min\{\mathcal{N}_T(p_0 \diamond r_0), \mathcal{N}_T(r_0)\}$.
(IFI-3) $\mathcal{N}_F(p_0) \leq max\{\mathcal{N}_F(p_0 \diamond r_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.12. [1] An intuitionistic fuzzy ideal $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ of a commutative BCK-algebra $\mathcal{K}$ is an intuitionistic fuzzy prime ideal of $\mathcal{K}$ if it follows the conditions:
(IFPI-1) $\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$.
(IFPI-2) $\mathcal{N}_F(p_0 \wedge r_0) \geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.13. [2] A Fuzzy Neutrosophic Set (FNS) in a non-empty set $\mathcal{K}$ is a structure of the form

$$\mathcal{N} = \{\langle p_0, \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle / p_0 \in \mathcal{K}\}.$$

Where $\mathcal{N}_T : \mathcal{K} \to [0,1], \mathcal{N}_I : \mathcal{K} \to [0,1]$ and $\mathcal{N}_F : \mathcal{K} \to [0,1]$ represents grade of belongingness, grade of indeterminacy and grade of non- belongingness of each element $p_0 \in \mathcal{K}$ to the set $\mathcal{N}$ respectively and

$$0 \leq \mathcal{N}_T(p_0) + \mathcal{N}_I(p_0) + \mathcal{N}_F(p_0) \leq 3.$$

We shall use the symbol $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ for the FNS

$$\mathcal{N} = \{\langle p_0, \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle / p_0 \in \mathcal{K}\}.$$

Definition 2.14. [10] A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{K}$) is a fuzzy neutrosophic sub–algebra (FNSA) of $\mathcal{K}$ if it follows the conditions:
(FNSA-1) $\mathcal{N}_T(p_0 \diamond r_0) \geq min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$.
(FNSA-2) $\mathcal{N}_I(p_0 \diamond r_0) \geq min\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$.
(FNSA-3) $\mathcal{N}_F(p_0 \diamond r_0) \leq max\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.15. [10] A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F) \in \mathcal{K}$ is a fuzzy neutrosophic ideal (FNI) of $\mathcal{K}$ if it follows the conditions:
(FNI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0), \mathcal{N}_I(0) \geq \mathcal{N}_I(p_0)$ *and* $\mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$.
(FNI-2) $\mathcal{N}_T(p_0) \geq min\{\mathcal{N}_T(p_0 \diamond r_0), \mathcal{N}_T(r_0)\}$.
(FNI-3) $\mathcal{N}_I(p_0) \geq min\{\mathcal{N}_I(p_0 \diamond r_0), \mathcal{N}_I(r_0)\}$.
(FNI-4) $\mathcal{N}_F(p_0) \leq max\{\mathcal{N}_F(p_0 \diamond r_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.16. [10] Every FNI of a BCK-algebra $\mathcal{K}$ is a fuzzy neutrosophic subalgebra of a BCK-algebra $\mathcal{K}$.

3. Main results

For the purpose of this article, we will use the notation $\mathcal{K}$ to refer to a commutative BCK-algebra.

Definition 3.1. A fuzzy neutrosophic ideal $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of $\mathcal{K}$ is called a fuzzy neutrosophic prime ideal (FNPI) of $\mathcal{K}$ if it satisfies:
(FNPI-1) $\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$.
(FNPI-2) $\mathcal{N}_I(p_0 \wedge r_0) \leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$.
(FNPI-3) $\mathcal{N}_F(p_0 \wedge r_0) \geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Example 3.2. Let $\mathcal{K} = \{0, 1, 2, 3\}$ with the Cayley table as defined in Table 1.

TABLE 1. Commutative BCK-algebra

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

It is easy to verify that $(\mathcal{K}, \diamond, 0)$ is a commutative BCK-algebra. Now, let's define a fuzzy neutrosophic set (FNS) $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ as shown in Table 2. By routine calculations, we can conclude that $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is an FNPI of $\mathcal{K}$.

TABLE 2. Fuzzy Neutrosophic Set

$\mathcal{K}$	$\mathcal{N}_T$	$\mathcal{N}_I$	$\mathcal{N}_F$
0	0.9	0.9	0
1	0.8	0.7	0.2
2	0.4	0.3	0.6
3	0	0	0.9

Theorem 3.3. *If $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of a $\mathcal{K}$, then the sets*

$$K_1 = \{p_0 \in \mathcal{K} : \mathcal{N}_T(p_0) = \mathcal{N}_T(0)\}, \quad K_2 = \{p_0 \in \mathcal{K} : \mathcal{N}_I(p_0) = \mathcal{N}_I(0)\},$$

and $K_3 = \{p_0 \in \mathcal{K} : \mathcal{N}_F(p_0) = \mathcal{N}_F(0)\}$ are prime ideals of $\mathcal{K}$.

Proof. Straightforward. □

Corollary 3.4. *If $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of a $\mathcal{K}$, then the sets $M_1 = \{p_0 \in \mathcal{K} : \mathcal{N}_T(p_0) = 0\}$, $M_2 = \{p_0 \in \mathcal{K} : \mathcal{N}_I(p_0) = 0\}$, and $M_3 = \{p_0 \in \mathcal{K} : \mathcal{N}_F(p_0) = 0\}$ are either empty or prime ideals of $\mathcal{K}$.*

Proposition 3.5. *Every FNPI of a $\mathcal{K}$ is a FNI of a $\mathcal{K}$.*

Remark 3.6. A FNI of a commutative BCK-algebra $\mathcal{K}$ need not be a FNPI of $\mathcal{K}$ as shown in the following Example 3.7.

TABLE 3. Commutative BCK-algebra

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Example 3.7. Let $\mathcal{K} = \{0, 1, 2, 3\}$ be a BCK-algebra with the Cayley table as following Table 3. Then the BCK-algebra $\mathcal{K}$ is commutative. Define a fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ as shown in Table 4.

TABLE 4. Fuzzy Neutrosophic Set

$\mathcal{K}$	$\mathcal{N}_T$	$\mathcal{N}_I$	$\mathcal{N}_F$
0	0.97	1	0
1	0.45	0.67	0.53
2	0.45	0.67	0.53
3	0	0	0.93

$\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNI of $\mathcal{K}$ but is not a FNPI of $\mathcal{K}$ because

$$\mathcal{N}_T(2 \wedge 3) = \mathcal{N}_T(0) = 0.97 > 0.45 = max\{\mathcal{N}_T(2), \mathcal{N}_T(3)\},$$
$$\mathcal{N}_I(2 \wedge 3) = \mathcal{N}_I(0) = 1 > 0.67 = max\{\mathcal{N}_I(2), \mathcal{N}_I(3)\},$$
$$\mathcal{N}_F(2 \wedge 3) = \mathcal{N}_F(0) = 0 < 0.53 = min\{\mathcal{N}_F(2), \mathcal{N}_F(3)\}.$$

Proposition 3.8. [10] *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in a $\mathcal{K}$ is a FNI of $\mathcal{K}$ iff $\mathcal{N}_T$, $\mathcal{N}_I$ and ${\mathcal{N}_F}^c$ are fuzzy ideals of $\mathcal{K}$.*

Proposition 3.9. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic set of a $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$ iff $\mathcal{N}_T$, $\mathcal{N}_I$, and ${\mathcal{N}_F}^c$ are fuzzy prime ideals of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of $\mathcal{K}$. Then, by Proposition 3.5, $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is an FNI. So, by Proposition 3.8, $\mathcal{N}_T$, $\mathcal{N}_I$, and $\mathcal{N}_F^c$ are fuzzy ideals of $\mathcal{K}$. Therefore, for any $p_0, r_0 \in \mathcal{K}$, from Definition 3.1, we have

$$\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$$
$$\mathcal{N}_I(p_0 \wedge r_0) \leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$$
$$\mathcal{N}_F(p_0 \wedge r_0) \geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$$

Clearly $\mathcal{N}_T$ and $\mathcal{N}_I$ are fuzzy prime ideals of $\mathcal{K}$. Now

$$1 - \mathcal{N}_F(p_0 \wedge r_0) \leq 1 - min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$$
$$[\mathcal{N}_F(p_0 \wedge r_0)]^c \leq max\{1 - \mathcal{N}_F(p_0), 1 - \mathcal{N}_F(r_0)\}$$

$$= max\{[\mathcal{N}_F(p_0)]^c, [\mathcal{N}_F(r_0)]^c\}.$$

Hence ${\mathcal{N}_F}^c$ is a fuzzy prime ideal of $\mathcal{K}$.
Conversely, assume that $\mathcal{N}_T$, $\mathcal{N}_I$, and ${\mathcal{N}_F}^c$ are fuzzy prime ideals of $\mathcal{K}$. Then

$$\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$$
$$\mathcal{N}_I(p_0 \wedge r_0) \leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$$
$$[\mathcal{N}_F(p_0 \wedge r_0)]^c \leq max\{[\mathcal{N}_F(p_0)]^c, [\mathcal{N}_F(r_0)]^c\}.$$

$$Now \quad 1 - [\mathcal{N}_F(p_0 \wedge r_0)]^c \geq 1 - max\{[\mathcal{N}_F(p_0)]^c, [\mathcal{N}_F(r_0)]^c\}$$
$$\{[\mathcal{N}_F(p_0 \wedge r_0)]^c\}^c \geq min\{1 - [\mathcal{N}_F(p_0)]^c, 1 - [\mathcal{N}_F(r_0)]^c\}$$
$$= min\{\{[\mathcal{N}_F(p_0)]^c\}^c, \{[\mathcal{N}_F(r_0)]^c\}^c\}$$
$$\Rightarrow \mathcal{N}_F(p_0 \wedge r_0) \geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}.$$

Hence $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$. □

Proposition 3.10. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNI of $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$ iff ${\mathcal{N}_T}^c$, ${\mathcal{N}_I}^c$, and $\mathcal{N}_F$ are anti-fuzzy prime ideals of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of a $\mathcal{K}$. Since, by Proposition 3.5, $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNI. Then ${\mathcal{N}_T}^c$, ${\mathcal{N}_I}^c$, and $\mathcal{N}_F$ are anti-fuzzy ideals of $\mathcal{K}$. Therefore, for any $p_0, r_0 \in \mathcal{K}$, from Definition 3.1, we have

$$\mathcal{N}_T(p_0 \wedge r_0) \leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$$
$$\mathcal{N}_I(p_0 \wedge r_0) \leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$$
$$\mathcal{N}_F(p_0 \wedge r_0) \geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}.$$

Clearly, $\mathcal{N}_F$ is an anti-fuzzy prime ideal of $\mathcal{K}$. Now

$$1 - \mathcal{N}_T(p_0 \wedge r_0) \geq 1 - max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$$
$$[\mathcal{N}_T(p_0 \wedge r_0)]^c \geq min\{1 - \mathcal{N}_T(p_0), 1 - \mathcal{N}_T(r_0)\}$$
$$[\mathcal{N}_T(p_0 \wedge r_0)]^c \geq min\{[\mathcal{N}_T(p_0)]^c, [\mathcal{N}_T(r_0)]^c\}$$

Also

$$1 - \mathcal{N}_I(p_0 \wedge r_0) \geq 1 - max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$$
$$[\mathcal{N}_I(p_0 \wedge r_0)]^c \geq min\{1 - \mathcal{N}_I(p_0), 1 - \mathcal{N}_I(r_0)\}$$
$$[\mathcal{N}_I(p_0 \wedge r_0)]^c \geq min\{[\mathcal{N}_I(p_0)]^c, [\mathcal{N}_I(r_0)]^c\}$$

Therefore, ${\mathcal{N}_T}^c$ and ${\mathcal{N}_I}^c$ are anti-fuzzy prime ideals of $\mathcal{K}$. The proof of converse part is easy and we omit it. □

Theorem 3.11. [10] *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{K}$ is a FNI of a $\mathcal{K}$ if and only if $\boxdot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ and $\odot\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ are FNIs of $\mathcal{K}$.*

Proposition 3.12. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of $\mathcal{K}$. Then*

$$\boxdot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$$

is a FNPI of $\mathcal{K}$.

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of $\mathcal{K}$. Then by Proposition 3.5 $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNI. So, by Theorem 3.11 $\boxdot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ is a FNI of $\mathcal{K}$. Therefore, for any $p_0, r_0 \in \mathcal{K}$, from Definition 3.1, we have

$$\begin{aligned} \mathcal{N}_T(p_0 \wedge r_0) &\leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\} \\ \mathcal{N}_I(p_0 \wedge r_0) &\leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\} \\ \mathcal{N}_F(p_0 \wedge r_0) &\geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}. \end{aligned}$$

Now

$$\begin{aligned} 1 - \mathcal{N}_T(p_0 \wedge r_0) &\geq 1 - max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\} \\ [\mathcal{N}_T(p_0 \wedge r_0)]^c &\geq min\{1 - \mathcal{N}_T(p_0), 1 - \mathcal{N}_T(r_0)\} \\ [\mathcal{N}_T(p_0 \wedge r_0)]^c &\geq min\{[\mathcal{N}_T(p_0)]^c, [\mathcal{N}_T(r_0)]^c\} \end{aligned}$$

Hence $\boxdot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ is a FNPI of $\mathcal{K}$. □

Proposition 3.13. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of $\mathcal{K}$. Then $\odot\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNPI of $\mathcal{K}$. Then by Proposition 3.5 $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNI. So, by Theorem 3.11 $\odot\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ is a FNI of $\mathcal{K}$. Therefore, for any $p_0, r_0 \in \mathcal{K}$, from Definition 3.1, we have

$$\begin{aligned} \mathcal{N}_T(p_0 \wedge r_0) &\leq max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\} \\ \mathcal{N}_I(p_0 \wedge r_0) &\leq max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\} \\ \mathcal{N}_F(p_0 \wedge r_0) &\geq min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\} \end{aligned}$$

Now

$$\begin{aligned} 1 - \mathcal{N}_F(p_0 \wedge r_0) &\leq 1 - min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\} \\ [\mathcal{N}_F(p_0 \wedge r_0)]^c &\leq max\{1 - \mathcal{N}_F(p_0), 1 - \mathcal{N}_F(r_0)\} \\ [\mathcal{N}_F(p_0 \wedge r_0)]^c &\leq max\{[\mathcal{N}_F(p_0)]^c, [\mathcal{N}_F(r_0)]^c\}. \end{aligned}$$

Hence, $\odot\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$. □

Theorem 3.14. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic set in a commutative BCK-algebra $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of a $\mathcal{K}$ if and only if $\boxdot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ and $\odot\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ are FNPI of $\mathcal{K}$.*

Definition 3.15. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic set in a $\mathcal{K}$. Then for $a, b, c \in [0, 1]$, the set $U(\mathcal{N}; \langle a, b\rangle)$ is defined as

$$U(\mathcal{N}; \langle a, b\rangle) = \{p_0 \in \mathcal{K} : \mathcal{N}_T(p_0) \geq a, \mathcal{N}_I(p_0) \geq b\},$$

and it is called the upper (a, b)-level cut of $\mathcal{N}$. Similarly, the set $L(\mathcal{N}; c)$ is defined as $L(\mathcal{N}; c) = \{p_0 \in \mathcal{K} : \mathcal{N}_F(p_0) \leq c\}$, and it is called the lower c-level cut of $\mathcal{N}$.

Theorem 3.16. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic set in a $\mathcal{K}$.Then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$ if, for all $a, b, c \in [0, 1]$, the sets $U(\mathcal{N}; \langle a, b\rangle)$ and $L(\mathcal{N}; c)$ are prime ideals of $\mathcal{K}$.*

Proof. Suppose the sets $U(\mathcal{N}; \langle a, b\rangle)$ and $L(\mathcal{N}; c)$ are prime ideals of $\mathcal{K}$. Then $U(\mathcal{N}; \langle a, b\rangle)$ and $L(\mathcal{N}; c)$ are ideals of $\mathcal{K}$ (see [4]). On contrary $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is not a FNPI of $\mathcal{K}$. Then there exists $p_0, r_0 \in \mathcal{K}$ such that

$$\mathcal{N}_T(p_0 \wedge r_0) > max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$$
$$\mathcal{N}_I(p_0 \wedge r_0) > max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$$
$$\mathcal{N}_F(p_0 \wedge r_0) < min\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$$

Let $a = \frac{1}{2}\{\mathcal{N}_T(p_0 \wedge r_0) + max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}\}$ and

$$b = \frac{1}{2}\{\mathcal{N}_I(p_0 \wedge r_0) + max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}\}$$

$\Rightarrow \mathcal{N}_T(p_0 \wedge r_0) > a > max\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$ and

$$\mathcal{N}_I(p_0 \wedge r_0) > b > max\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}.$$

$\Rightarrow p_0 \wedge r_0 \in U(\mathcal{N}; \langle a, b\rangle)$ but $p_0 \notin U(\mathcal{N}; \langle a, b\rangle)$ *and* $r_0 \notin U(\mathcal{N}; \langle a, b\rangle)$. This is a contradiction. Hence $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNPI of $\mathcal{K}$. □

4. CONCLUSION

The fuzzy neutrosophic prime ideals concept expands the field of algebraic structures by introducing a new dimension to prime ideals. By extending these ideals within the realm of fuzzy neutrosophic sets, we have also opened up exciting possibilities for additional investigation and study.

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