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A NON-COMMUTATIVE GENERALIZATION OF MTL-RINGS

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Abstract. The current work extends the class of commutative MTL-rings established by the authors to the non-commutative ones. That class of rings will be named generalized MTL-rings since they are not necessary commutative. We show that in the non-commutative case, a local ring with identity is a generalized MTL-ring if and only if it is an ideal chain ring. We prove that the ring of matrices over an MTL-ring is a non-commutative MTL-ring. We also study their representation in terms of subdirect irreducibility.

1. Introduction

In recent articles, several authors have investigated classes of rings for which the semiring of ideals is an algebra of a well-known subvariety of residuated lattices. In 2009, Belluce and Di Nola started the study of commutative rings whose ideals form an MV-algebra [2]. Years after, that study was extended to Gödel algebras as it can be seen in [3]. In 2018 Heubo-kwegna et al. [13] pushed that study futher to BL-algebras. Then a year later, Chajda and Langer investigated commutative rings whose ideals are complemented [5]. It was in 2016 and 2022 that Kadji et al. and Atamewoue et al.[1, 18] extended the works [2] and [13] to their non-commutative perspective (pseudo BL-algebras).

In [10] Flondor et al. introduced the notion of *pseudo MTL-algebras* under the name of *weak-pseudo BL-algebras* through the concept of pseudo t-norm which is not necessary a commutative t-norm. Since a continuous pseudo t-nom on the interval [0, 1] is always commutative as it can be seen in [10], then no strict pseudo BL-algebra can be constructed on [0, 1]. However, dropping the pseudo-divisibility property of pseudo BL-algebras leads to *weak-pseudo BL-algebras*, also known as *pseudo MTL-algebras*.

In the current work, we study and characterize rings whose lattice of ideals forms a pseudo MTL-algebra. That class of rings will be named generalized MTL-rings since they are not necessary commutative.

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We start by some preliminaries on pseudo MTL-algebras. Then we continue by showing that, in the non-commutative case, a local ring with identity is a generalized MTL-ring if and only if it is an ideal chain ring, as it is proved for commutative MTL-rings [19]. But this result is not true when rings have no identity. We also realize that the ring of matrices on MTL-rings is a generalized MTL-ring without being a generalized BL-ring. Some other important examples and properties of generalized MTL-rings are given. We end up studying their representation in terms of subdirect irreducibility.

2. Foundations on pseudo MTL-algebras

In this section we give definitions, examples and some properties of pseudo MTL-algebras.

Definition 2.1. An algebraic structure $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ is called a pseudo MTL-algebra if the following conditions are satisfied:

(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.
(ii) $(A, \odot, 1)$ is a monoid, that is, $\odot$ is associative and $x \odot 1 = 1 \odot x = x$.
(ii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$, $\forall x, y, z \in A$.
(iv) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, $\forall x, y, z \in A$, (*pseudo-prelinearity property*).

Definition 2.2. A pseudo MTL-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is called a *pseudo BL-algebra* if $x \wedge y = x \odot (x \rightarrow y) = (x \rightsquigarrow y) \odot x$ (pseudo-divisibility property).

Before giving some examples of pseudo MTL-algebras, let us recall the notion of *t-norm*.

Definition 2.3. A *t-norm* on the segment $[0, 1]$ is a function $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which satisfies the following properties.

(i) *Commutativity:* $T(a, b) = T(b, a)$.
(ii) *Monotonicity:* $T(a, b) \leq T(c, d)$ if $a \leq c$ and $b \leq d$.
(iii) *Associativity:* $T(a, T(b, c)) = T(T(a, b), c)$.
(iv) The number 1 acts as identity element: $T(a, 1) = a$.

Without the commutativity, T is called a *pseudo t-norm*. A *proper pseudo t-norm* is a pseudo t-norm which is not commutative.

The following examples from [7] and [10] show that there are pseudo MTL-algebras which are not pseudo BL-algebras.

Example 2.4. Let $A = \{0, a, b, c, 1\}$ be a chain: $0 < a < b < c < 1$ and the operations $\odot$, $\rightarrow$ and $\rightsquigarrow$ are defined by:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

Then $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo MTL-algebra. But it is not a pseudo BL-algebra because: $(b \rightarrow a) \odot b = a \neq 0 = b \odot (b \rightsquigarrow a)$.

Example 2.5. Let $0 < a_1 < b_1 < 1$. Define $T_{0,1} : [0,1] \times [0,1] \longrightarrow [0,1]$ by

$$T_{0,1}(x,y) = \begin{cases} 0, & if\ 0 \leq x \leq a_1,\ 0 \leq y \leq b_1; \\ \min(x,y), & otherwise. \end{cases}$$

Then $T_{0,1}$ is a pseudo t-norm on $[0,1]$. It is not commutative since $0 = T_{0,1}(a_1, b_1) \neq T_{0,1}(b_1, a_1) = \min(b_1, a_1) = a_1$.
The two implications associated to $T_{0,1}$ are defined by:

$$x \rightarrow y = \begin{cases} \max(a_1, y), & if\ x \leq b_1,\ (x > y); \\ y, & if\ x > b_1,\ (x > y); \\ 1, & if\ (x \leq y). \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} b_1, & if\ x \leq a_1,\ (x > y); \\ y, & if\ x > a_1,\ (x > y); \\ 1, & if\ (x \leq y). \end{cases}$$

Then the algebraic structure $\mathcal{A} = ([0,1], \wedge, \vee, T_{0,1}, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo MTL-algebra which is not a pseudo BL-algebra.

Here below are some essential rules of calculus in pseudo MTL-algebras.

Proposition 2.6 ([10])**.** *The following properties hold in any pseudo MTL-algebra $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ and for all $x, y, z \in A$.*

(1) $x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y)$ *and* $x \odot (x \rightsquigarrow y) \leq x \leq y \rightsquigarrow (y \odot x)$.
(2) $(x \rightarrow y) \odot x \leq x \leq y \rightarrow (x \odot y)$ *and* $(x \rightarrow y) \odot x \leq y \leq x \rightarrow (y \odot x)$.
(3) *If* $x \leq y$*, then* $z \rightsquigarrow x \leq z \rightsquigarrow y$ *and* $z \rightarrow x \leq z \rightarrow y$.
(4) *If* $x \leq y$*, then* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$.
(5) *If* $x \leq y$*, then* $y \rightsquigarrow z \leq x \rightsquigarrow z$ *and* $y \rightarrow z \leq x \rightarrow z$.
(6) $x \leq y$ *iff* $x \rightsquigarrow y = 1$ *iff* $x \rightarrow y = 1$.

(7) $x \rightsquigarrow x = x \to x = 1$.
(8) $1 \rightsquigarrow x = 1 \to x = x$.
(9) $y \leq x \rightsquigarrow y$ *and* $y \leq x \to y$.

In this work, a ring R is not necessary a commutative ring. An ideal of R without any other precision refers to a two-sided ideal of R. Noetherian refers to left and right Noetherian. Without any precision rings use in this work have an identity.

3. Generalized MTL-rings, definitions and first properties

In this section, we generalize the notion of MTL-rings [19] to non-commutative rings by introducing the so-called generalized MTL-rings. Considering that this generalized MTL-rings are not necessary commutative. Main properties and some connections to other classes of rings are presented.

Note 3.1. Let R be a ring, 0_R its neutral element with respect to addition. Let A be an ideal of R.
$O := \{0_R\}$.
$\mathcal{I}d(R)$ denotes the set of all ideals of R.
$A^\bullet := \{x \in R \,/\, xA = O\}$ denotes the right-annihilator of R.
$A^- := \{x \in R \,/\, Ax = O\}$ denotes the left annihilator of R.

A ring R is said to have the condition (*) if for every $x \in R$, there exists an element $e \in R$ such that $e \cdot x = x \cdot e = x$. All the rings used in this paper are supposed to have the condition (*). From [11], there is a residuated lattice formed by the two-sided ideals of the ring R which is the following: $\mathcal{A}(R) := \langle \mathcal{I}d(R), \wedge, \vee, \odot, \to, \rightsquigarrow, O, R\rangle$, where $A \wedge B := A \cap B$, $A \vee B := A + B$, $A \odot B := A \cdot B$, $A \to B := \{x \in R \,/\, xA \subseteq B\}$ and

$$A \rightsquigarrow B := \{x \in R \,/\, Ax \subseteq B\}.$$

Definition 3.2. A ring R is called a generalized MTL-ring if it satisfies the following pseudo-prelinearity condition (gMTL):

$$(A \to B) + (B \to A) = (A \rightsquigarrow B) + (B \rightsquigarrow A) = R,$$

for all $A, B \in \mathcal{A}(R)$.

Definition 3.3. A ring R is called a left MTL-ring if the pseudo-prelinearity condition holds for all the left-ideals of R.

In case the pseudo-prelinearity condition is satisfied for all the right-ideals of a ring, the ring will be named right MTL-ring.

Remark 3.4. As explored in [1] and [13],

(a): the condition (gMTL) in the ring R is equivalent to each of the following conditions:
(gMTL-1):
$$(A \cap B) \rightsquigarrow C = (A \rightsquigarrow C) + (B \rightsquigarrow C) \text{ for all } A, B, C \in \mathcal{I}d(R),$$
(gMTL-2):
$$A \rightsquigarrow (B + C) = (A \rightsquigarrow B) + (A \rightsquigarrow C) \text{ for all } A, B, C \in \mathcal{I}d(R);$$
(b): every pseudo BL-ring is a generalized MTL-ring.

Theorem 3.5. *A ring R is a generalized MTL-ring if and only if $\mathcal{A}(R)$, the lattice of ideals of R, is a pseudo MTL-algebra.*

Proof. The proof follows the same path as the one in [19]. □

Lemma 3.6 ([1])**.** *Let R be a ring. The following conditions hold for all ideals A, B and C such that $A \subseteq B$ and $A \subseteq C$.*

(1) $A \subseteq (A^{\bullet} \cdot B)^{\bullet}$, $A \subseteq B \to A$, $A \subseteq B \to C$, $A \subseteq C \to B$;
(1') $A \subseteq (A^{-} \cdot B)^{-}$, $A \subseteq B \rightsquigarrow A$, $A \subseteq B \rightsquigarrow C$, $A \subseteq C \rightsquigarrow B$;
(2) $(B/A)^{\bullet} = (B \to A)/A$ *and* $(B/A)^{-} = (B \rightsquigarrow A)/A$;
(3) $(B/A) \to (C/A) = (B \to C)/A$ *and* $(B/A) \rightsquigarrow (C/A) = (B \rightsquigarrow C)/A$.

Let us consider the following condition
(gMTL^{*-}): for all ideals A, B of R, if $A \cap B = \{0\}$, then
$$A^{\bullet} + B^{\bullet} = A^{-} + B^{-} = R.$$

Remark 3.7. Every generalized MTL-ring satisfies the condition (gMTL^{*-}). Actually, since $A \to B = A \to (A \cap B)$ and $B \to A = B \to (A \cap B)$, then $A \cap B = O$ implies $A \to B = A \to O = A^{\bullet}$ and $B \to A = B \to O = B^{\bullet}$. Moreover, $A \rightsquigarrow B = A \rightsquigarrow (A \cap B)$ and $B \rightsquigarrow A = B \rightsquigarrow (A \cap B)$, then $A \cap B = O$ implies $A \rightsquigarrow B = A \rightsquigarrow O = A^{-}$ and $B \rightsquigarrow A = B \rightsquigarrow O = B^{-}$. Therefore,
$$A^{\bullet} + B^{\bullet} = (A \to B) + (B \to A) = (A \rightsquigarrow B) + (B \rightsquigarrow A) = A^{-} + B^{-} = R.$$

Proposition 3.8. *A ring R is a generalized MTL-ring if and only if every quotient (by an ideal) of R satisfies the condition (gMTL^{*-}).*

Proof. Same as for commutative MTL-rings [19]. □

Proposition 3.9. *Left(right) MTL-rings are left(right) arithmetical rings.*

Proof. Just notice that the lattice of left(right) ideals of a left(right) MTL-ring is distributive. □

Remark 3.10. Noetherian generalized MTL-rings are pseudo BL-rings. We recall that the same remark is done for the commutative case.

Theorem 3.11. *A left(right) Noetherian ring is a left(right) MTL-ring if and only if it is left(right) arithmetical ring.*

Proof. The proof of this theorem is similar to the one done for the commutative case. Actually, Jensen in [17] characterizes arithmetical rings to be rings for which the prelinearity condition holds for all its finitely generated ideals. Since the ring is Noetherian, all its ideals are finitely generated, which means that the prelinearity condition holds for all its ideals, that is, the ring is an MTL-ring. By replacing ideals by left or right ideals, we still have the similar conclusion. □

The following paragraphs focus on the construction of some generalized MTL-rings.

Definition 3.12. A *non-commutative valuation ring* is a non-commutative ring whose ideals are totally ordered by the set-inclusion.

Proposition 3.13. *A ring R with an identity is a local generalized MTL-ring if and only if R is a valuation ring.*

Proof. Let R be a local generalized MTL-ring not necessary commutative with identity; M the unique maximal ideal of R; and $A, B \in \mathcal{I}d(R)$.

We want to show that $A \subseteq B$ or $B \subseteq A$. It suffices to show that $A \to B = R$ or $B \to A = R$ (since the ring R has an identity).

Suppose that $A \to B \neq R$ and $B \to A \neq R$, then $A \to B \subseteq M$ and $B \to A \subseteq M$, since M is the unique maximal ideal of R. Hence, $R = (A \to B) + (B \to A) \subseteq M$: contradiction because M is a maximal ideal of the generalized MTL-ring R. So $A \subseteq B$ or $B \subseteq A$, that is, R is a valuation ring.

The converse is obvious. □

Remark 3.14. If we drop the hypothesis that the ring has an identity, then the Proposition 3.13 will not be true anymore. It is the case when we consider the example.

Example 3.15. For each $n \geq 2$, $\mathcal{V}_n$ denotes the linearly ordered set of n elements, $\mathcal{R}_n$ denotes the class of rings R such that the lattice of ideals $\mathcal{A}(R)$ is isomorphic to $\mathcal{V}_n$. Let $R \in \mathcal{R}_n$ such that R is commutative with an identity and let M be a simple R-module. Defining the addition and the multiplication on $\hat{R} = R \times M$ by:

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2);\ (r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, m_1 m_2).$$

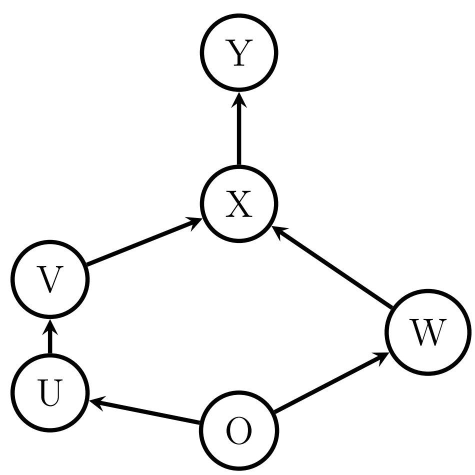
It follows from [20] that $(\hat{R}, +, \cdot)$ is a non-commutative ring with no identity which is an arithmetical local ring. The left ideals of $\hat{R}$ are: $B_k \times 0$, $B_k \times M$, $(1 \leq k \leq n)$, where $0 = B_1 \subset B_2 \subset \cdots \subset B_{n-1} \subset B_n = R$ are ideals of R. It is noticed that all the left ideals of $\hat{R}$ are also right ideals of $\hat{R}$ except the left ideal $B_n \times 0$. So the only proper ideals of $\hat{R}$ are: $B_k \times 0$ and $B_k \times M$, $1 \leq k < n$, with $B_{n-1} \times M$ as the maximal ideal.

The previous Remark and Example 3.15 yield the following proposition.

Proposition 3.16. *For $n \geq 3$, $\hat{R}$ is a local generalized MTL-ring whose lattice of ideals $\mathcal{A}(\hat{R})$ does not form a chain.*

Proof. To prove that $\mathcal{A}(\hat{R})$ is not a chain, it suffices to prove it for $n = 3$. The elements of $\mathcal{A}(\hat{R})$ are of the form:

$$\underbrace{(0) \times (0)}_{O};\quad \underbrace{B_1 \times (0)}_{U};\quad \underbrace{B_2 \times (0)}_{V};\quad \underbrace{B_1 \times M}_{W};\quad \underbrace{B_2 \times M}_{X};\quad \underbrace{(1) \times M}_{Y}:$$



So $\mathcal{A}(\hat{R})$ is not a chain as it is seen from the picture above. (the arrows mean "it is included")

The arithmetical ring $\hat{R}$ is Noetherian since it is the product of Noetherian R-modules R and M. According to Theorem 3.11, $\hat{R}$ is a generalized MTL-ring. □

The following theorem generalizes Proposition 3.13.

Theorem 3.17. *Let R be a ring with an identity such that:*

(1) *R is left(right) local, that is, it has only one maximal left(right) ideal; and*

(2) *R is left(right) MTL-ring.*

Then $\mathcal{A}(R)$, the lattice of ideals of R, is a chain.

Proof. The proof is similar to the proof given for Proposition 3.13. □

The next result is very important because it gives a practical tool for the construction of generalized MTL-rings.

Theorem 3.18. *Let R be an MTL-ring with identity and n a non-negative integer. Then the ring of square matrices $\mathcal{M}_n(R)$ is a generalized MTL-ring.*

Proof. Let R be an MTL-ring and $\mathcal{M}_n(R)$ the ring of square matrices on R ($n \in \mathbb{N}^*$).

(i) $\mathcal{M}_n(R)$ is already non-commutative.

(ii) Ideals of $\mathcal{M}_n(R)$ are of the form $\mathcal{I} = \mathcal{M}_n(I)$ where I is an ideal of R. They satisfy the prelinearity condition. Indeed, let $\mathcal{I}$, $\mathcal{J} \in \mathcal{I}d(\mathcal{M}_n(R))$. $\mathcal{I} = \mathcal{M}_n(I)$ and $\mathcal{J} = \mathcal{M}_n(J)$, where I and J are ideals of R.

$$\mathcal{I} \to \mathcal{J} = \{A \in \mathcal{M}_n(R) \,/\, A \cdot B \in \mathcal{J},\, \forall B \in \mathcal{I}\}.$$

Let I_n be the identity matrix of $\mathcal{M}_n(R)$. To show that

$$(\mathcal{I} \to \mathcal{J}) + (\mathcal{J} \to \mathcal{I}) = \mathcal{M}_n(R),$$

it suffices to show that $I_n \in (\mathcal{I} \to \mathcal{J}) + (\mathcal{J} \to \mathcal{I})$.

Since R is an MTL-ring, then $(I \to J) + (J \to I) = R$. Then $1 = x_0 + y_0$ for certain $x_0 \in I \to J$ and $y_0 \in J \to I$.

$$I_n = \begin{pmatrix} x_0 + y_0 & 0 & \cdots & 0 \\ 0 & x_0 + y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & x_0 + y_0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x_0 & 0 & \cdots & 0 \\ 0 & x_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & x_0 \end{pmatrix}}_{D_{x_0}} + \underbrace{\begin{pmatrix} y_0 & 0 & \cdots & 0 \\ 0 & y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & y_0 \end{pmatrix}}_{D_{y_0}}$$

The matrix D_{x_0} belongs to $\mathcal{I} \to \mathcal{J}$ and D_{y_0} belongs to $\mathcal{J} \to \mathcal{I}$. Indeed, let $A = (a_{ij}) \in \mathcal{I}$, then $D_{x_0}A = x_0A \in \mathcal{J}$ because $x_0 \in I \to J$ and all the $a_{ij} \in I$. So $D_{x_0} \in \mathcal{I} \to \mathcal{J}$. Similarly, $D_{y_0} \in \mathcal{J} \to \mathcal{I}$. So, $I_n \in (\mathcal{I} \to \mathcal{J}) + (\mathcal{J} \to \mathcal{I})$ and this proves that $(\mathcal{I} \to \mathcal{J}) + (\mathcal{J} \to \mathcal{I}) = \mathcal{M}_n(R)$.

(iii) If R is not Noetherian, then $\mathcal{M}_n(R)$ is not Noetherian. Actually, suppose that $\mathcal{M}_n(R)$ is a Noetherian ring. Let $I \in \mathcal{I}d(R)$ be an ideal not finitely generated. Since $\mathcal{M}_n(R)$ is supposed to be a Noetherian ring, then $\mathcal{M}_n(I)$ is finitely generated. That is, there exist m matrices $A_k = (a_{ij}^k)_{1\leq i,j\leq n}$, $k = 1, \cdots, m$, $m \in \mathbb{N}^*$ such that $\mathcal{M}_n(I) = \langle A_1, A_2, \cdots, A_m\rangle$.
So for all $b \in I$,

$$\begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} = \sum_{k=1}^{m} r_k A_k = \sum_{k=1}^{m} r_k (a_{ij}^k) = \sum_{k=1}^{m} (r_k a_{ij}^k).$$

This implies $b = \sum_{k=1}^{m} r_k (a_{11}^k)$ and this is for all $b \in I$, which means that I is finitely generated: this yields a contradiction. Therefore, $\mathcal{M}_n(R)$ is a non-Noetherian ring.
This concludes the proof. □

Corollary 3.19. *Let R be a non-commutative non-Noetherian valuation ring with identity. Then, $\mathcal{M}_n(R)$ is a generalized MTL-ring which is not a pseudo BL-ring.*

Proof. Since the ring R is a non-Noetherian MTL-ring, then $\mathcal{M}_n(R)$ is non-Noetherian. The conclusion falls from the theorem above because $\mathcal{M}_n(R)$ is a generalized MTL-ring. □

Remark 3.20. In Theorem 3.18 above, $\mathcal{M}_n(R)$ in general, is a strict generalized MTL-ring in the sense that it is not a pseudo BL-ring.

We are now going to give another way to construct generalized MTL-rings using the non-commutative version of the Nagata's construction seen in [19]. Before we can do it, let us recall some important condition: the Ore condition (see [6] for more details), which is the condition for a ring to have a fraction field. More precisely, let R be a ring without non trivial zero-divisor.

Right Ore condition: the ring R is said to be *right Ore* if the following condition holds: $aR \cap bR \neq 0$, for all $a, b \in R\backslash\{0\}$, where aR and bR are right principal ideals of R.

Left Ore condition: the ring R is said to be *left Ore* if the following condition holds: $Ra \cap Rb \neq 0$, for all $a, b \in R\backslash\{0\}$, where Ra and Rb are left principal ideals of R.

The ring R is said to be *Ore* or *to have Ore condition* if it is left and right Ore.

Theorem 3.21. *Let V_i, $i = 1, \cdots, n$; $(n \in \mathbb{N})$ be non-Noetherian non-commutative valuation rings without non trivial zero-divisor, all of them with the Ore condition and with the same fraction field. Then the ring $R = \bigcap_{i=1}^{n} V_i$ is a generalized MTL-ring which is not a pseudo BL-ring.*

Proof. The proof follows exactly the same path as the one done for the commutative MTL-ring which can be seen in [19]; just replace ideals by two-sided ideals to have the result. □

Other examples of generalized MTL-rings are the following:
(i) Polynomial rings over non-commutative non-Noetherian valuation rings (or in general, polynomial rings over generalized MTL-rings). Shortly a polynomial ring over a generalized MTL-ring $R[X]$ is a generalized MTL-ring.

(ii) Skew-polynomial rings over a commutative non-Noetherian valuation rings (or in general, skew-polynomial rings over MTL-rings).

Let us now deal with ideals in generalized MTL-rings.

Remark 3.22. In every ring R, the following properties are true for all ideals A, B and C in R.

(a): $A \rightsquigarrow B = A \rightsquigarrow (A \cap B)$;
(b): $(A + B) \rightsquigarrow C = (A \rightsquigarrow C) \cap (B \rightsquigarrow C)$;
(c): $A \rightsquigarrow (B \cap C) = (A \rightsquigarrow B) \cap (A \rightsquigarrow C)$.

Proposition 3.23. *Let R be a generalized MTL-ring with identity, M be a maximal ideal of R and A, B and C ideals of R. One has the following properties:*

(1) *$S^{-1}A \subseteq S^{-1}B$ or $S^{-1}B \subseteq S^{-1}A$, where $S = R \backslash M$ is a multiplication set and $S^{-1}R$ is the localization at M.*
(2) *For all ideal C of R,*
   (i) *$A^aB^b \subseteq A^{a+b} + B^{a+b}$, for all natural integers a and b. So*
   (ii) *$(A + B)^n = A^n + B^n$, for all natural integer n.*
   (iii) *$(A + B)(A \cap B) = AB$.*
   (iv) *$C(A \cap B) = CA \cap CB$.*
   (v) *$C + (A \cap B) = (C + A) \cap (C + A)$*
   (vi) *$C \cap (A + B) = (C \cap A) + (C \cap B)$.*
   (vii) *$C \rightsquigarrow (A + B) = (C \rightsquigarrow A) + (C \rightsquigarrow B)$.*
   (viii) *$(A \cap B) \rightsquigarrow C = (A \rightsquigarrow C) + (B \rightsquigarrow C)$.*

(3) $\sqrt{A+B}=\sqrt{A}+\sqrt{B}$, *where* $\sqrt{A}$ *is the radical of the ideal* A.

(4) *There is an endomorphism* ϕ *of determinant 1 such that :*

$$\begin{array}{rccc}\phi: & R\times R & \longrightarrow & R\times R\\ & (A,B) & \longmapsto & (A+B, A\cap B)\end{array}$$

(5) *Distorted Chinese Remainder Theorem:*

$$R/A\times R/B\cong R/(A+B)\times R/(A\cap B).$$

(6) *More generally, for all* $n\geq 1$, *there is an endomorphism* ϕ *of determinant 1 such that:*

$$\begin{array}{rccc}\phi: & R^n & \longrightarrow & R^n\\ & A=(A_j)_{1\leq j\leq n} & \longmapsto & (\sigma_j(A)),\end{array}$$

where $\sigma_j(A)$ *is homogeneous elementary symmetric polynomial of degree* j *in* A_k*:*

$$\sigma_j(A)=\sum_{k_1<\cdots<k_j}A_{k_1}\cap\cdots\cap A_{k_j}=\bigcap_{k_0<\cdots<k_{n-1}}A_{k_0}+\cdots+A_{k_{n-j}}.$$

In the above properties, the operator $\rightsquigarrow$ can be replaced by $\to$ and the proof follows the same path as the one done in [19] with the operator $\to$.

4. Further on generalized MTL-rings

Proposition 4.1 ([1]). *Generalized MTL-rings are closed under*

(i) *finite direct products,*

(ii) *arbitrary direct sums,*

(iii) *homomorphic images.*

We recall that ([4]) an algebra $\mathcal{A}$ is a subdirect product of an indexed family $(\mathcal{A}_i)_{i\in I}$ of algebras if:

(i) $\mathcal{A}\leq\prod\limits_{i\in I}\mathcal{A}_i$ and

(ii) $\pi_i(\mathcal{A})=\mathcal{A}_i$, for each $i\in I$.

An embedding $\alpha:\mathcal{A}\to\prod\limits_{i\in I}\mathcal{A}_i$ is subdirect if $\alpha(\mathcal{A})$ is a subdirect product of the $\mathcal{A}_i$.

An algebra $\mathcal{A}$ is subdirectly irreducible if for every subdirect embedding $\alpha:\mathcal{A}\to\prod\limits_{i\in I}\mathcal{A}_i$, there is an $i\in I$ such that $\pi_i\circ\alpha:\mathcal{A}\to\mathcal{A}_i$ is an isomorphism.

The famous Birkhoff subdirectly irreducible representation theorem of algebras says that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras (which are its homomorphic images).

Since we are dealing with two-sided ideals of generalized MTL-rings, we have a following representation theorem whose proof is similar to the one for commutative MTL-rings.

Proposition 4.2. *Every generalized MTL-ring R is a subdirect product of a family $\{R_r : r \in R\backslash\{0\}\}$ of subdirectly irreducible generalized MTL-rings satisfying:*

1. *$\mathcal{A}(R)$ is a subdirect product of $\{\mathcal{A}(R_r) : r \in R\backslash\{0\}\}$.*
2. *$\mathcal{A}(R_r)$ is a pseudo MTL-algebra with a unique atom.*

Proof. The proof follows the same path as the one done in [19], Theorem 7.1. □

Example 4.3. Consider $R = \mathcal{M}_2(\mathbb{Z}_4)$ the ring of 2×2 square matrices over the ring $\mathbb{Z}_4$.

R is a generalized MTL-ring since $\mathbb{Z}_4$ is an MTL-ring. Let $r \in R\backslash\{0\}$; let I_r be a maximal ideal among ideals which do not contain r. So, $\bigcap\limits_{r\neq 0} I_r = \{0\}$, every factor R/I_r is subdirectly irreducible and R is a subdirect product of the family $\{R/I_r : r \in R\backslash\{0\}\}$. It can also be seen that, every quotient R/I_r is an MTL-ring. Set $R_r = R/I_r$.

(1) To show that R is a subdirect product of $\{R/I_r : r \in R\backslash\{0\}\}$, consider $\alpha : \mathcal{A}(R) \to \prod\limits_{r\neq 0} \mathcal{A}(R/I_r)$ defined by $\alpha(I)(r) = (I + I_r) \mod I_r$.

(2) Since R_r is subdirectly irreducible, each $\mathcal{A}(R_r)$ is an MTL-algebra with a unique atom.

5. Conclusion

In this work, we extended the notion of MTL-ring to the non-commutative case by defining rings whose ideals form pseudo MTL-algebras.
Useful properties have been presented. For instance, it has been found that in the non-commutative case, a local ring with identity is a generalized MTL-ring if and only if it is an ideal chain ring. But this is not true with non-commutative rings without identity. Some examples of generalized MTL-rings have been studied. We found that non-commutative non-Noetherian valuation rings with identity and $\mathcal{M}_n(R)$, the ring of matrices over R, R being an MTL-ring ($n \in \mathbb{N}^*$), are significant examples of generalized MTL-rings. Further work related to this topic can be the study of rings whose fuzzy ideals form an MTL-algebra as introduced in [21].

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A NON-COMMUTATIVE GENERALIZATION OF MTL-RINGS

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یک تعمیم ناجابجایی از MTL-حلقه‌ها

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در این مقاله، کلاس MTL-حلقه‌های جابجایی که توسط نویسندگان معرفی شده‌اند را به حالت ناجابجایی تعمیم داده می‌شود. این کلاس از حلقه‌ها، MTL-حلقه‌های تعمیم‌یافته نامیده خواهند شد زیرا ضرورتاً جابجایی نیستند. ما نشان می‌دهیم که در حالت ناجابجایی، یک حلقه موضعی یکدار، یک MTL-حلقه تعمیم‌یافته است اگر و تنها اگر یک حلقه زنجیری ایده‌آل باشد. همچنین اثبات می‌کنیم که حلقه ماتریس‌ها روی یک MTL-حلقه، یک MTL-حلقه ناجابجایی است. به علاوه، نمایش آن‌ها را از نظر تحویل‌ناپذیری زیرمستقیم مطالعه می‌کنیم.

کلمات کلیدی: شبه MTL-جبر، MTL-حلقه، حلقه ناجابجایی، حلقه غیر نوتری، حلقه ارزیاب.