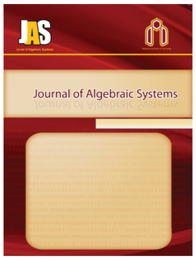
Journal of Algebraic Systems



ISSN:(2345-5128)(2345-511X)
Journal Homepage: www.jas.shahroodut.ac.ir

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To cite this article: B. P. Minh* and N. V. Sanh (15 October 2024): On $\mathscr{Z}$-symmetric modules, Journal of Algebraic Systems,
DOI: 10.22044/jas.2023.13005.1711

To link to this article: https://doi.org/10.22044/jas.2023.13005.1711

Published online: 15 October 2024

Journal of Algebraic Systems, vol. xx, no. xx, (202x), pp xx-xx
https://doi.org/10.22044/jas.2023.13005.1711

ON $\mathcal{Z}$-SYMMETRIC MODULES

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Abstract. A ring R is called a left $\mathcal{Z}$-symmetric ring if $ab \in \mathcal{Z}_l(R)$ implies $ba \in \mathcal{Z}_l(R)$, where $\mathcal{Z}_l(R)$ is the set of left zero-divisors. A right $\mathcal{Z}$-symmetric ring is defined similarly, and a $\mathcal{Z}$-symmetric ring is one that is both left and right $\mathcal{Z}$-symmetric. In this paper, we introduce the concept of $\mathcal{Z}$-symmetric modules as a generalization of $\mathcal{Z}$-symmetric ring. Additionally, we introduce the concept of an eversible module as an analogy to eversible rings and prove that every eversible module is also a $\mathcal{Z}$-symmetric module, like in the case of rings.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules. Let $S = \mathrm{End}(M_R)$ denote the *endomorphism ring* of M. With this notation, M_R becomes an S-R bimodule. A submodule X of M is called a *fully invariant submodule* if $f(X) \subset X$ for any $f \in S$. The class of fully invariant submodules of M is nonempty and closed under intersections and sums. For a submodule X of M, we denote $I_X = \{f \in S \mid f(M) \subset X\}$–the right ideal of S related to X. Following [13], a fully invariant proper submodule X is called a *prime submodule* if $\varphi S m \subset X$, then either $\varphi(M) \subset X$ or $m \in X$ for any $\varphi \in S$ and any $m \in M$ (X is called a *strongly prime submodule* of M if for any $\varphi \in S = \mathrm{End}(M_R)$ and any $m \in M, \varphi(m) \in X$ implies that $\varphi(M) \subset X$ or $m \in X$). Especially, M is *prime* (resp. *strongly prime*) if 0 is the prime (resp. strongly prime) submodule of M. A right R-module M is called a *self-generator* if it generates all its submodules. A nonzero submodule U of M is essential in M if the intersection of U with any nonzero submodule of M is nonzero. We also denote $l_S(m) = \{\varphi \in S \mid \varphi(m) = 0\}$, $r_R(m) = \{r \in R \mid mr = 0\}$ and $l_S(X) = \{\varphi \in S \mid \varphi(x) = 0 \, \forall x \in X\}$, where $m \in M$, $X \subset M$. The readers are referred to [5] and [13] for all undefined concepts and terminologies.

It is well known that a ring R is reduced (= having no nonzero nilpotent elements) if and only if $a^2 = 0$ implies $a = 0$ for all $a \in R$. The concept of

Published online: 15 October 2024
MSC(2020): Primary: 16D10; Secondary: 16D70, 16D80.
Keywords: $\mathcal{Z}$-symmetric ring; $\mathcal{Z}$-symmetric module; Eversible ring; Eversible module.
Received: 25 April 2023, Accepted: 30 September 2023.
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reduced rings was extended to reduced modules by Zhou and Lee [9], that is, a right R-module M is called a *reduced module* if for any $r \in R$ and any $m \in M$, $mr = 0$ implies $mR \cap Mr = 0$. A module M is called *symmetric* [1] if whenever $r, s \in R$ and $m \in M$ satisfy $m(rs) = 0$, then we have $m(sr) = 0$. Following [1] and [2], a module M is said to be *semi-commutative* if $mr = 0$ implies $mRr = 0$ for any $m \in M$ and any $r \in R$. Following Sanh, a submodule X of a right R-module M is *IFP* if for any $m \in M$ and any $\varphi \in S$, if $\varphi(m) \in X$, then $\varphi Sm \subset X$. M is an IFP module if 0 is an IFP submodule. For any ring R, R being IFP is equivalent to R being semi-commutative, but in the case of modules two notions are different.

By [5] an element a of the ring R is a *left (resp. right) zero-divisor* if there exists $0 \neq b \in R$ such that $ab = 0$ (resp. $ba = 0$). If a is both a left and a right zero-divisor, then it is called a zero-divisor. The sets of all left and right zero-divisors in a ring R are denoted by $\mathcal{Z}_l(R)$ and $\mathcal{Z}_r(R)$, respectively. Note that an element $a \in R$ is *regular* if it is not a zero-divisor. A ring in which every left zero-divisor is also a right zero-divisor is called an *eversible ring* [7]. From [6], a ring R is said to be *reversible* if $ab = 0$ implies that $ba = 0$ for any $a, b \in R$.

Recently, A. K. Chaturvedi et al.[5] defined the notion of *$\mathcal{Z}$-symmetric rings*. A ring R is called *left $\mathcal{Z}$-symmetric* if $ab \in \mathcal{Z}_l(R)$, then $ba \in \mathcal{Z}_l(R)$ for any $a, b \in R$. Similarly, a ring R is said to be *right $\mathcal{Z}$-symmetric* if $ab \in \mathcal{Z}_r(R)$, then $ba \in \mathcal{Z}_r(R)$ for any $a, b \in R$. Naturally, R is $\mathcal{Z}$-symmetric if it is both left and right $\mathcal{Z}$-symmetric. It has been shown in [5] that all commutative rings, nil rings, reduced rings and Artinian rings are $\mathcal{Z}$-symmetric rings.

By a mild motivation, we introduce the concept of $\mathcal{Z}$-symmetric modules. In Section 2, we present a modified version of reduced modules that includes left and right-sidedness. We also introduce the definition of reversible modules as a generalization of reversible rings and establish some connections between the concepts introduced in this section. Section 3 starts with the concepts of left and right zero-divisors in modules, which serve as the foundation for the concept of $\mathcal{Z}$-symmetric modules. We then establish some conditions under which the endomorphism ring of a right $\mathcal{Z}$-symmetric module is right $\mathcal{Z}$-symmetric. In Section 4, we introduce the concept of eversible modules, which is an extension of eversible rings. We prove in general that if a module is eversible, then it is $\mathcal{Z}$-symmetric, analogous to the case of rings. Furthermore, we provide an example illustrating that the relationship between reversible modules and eversible modules may not necessarily be the same as in the ring context, where every reversible ring is also an eversible ring. We also show that if a quasi-projective module M_R has a nil endomorphism ring, then its factor module M/X is a right $\mathcal{Z}$-symmetric module. Here, X is a fully invariant proper submodule of M. We finally demonstrate that when the sets of left and right zero-divisors in a $\mathcal{Z}$-symmetric module M_R are proper subsets of it, both the ring R and the endomorphism ring S are Dedekind-finite.

2. Preliminaries

Definition 2.1. We call a right R-module M a *left-reduced module* if for any $\varphi \in S$ and any $m \in M$, $\varphi(m) = 0$ implies that $Sm \cap \varphi(M) = 0$, where $S = \mathrm{End}(M_R)$. A right R-module M is a *right-reduced module* if for any $m \in M$ and any $r \in R, mr = 0$ implies that $mR \cap Mr = 0$.

Example 2.2. We claim that $\mathbb{Z}_p$ is a left-reduced $\mathbb{Z}$-module, where p is a prime. Let $\varphi_k \in S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_p), k = 0, \dots, p-1$ and $x \in \mathbb{Z}_p$ with $\varphi_k(x) = 0$. Note that φ_k is the $\mathbb{Z}$-endomorphism sending 1 to $k \in \mathbb{Z}_p$. One has $\varphi_k(x) = \varphi_k(1)x = kx = 0$. Therefore, either $k = 0$ or $x = 0$. Thus $\varphi_k(\mathbb{Z}_p) \cap Sx = 0$.

Remark 2.3.
- (a) R is a reduced ring iff R_R is left-reduced as a right R-module. Similarly, R is reduced iff R_R is right-reduced.
- (b) It is not difficult to verify that if M_R is left-reduced, then $S = \mathrm{End}(M_R)$ is a reduced ring. Note that if M_R is right-reduced, then we might not conclude that R is a reduced ring.

With a mild modification (see also [9, Lemma 1.2]), we have the following lemma:

Lemma 2.4. *The following are equivalent:*
1. *M_R is left-reduced.*
2. *For any $m \in M$ and any $\varphi \in S$, the following conditions hold:*
   (a) *$\varphi(m) = 0$ implies $\varphi Sm = 0$.*
   (b) *$\varphi^2(m) = 0$ then $\varphi(m) = 0$.*

For the sake of completeness, we present shortly the proof here.

Proof. Assume that (1) holds, and suppose that $\varphi(m) = 0$. It then follows $\varphi\phi m \in Sm \cap \varphi(M) = 0$ for all $\phi \in S$. Thus $\varphi Sm = 0$ and (a) follows. Let $\varphi^2(m) = 0$, then $S\varphi(m) \cap \varphi(M) = 0$. From $\varphi(m) \in S\varphi(m) \cap \varphi(M)$, it follows that $\varphi(m) = 0$, proving (b).

The proof of (2) $\implies$ (1) is routine, hence is omitted. □

Remark 2.5. The class of modules satisfying 2(a) in Lemma 2.4 is said to have *insertion factor property*, in short, *IFP module* (see [3]). Note that if $\varphi^2(m) = 0$ implies that $\varphi Sm = 0$, then M is said to be *strongly IFP* ([3]).

Definition 2.6. An M_R module is called *left-reversible* if S is a reversible ring. If R is a reversible ring, then we call M_R a *right-reversible module.* If M_R is both left and right-reversible, then we call M_R a *reversible module.*

Definition 2.7. Let M be a right R-module. M is called a *left-symmetric module* if for any $f, g \in S$ and any $m \in M$, $fg(m) = 0$ implies that $gf(m) = 0$. M_R is *right-symmetric* if $m(ab) = 0$ implies that $m(ba) = 0$ for any $a, b \in R$ and any $m \in M$. If M_R is both left and right-symmetric, then we call M_R a *symmetric module.*
In case $M_R = R_R$, if R_R is left-symmetric, then R_R is right-symmetric, and vice versa.

Proposition 2.8. *If M is strongly IFP, then M is left-symmetric (see also* [3, Proposition 2.3]*).*

Proof. Let $f, g \in S$ and $m \in M$ such that $fg(m) = 0$. Since M is an IFP module, we have $fgSm = 0$. Hence $fgf(m) = 0$. It then follows that $(gf)^2(m) = 0$. As M is strongly IFP, $gfSm = 0$. Thus $gf(m) = 0$, proving our proposition. □

Proposition 2.9. *If M is left-symmetric, then M is left-reversible.*

Proof. Observe that for $f, g \in S$, $fg = 0$ implies that $fg(m) = 0$ for all $m \in M$. By assumption, $gf(m) = 0$ for all $m \in M$. Thus $gf = 0$ as required. □

Example 2.10.
(a) Clearly, all M_R modules whose endomorphism rings are commutative (e.g. $\mathbb{Z}_n$) are all left-symmetric, left-reversible and IFP.
(b) It is straightforward to show that all *left-reduced modules are strongly IFP* (see Lemma 2.4), hence are IFP. The converse needs not be true. Take $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module. Apparently, $\mathbb{Z}_{12}$ is IFP, however, $\mathbb{Z}_{12}$ is not a left-reduced module. To see this, take $\varphi_6 \in S$ and $2 \in \mathbb{Z}_{12}$. We have $\varphi_6(2) = 0$, but $0 \neq 6 \in \varphi_6(\mathbb{Z}_{12}) \cap S(2)$.

Remark 2.11. The following implications hold for any module M_R: "M_R is left-reduced $\Longrightarrow$ M_R is strongly IFP $\Longrightarrow$ M_R is left-symmetric $\Longrightarrow$ M_R is left-reversible". For the right side, we only have "M_R is right-reduced $\Longrightarrow$ M_R is strongly semi-commutative $\Longrightarrow$ M_R is right-symmetric".

Proposition 2.12. *Let M_R be a self-generator. If M is a left-reversible module, then M is an IFP module.*

Proof. Since M is left-reversible, S is a reversible ring by definition. Hence S is IFP. It is clear that if M is IFP, then S is an IFP ring. Thus it suffices to show that for a self-generator module M, if $S = \mathrm{End}(M_R)$ is IFP, then M is IFP.

Let $\varphi(m) = 0$ with arbitrary $\varphi \in S, m \in M$. Since M is a self-generator, $mR = \sum_{i\in I} \phi_i(M)$, where $\phi_i \in S$ for some set I. It then follows that $0 = \varphi(mR) = \varphi\left(\sum_{i\in I} \phi_i(M)\right) = \sum_{i\in I} \varphi\phi_i(M)$. Consequently, $\varphi\phi_i(M) = 0$ for all i. Then by hypothesis, $\varphi S\phi_i = 0$ for all $i \in I$. Hence $\varphi S(mR) = 0$, proving that $\varphi Sm = 0$. This completes our proof. □

Remark 2.13. Recall that M_R is *faithful* if $r_R(M) = 0$. If M_R is faithful, then we have "M_R is right-reduced $\Longrightarrow$ M_R is strongly semi-commutative $\Longrightarrow$ M_R is right-symmetric $\Longrightarrow$ M_R is right-reversible".

3. $\mathcal{Z}$-symmetric modules

Definition 3.1. An element $m \in M$ is said to be a *right zero-divisor* if there exists a nonzero $\varphi \in S$ such that $\varphi(m) = 0$. It is called a *left zero-divisor* if there is $0 \neq r \in R$ such that $mr = 0$. The set of all right (resp. left) zero-divisors of M is denoted by $\mathcal{Z}_r(M)$ (resp. $\mathcal{Z}_l(M)$). An element $m \in M$ is called a *zero-divisor* if it is both a left and a right zero-divisor.

Example 3.2. (a) Clearly, if $M \neq 0$, then $\mathcal{Z}_r(M)$ and $\mathcal{Z}_l(M)$ are non empty since 0 belongs to these sets.

(b) Consider $\mathbb{Z}_5$ as a $\mathbb{Z}$-module. Nonzero elements of $S = \mathrm{End}(\mathbb{Z}_5)$ are $\varphi_k, k = 1, 2, 3, 4$, where φ_k maps 1 to $k \in \mathbb{Z}_5$. For any $x \in \mathbb{Z}_5$, $\varphi_k(x) = kx$, which is zero if and only if x is zero. Therefore, a nonzero element of $\mathbb{Z}_5$ is not a right zero-divisor. Apparently, every nonzero element of $\mathbb{Z}_5$ is a left zero-divisor. Note also that any prime $p > 1$ serves our purpose.

(c) For an element m of M, $m \in \mathcal{Z}_r(M) \Leftrightarrow l_S(m) \neq 0$. Analogously, $m \in \mathcal{Z}_l(M) \Leftrightarrow r_R(m) \neq 0$.

We are now in a position to define the concept of $\mathcal{Z}$-symmetric modules.

Definition 3.3. An M_R module is said to be *right $\mathcal{Z}$-symmetric* if for any $f, g \in S$ with $fg \in \mathcal{Z}_r(S)$, then $gf(M) \subset \mathcal{Z}_r(M)$. Similarly, we call a module M_R, *left $\mathcal{Z}$-symmetric* if for any r_1, r_2 in R such that $r_1r_2 \in \mathcal{Z}_l(R)$, then

$Mr_2r_1 \subset \mathcal{Z}_l(M)$. A module M_R is called a *$\mathcal{Z}$-symmetric module* if it is both left and right $\mathcal{Z}$-symmetric.

A ring R is *right $\mathcal{Z}$-symmetric* if for any $a, b \in R$, whenever $ab \in \mathcal{Z}_r(R)$, then $ba \in \mathcal{Z}_r(R)$. R is *left $\mathcal{Z}$-symmetric* if for any $a, b \in R$, whenever $ab \in \mathcal{Z}_l(R)$, then $ba \in \mathcal{Z}_l(R)$. R is *$\mathcal{Z}$-symmetric* if it is both left and right $\mathcal{Z}$-symmetric (see [5]).

Remark 3.4.
(a) A ring R is left (resp. right) $\mathcal{Z}$-symmetric iff the module R_R is left (resp. right) $\mathcal{Z}$-symmetric.
(b) In the above definition, $gf(M) \subset \mathcal{Z}_r(M)$ implies $l_S(gf(m)) \neq 0$ for all $m \in M$. Similarly, $Mr_2r_1 \subset \mathcal{Z}_l(M)$ if $r_R(mr_2r_1) \neq 0$ for all $m \in M$. Note that $l_S(gf(M))$ and $r_R(Mr_2r_1)$ need not necessarily be nonzero.
(c) It is clear from the definition that if the ring S (resp. R) is right (resp. left) $\mathcal{Z}$-symmetric, then M_R is right (resp. left) $\mathcal{Z}$-symmetric.

Next, we find some conditions for the right $\mathcal{Z}$-symmetricity of M to be the right $\mathcal{Z}$-symmetricity of S.

Theorem 3.5. *Let M_R be a cyclic module. Then M is right $\mathcal{Z}$-symmetric if and only if S is right $\mathcal{Z}$-symmetric.*

Proof. Let $f, g \in S$, $fg \in \mathcal{Z}_r(S)$. Then by assumption,

$$gf(m)R = gf(mR) = gf(M) \subset \mathcal{Z}_r(M).$$

Since $0 \neq l_S(gf(m))$, then $0 \neq l_S(gf(mR)) = l_S(gf(M))$, proving that $gf \in \mathcal{Z}_r(S)$, hence S is right $\mathcal{Z}$-symmetric. This completes our proof. □

In general, if M_R is a *finitely generated module*, which is a *right $\mathcal{Z}$-symmetric* module, we do not know if the ring S is *right $\mathcal{Z}$-symmetric*.

Recall that a nonzero M_R module is said to be *uniform* if every nonzero submodule of M is *essential* in M. Note that any nonzero submodule of a uniform module is uniform and that any *essential extension* of a uniform module is again uniform. A submodule X of M is called an *M-annihilator* if $X = r_M(T) = \mathrm{Ker}(T)$ for some $T \subset S$.

The *singular submodule* of a right R-module M is defined as follow:

$$Z(M) = \{m \in M \mid mK = 0 \text{ for some essential right ideal } K \text{ of } R\}$$

It is equivalent to say that if $m \in Z(M)$, then $r_R(m) = \{r \in R \mid mr = 0\}$ is an essential right ideal of R. A right R-module M is called a *non-singular module* if $Z(M) = 0$. At the other extreme, M is called a *singular module* if $Z(M) = M$.

Lemma 3.6. *Let M be a non-singular right R-module, and let A, B be M-annihilators of M with A being essential in B. Then $A = B$.*

Proof. The proof can be found in [12], and we present shortly here for the sake of completeness. Let $0 \neq b \in B$, by [4, Lemma 1.1], there is an essential right ideal K of R such that $0 \neq bK \subset A$. Therefore,

$$l_S(A)(bK) \subset l_S(A)A = 0.$$

Since M is non-singular, $l_S(A)(b) \subset Z(M) = 0$, then it follows

$$b \in r_M(l_S(A)) = r_M(l_S(r_M(T)) = r_M(T) = A$$

for some $T \subset S$. Thus $B = A$. □

Theorem 3.7. *Let M be a uniform non-singular right R-module. Then M is right $\mathcal{Z}$-symmetric iff S is right $\mathcal{Z}$-symmetric.*

Proof. It suffices to show the "only if" part. For convenience, we write $r_M l_S(fX)$ instead of $r_M(l_S(f(X))$ and write fm where it is appropriate to refer to $f(m)$ for any $f \in S$, $m \in M$.

Let M be right $\mathcal{Z}$-symmetric, and let $fg \in \mathcal{Z}_r(S)$. Then $gf(M) \subset \mathcal{Z}_r(M)$. We can assume that $gf \neq 0$. Pick an element $a \in M$ with $gf(a) \neq 0$. Because $gf(a)$ is a right zero-divisor, $l_S(gfa) \neq 0$, hence $l_S(gfaR) \neq 0$. Obviously, $l_S(gfaR) \supset l_S(gfM)$, it follows that $r_M l_S(gfaR) \subset r_M l_S(gfM)$. Note also that $r_M l_S(gfaR) \neq 0$, since $l_S(gfaR)(gfaR) = 0$. It follows from M being uniform that $r_M l_S(gfaR)$ is an essential submodule of $r_M l_S(gfM)$. These two sets are clearly M-annihilators. Hence, by Lemma 3.6, $r_M l_S(gfaR) = r_M l_S(gfM)$. One has

$$0 \neq l_S(gfaR) = l_S r_M l_S(gfaR) = l_S r_M l_S(gfM) = l_S(gfM).$$

Therefore, there is $0 \neq h \in S$ such that $hgf(m) = 0$ for all m. Thus $hgf = 0$, proving that $gf \in \mathcal{Z}_r(S)$, and S is right $\mathcal{Z}$-symmetric. □

We restrict ourselves to the commutative case for a while. Let R be a commutative ring, M_R is called a *multiplication module* if every submodule of M is of the form MI for some ideal I of R. It turns out to be that multiplication modules have some nice properties.

Proposition 3.8. *Let M_R be a multiplication right R-module, and let $\varphi \in S$. Then $\forall m \in M, \exists r \in R$ such that $\varphi(m) = mr$. Therefore, $S = End(M_R)$ is commutative, and each submodule of M is fully invariant.*

Proof. This can be proven using similar arguments as in [10, Proposition 1.1] with a note that MI is a fully invariant submodule of M for any ideal I of R. □

Theorem 3.9. *Let M_R be a nonzero multiplication right R-module. Then M is a $\mathcal{Z}$-symmetric module.*

Proof. Since R is commutative, R is a left $\mathcal{Z}$-symmetric module. Similarly, S is commutative [Proposition 3.8], then S is a right $\mathcal{Z}$-symmetric module. Thus M is a $\mathcal{Z}$-symmetric module. □

4. Some properties

In this section, we first define the concept of *eversible modules*. After that, various connections between left (resp. right) $\mathcal{Z}$-symmetric modules and the other types of modules are investigated.

Definition 4.1. A right R-module M is called an *eversible module* if every left zero-divisor is a right zero-divisor, and vice versa. In other words, M is eversible if $\mathcal{Z}_r(M) = \mathcal{Z}_l(M)$. A ring R is *eversible* iff R_R is an eversible module.

Example 4.2. (a) Consider the $\mathbb{Z}$-module $\mathbb{Z}_2$. Clearly, $\mathrm{End}(\mathbb{Z}_2) = \{0, \mathrm{id}\}$. Therefore, $0 \in \mathbb{Z}_2$ is the only right zero-divisor. On the other hand, $1 \in \mathbb{Z}_2$ is clearly a nonzero left zero-divisor of $\mathbb{Z}_2$. Thus $\mathbb{Z}_2$ is not eversible.

(b) In general, if $S = \mathrm{End}(M_R)$ is eversible, we could not conclude that M_R is eversible (see Example 4.2 (a)).

Theorem 4.3. *Let M be a strongly prime module. Then $S = End(M_R)$ is eversible with $\mathcal{Z}_l(S) = \mathcal{Z}_r(S) = 0$.*

Proof. Let $f \in \mathcal{Z}_r(S)$, there exists $0 \neq h$ such that $hf(M) = 0$, that is, $hf(m) = 0$ for all $m \in M$. Since M is strongly prime and $h(M) \neq 0$, it follows that $f(m) = 0$ for all $m \in M$, hence $\mathcal{Z}_r(S) = 0$. Now, let $f \in \mathcal{Z}_l(S)$, $fh(M) = 0$ for some $h \neq 0$. Take an $a \in M$ such that $h(a) \neq 0$. By the same reasoning, $fh(a) = 0$, which implies that $f(M) = 0$ or $h(a) = 0$. Therefore, $f(M) = 0$, proving that $\mathcal{Z}_l(S) = 0$. □

From Theorem 4.3, we can deduce that *any strongly prime module is a right $\mathcal{Z}$-symmetric module.* Before going any further, we need the following lemma.

Lemma 4.4. *Let $f, g \in S$ with $fg \in \mathcal{Z}_r(S)$. Then either $gf(M) \subset \mathcal{Z}_r(M)$ or $f(M) \subset \mathcal{Z}_r(M)$.*

Proof. Let $fg \in \mathcal{Z}_r(S)$ such that $gf(M) \not\subset \mathcal{Z}_r(M)$. Then there exists an $a \in M$ such that $gf(a) \notin \mathcal{Z}_r(M)$, that is $\varphi gf(a) \neq 0$ for all $0 \neq \varphi \in S$. Since $fg \in \mathcal{Z}_r(S), fg(M) \subset \mathcal{Z}_r(M)$. In particular, $fg(f(a)) \in \mathcal{Z}_r(M)$. Hence, there exists $0 \neq h$ such that $(hf)gf(a) = 0$. As we pointed out previously that $l_S(gf(a)) = 0$, then hf must be zero, hence $f \in \mathcal{Z}_r(S)$. Therefore, $f(M) \subset \mathcal{Z}_r(M)$. □

Lemma 4.5. *Let $r_1, r_2 \in R$ with $r_1r_2 \in \mathcal{Z}_l(R)$. Then either $Mr_2r_1 \subset \mathcal{Z}_l(M)$ or $Mr_2 \subset \mathcal{Z}_l(M)$.*

Following [5], *every eversible ring is $\mathcal{Z}$-symmetric.* We now generalize this result.

Theorem 4.6. *Let M_R be eversible, then M_R is a $\mathcal{Z}$-symmetric module.*

Proof. We give only the proof for the case concerning right $\mathcal{Z}$-symmetric module. The left $\mathcal{Z}$-symmetric case can be obtained in a similar fashion. This is a proof by way of contrapositive.

Suppose that M is not right $\mathcal{Z}$-symmetric. Then we could find a pair $f, g \in S$ with $fg \in \mathcal{Z}_r(S)$ such that $gf(M) \not\subset \mathcal{Z}_r(M)$. Therefore, there exists an $x \in M$ such that $gf(x) \notin \mathcal{Z}_r(M)$. Moreover, it follows from Lemma 4.4 that $f(M) \subset \mathcal{Z}_r(M)$, which implies that $f(m) \in \mathcal{Z}_r(M)$ for all $m \in M$. In particular, $f(x) \in \mathcal{Z}_r(M)$. Now, if $gf(x) \in \mathcal{Z}_l(M)$, we are done since $\mathcal{Z}_r(M) \neq \mathcal{Z}_l(M)$. For otherwise, suppose that $gf(x) \notin \mathcal{Z}_l(M)$, it then follows that $gf(x)r \neq 0$ for all nonzero $r \in R$. Consequently, $f(x)r \neq 0$ for all $0 \neq r \in R$, proving that $f(x) \notin \mathcal{Z}_l(M)$. As mentioned earlier, from $f(x) \in \mathcal{Z}_r(M)$, we see that $\mathcal{Z}_l(M) \neq \mathcal{Z}_r(M)$. Therefore, M is not eversible. □

Lemma 4.7. *In an Artinian ring R, if an element $a \notin \mathcal{Z}_l(R)$ (resp. $\mathcal{Z}_r(R)$), then it is right (resp. left) invertible.*

Proof. Consider a stationary descending chain

$$aR \supset \ldots \supset a^nR \supset a^{n+1}R \supset \ldots,$$

where $n \geq 2$. Since a^2 is clearly nonzero, we can choose an $m \geq 1$ such that $0 \neq a^mR = a^{m+1}R$. One has $a^m = a^{m+1}r$ for some r, hence

$$a[a^{m-1}(ar - 1)] = 0.$$

Therefore, $a^{m-1}(ar-1)$ must be zero. Continuing in this process, at the very last step, we get $ar-1=0$ implying that $ar=1$, proving that a is right invertible.

The other case can be obtained similarly. □

Following lemma 4.7, we can deduce that *every Artinian ring is eversible*, and from [5] *every reversible ring is eversible*. Interestingly, these are not true in module cases (see examples below).

Example 4.8. (a) Recall from Example 4.2 that $\mathbb{Z}_2$ is not eversible, which is clearly a reversible $\mathbb{Z}$-module.

(b) Consider $\mathbb{R}^n$ as a vector space over the reals. Clearly, $\mathbb{R}^n$ is an Artinian $\mathbb{R}$-module. For $x \in \mathbb{R}^n$ and $k \in \mathbb{R}$, $xk = 0$ implies that $x = 0$ or $k = 0$. Therefore, $\mathcal{Z}_l(\mathbb{R}^n) = 0$. On the other hand, since $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^n) \cong M_n(\mathbb{R})$, if we pick a nonzero $A \in M_n(\mathbb{R})$ such that A has nonzero null space, then it follows that $\mathcal{Z}_r(\mathbb{R}^n) \neq 0$, proving that $\mathbb{R}^n$ is not eversible.

Recall that *every nil ring is $\mathcal{Z}$-symmetric* [5]. We have the following theorem:

Theorem 4.9. *Let M be a nonzero right R-module. If $S = End(M_R)$ is a nil ring, then M is a right $\mathcal{Z}$-symmetric module. Similarly, if R is a nil ring, then M is a left $\mathcal{Z}$-symmetric module.*

Proof. The proof is straightforward, since every nil ring is $\mathcal{Z}$-symmetric. □

Theorem 4.10. *Let M be a nonzero IFP module. Then M is a right $\mathcal{Z}$-symmetric module. If M is a faithful semi-commutative module, then M is a left $\mathcal{Z}$-symmetric module.*

Proof. We prove only the case of right $\mathcal{Z}$-symmetric module. Let $fg \in \mathcal{Z}_r(S)$. There exists $0 \neq h \in S$ such that $hfg(m) = 0$ for all $m \in M$. From M being IFP, it follows that $hfgS(m) = 0$, particularly $hfgf(m) = 0$ for all $m \in M$. If $hf \neq 0$, then we are done. Otherwise, assuming $hf = 0$, recall that if M is IFP, then S is an IFP ring. It then follows $hgf = 0$, which implies that $hgf(m) = 0$ for all m. Therefore, $gf(M) \subset \mathcal{Z}_r(M)$, proving that M_R is right $\mathcal{Z}$-symmetric. □

Theorem 4.11. *If $M \neq 0$ is left-reversible, then M is right $\mathcal{Z}$-symmetric. If M is right-reversible, then M is left $\mathcal{Z}$-symmetric.*

Proof. Note that M is left-reversible if the ring S is reversible. Let $fg \in \mathcal{Z}_r(S)$, that is, $l_S(fg) \neq 0$. There exists a nonzero h such that

$h(fg) = (hf)g = 0$. Since $(hf)gf = (hfg)f = 0$, if $hf \neq 0$, we are done. For otherwise, $hf = 0 \implies fh = 0 \implies gfh = 0$, hence $hgf = 0$. We conclude that M is right $\mathcal{Z}$-symmetric as desired. □

Since *"M is left-reduced $\implies$ M is strongly IFP $\implies$ M is left-symmetric $\implies$ M is left-reversible $\implies$ M is right $\mathcal{Z}$-symmetric module"*. Therefore, all nonzero left-reduced, strongly IFP, left-symmetric and left-reversible modules are right $\mathcal{Z}$-symmetric modules. On the other hand, if M is faithful, then we have the same results hold for left $\mathcal{Z}$-symmetric case. Note also that "left-reversible" might not imply "IFP". However, as shown in Proposition 2.12, if we were given further that M is a self-generator, then we would have *"M is left-reduced $\implies$ M is strongly IFP $\implies$ M is left-symmetric $\implies$ M is left-reversible $\implies$ M is IFP $\implies$ M is right $\mathcal{Z}$-symmetric module"*.

Next, we turn our attention to factor modules and submodules of M.

Lemma 4.12. *([11]) Let X be a fully invariant submodule of M, and let $\varphi \in End(M)$. Then there is a unique $\bar{\varphi} \in End(M/X)$ such that $\bar{\varphi}\nu = \nu\varphi$, where $\nu : M \to M/X$ is the natural projection.*

Lemma 4.13. *([11]) Let X be a submodule of a quasi-projective module M, and let $\varphi \in End(M/X)$. There is an $f \in End(M)$ such that $\varphi\nu = \nu f$, where ν is the natural projection.*

Lemma 4.14. *([11]) Let M be a quasi-projective right R-module, and let X be a fully invariant submodule of M. Then $End(M/X) \cong S/I_X$.*

Theorem 4.15. *Let M be quasi-projective right R-module whose endomorphism ring is a nil ring. Then M/X is right $\mathcal{Z}$-symmetric, where X is a fully invariant proper submodule of M.*

Proof. First of all, we show that if M is quasi-projective and $S = \text{End}(M_R)$ is a nil ring, then $\bar{S} = \text{End}(M/X)$ is also a nil ring, where X is a fully invariant proper submodule of M. For every $\varphi \in \bar{S} = \text{End}(M/X)$, there exists $f \in S$ such that $\varphi\nu = \nu f$ (Lemma 4.13). Furthermore, since $S = \text{End}(M)$ is nil, there is a positive integer n such that $f^n = 0$. One has $\forall \bar{m} \in M/X$,

$$\varphi^n(\bar{m}) = \varphi^n\nu(m) = \varphi^{n-1}(\varphi\nu)(m) = \varphi^{n-1}(\nu f)(m) = \cdots = \nu f^n(m) = 0.$$

Therefore, $\bar{S}$ is nil. Now since $\bar{S}$ is nil, $\bar{S}$ is a right $\mathcal{Z}$-symmetric ring (every nil ring is $\mathcal{Z}$-symmetric). It then follows that M/X is right $\mathcal{Z}$-symmetric. □

In [8], a ring R is *Dedekind-finite* if $ab = 1$ implies that $ba = 1$ for any $a, b \in R$.

Theorem 4.16. *Let M be a right R-module and suppose that $\mathcal{Z}_r(M) \neq M$. If M is right $\mathcal{Z}$-symmetric, then $S = End(M_R)$ is Dedekind-finite. Similarly, if $\mathcal{Z}_l(M) \neq M$ and M is a left $\mathcal{Z}$-symmetric module, then R is Dedekind-finite.*

Proof. Let $f, g \in S$ such that $fg = 1$. One has $0 = (gf - 1)gf$. If $gf - 1 \neq 0$, then $gf \in \mathcal{Z}_r(S)$. Consequently, $M = fg(M) \subset \mathcal{Z}_r(M)$, a contradiction. Thus $gf = 1$. Similarly, let $r_1 r_2 = 1$, then $r_2 r_1(1 - r_2 r_1) = 0$. If $r_2 r_1 \neq 1$, then $r_2 r_1 \in \mathcal{Z}_l(R)$. It now follows from M being left $\mathcal{Z}$-symmetric that $M = M r_1 r_2 \subset \mathcal{Z}_l(M)$, a contradiction. □

Proposition 4.17. *Let M be a quasi-projective right R-module, and let X be a fully invariant submodule of M. Then M/X is right $\mathcal{Z}$-symmetric if for any $f, g \in S$ and any $\varphi \in S \backslash I_X$ satisfying $\varphi fg \in I_X$, then $\phi gf \in I_X$ for some $\phi \in S \backslash I_X$.*

Proof. By [5, Proposition 2.12], S/I_X is right $\mathcal{Z}$-symmetric. It follows from [11, Lemma 2.13] that $S/I_X \cong \mathrm{End}(M/X)$. Consequently, M/X is right $\mathcal{Z}$-symmetric. □

Acknowledgments

The authors would like to extend sincere appreciation to the anonymous reviewers and the editor for their meticulous review and valuable comments, which greatly contributed to the enhancement of this manuscript. Their insights and suggestions have been instrumental in refining the paper.

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