Journal of Algebraic Systems, Vol. 13(No. 2): (2025), pp 77-94.
https://doi.org/10.22044/JAS.2023.13024.1713

MORE ON TOTAL DOMINATION POLYNOMIAL AND $\mathcal{D}_t$-EQUIVALENCE CLASSES OF SOME GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph of order n. A total dominating set of G is a subset D of V such that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to the minimum cardinality of a total dominating set in G and is denoted by $\gamma_t(G)$. The total domination polynomial of G is the polynomial $D_t(G,x) = \sum_{i=\gamma_t(G)}^{n} d_t(G,i)x^i$, where $d_t(G,i)$ is the number of total dominating sets of G of size i. Two graphs G and H are said to be total dominating equivalent or simply $\mathcal{D}_t$-equivalent, if $D_t(G,x) = D_t(H,x)$. The equivalence class of G, denoted $[G]$, is the set of all graphs $\mathcal{D}_t$-equivalent to G. A polynomial $\sum_{k=0}^{n} a_k x^k$ is called unimodal, if the sequence of its coefficients is unimodal, that means there is some $k \in \{0, 1, \ldots, n\}$, such that $a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$. In this paper, we investigate $\mathcal{D}_t$-equivalence classes of some graphs. Also, we introduce some families of graphs whose total domination polynomials are unimodal. The $\mathcal{D}_t$-equivalence classes of graphs of order ≤ 6 are presented in the appendix.

1. Introduction

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, *the open neighborhood of* v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is the set $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. The set $D \subset V$ is a *total dominating set* if every vertex of V is adjacent to some vertices of D, or equivalently, $N(D) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-set. An i-subset of V is a subset of V of cardinality i. Let $D_t(G,i)$ be the family of total dominating sets of G which are i-subsets and let $d_t(G,i) = |D_t(G,i)|$. The polynomial $D_t(G,x) = \sum_{i=1}^{n} d_t(G,i)x^i$ is defined as *total domination polynomial* of G. A root of $D_t(G,x)$ is called a *total domination root* of G. For many graph polynomials, their roots have attracted considerable attention.

Published online: 1 April 2024
MSC(2020): Primary: 05C30; Secondary: 05C69.
Keywords: Total domination polynomial; Equivalence class; Unimodal.
Received: 28 April 2023, Accepted: 12 September 2023.
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A natural question to ask is to what extent can a graph polynomial describe the underlying graph (for example, a survey of what is known with regards to chromatic polynomials can be found in Chapter 3 of [16]). We say that two graphs G and H are *total domination equivalent* or simply $\mathcal{D}_t$-equivalent (written $G \sim_t H$) if they have the same total domination polynomial. Similarly to domination polynomial [1, 22], we let $[G]$ denote the $\mathcal{D}_t$-equivalence class determined by G, that is $[G] = \{H|H \sim_t G\}$. A graph G is said to be *total dominating unique* or simply $\mathcal{D}_t$-unique if $[G] = \{G\}$. Two problems arise:

(i) Which graphs are $\mathcal{D}_t$-unique, that is, are completely determined by their total domination polynomials?
(ii) Can we determine the $\mathcal{D}_t$-equivalence class of a graph?

Both problems appear difficult, but there are some partial results known. Recurrence relations of graph polynomials have received considerable attention in the literature. It is well-known that the independence polynomial and matching polynomial of a graph satisfies a linear recurrence relation with respect to two vertex elimination operations, the deletion of a vertex and the deletion of vertex's closed neighborhood. Other graph polynomials in the literature satisfy similar recurrence relations with respect to vertex and edge elimination operations [24]. In contrast, it is significantly harder to find recurrence relations for the domination polynomial and the total domination polynomial. The easiest recurrence relation is to remove an edge and to compute the total domination polynomial of the graph arising instead of the one for the original graph. Indeed, for the total domination polynomial of a graph there might be such irrelevant edges, that can be deleted without changing the value of the total domination polynomial at all. An *irrelevant edge* is an edge $e \in E$ of G, such that $D_t(G, x) = D_t(G \setminus e, x)$. These edges can be useful to classify some graphs by their total domination polynomials.

The corona of two graphs G_1 and G_2, as defined by Frucht and Harary in [18], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i-th vertex of G_1 is adjacent to every vertex in the i-th copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be *unimodal*, if there is some $k \in \{0, 1, \ldots, n\}$, called the *mode* of sequence, such that

$$a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n.$$

The mode is unique if $a_{k-1} < a_k > a_{k+1}$. A polynomial is called unimodal if the sequence of its coefficients is unimodal. It is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$. It is *symmetric* if $a_k = a_{n-k}$ for $0 \leq k \leq n$. A log-concave sequence of positive numbers is unimodal (see, e.g., [11, 12, 27]). We say that a polynomial is unimodal (log-concave, symmetric, respectively) if the sequence of its coefficients is unimodal (log-concave, symmetric, respectively). A mode of the sequence $a_0, a_1, \ldots, a_n$ is also called a mode of the polynomial $\sum_{k=0}^{n} a_k x^k$.

Unimodality problems of graph polynomials have always been of great interest to researchers in graph theory [4, 26]. There are a number of results concerning the coefficients of independence polynomials, many of which consider graphs formed by applying some sort of operation to simpler graphs. In [29], for instance, Rosenfeld examines the independence polynomials of graphs formed by taking various rooted products of simpler graphs (see [19] for the definition of the rooted product of two graphs.) In particular, he has shown that the property of having only real roots is preserved under forming rooted products. Mandrescu in [28] has shown that the independence polynomial of corona product of any graph with 2 copies of K_1, i.e., $I(G \circ 2K_1, x)$ is unimodal. Levit and Mandrescu in [27] generalized this result and have shown that if $H = K_r - e$, $r \geq 2$, then the polynomial $I(G \circ H, x)$ is unimodal and symmetric for every graph G. In 2014, Alikhani and Peng conjectured that the domination polynomial is unimodal [9]. In [5], Alikhani and Jahari demonstrated unimodality of the domination polynomials for several families of graphs, including every friendship graph as well as the corona of any graph with P_3 or K_n. Recently, further conditions for unimodality of the domination polynomial have been proved in [10, 13, 25].

Although the unimodality of the independence polynomial and the domination polynomial has been actively studied, almost no attention has been given to the unimodality of the total domination polynomials. We checked the total domination polynomial of graphs of order at most six (see [8]) and observed that all of these polynomials are unimodal. As usual we denote the complete graph, path and cycle of order n by K_n, P_n and C_n, respectively. Also S_n is the star graph with n vertices.

In the next section, we study the total dominating equivalence classes of some graphs such as $G \circ \overline{K_m}$ and $K_{1,n}$. In Section 3, we consider some specific graphs and study the unimodality of their total domination polynomials. The $\mathcal{D}_t$-equivalence classes of graphs of order ≤ 6 are presented in the appendix.

2. $\mathcal{D}_t$-CLASSES OF SOME GRAPHS

Two graphs G and H are said to be total dominating equivalent or simply $\mathcal{D}_t$-equivalent, if $D_t(G,x) = D_t(H,x)$ and written $G \sim_t H$. It is evident that the relation $\sim$ of $\mathcal{D}_t$-equivalent is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $\mathcal{D}_t$-equivalence classes. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} \ : H \sim_t G\}.$$

If $[G] = \{G\}$, then G is said to be total dominating unique or simply $\mathcal{D}_t$-unique.

It is easy to see, if two graphs G and H are isomorphic, then $D_t(G,x) = D_t(H,x)$, but the reverse is not always true. We have shown all graphs of order less than or equal six that are not isomorphic but have the same total domination polynomial, as Figures 8, 9, and 10 in Appendix. Note that all graphs of order one, two and three are $\mathcal{D}_t$-unique. We need the following theorems to obtain more results on $\mathcal{D}_t$-equivalence classes of some graphs:

Theorem 2.1. [15] *Let $G = (V,E)$ be a graph. Then*

$$D_t(G,x) = D_t(G \setminus v, x) + D_t(G \odot v, x) - D_t(G \circledcirc v, x)$$

where $G \odot v$ denotes the graph obtained from G by removing all edges between vertices of $N(v)$ and $G \circledcirc v$ denotes the graph $G \odot v \setminus v$.

Theorem 2.2. [15] *If $G = (V,E)$ is a graph and $e = \{u,v\} \in E$ with $N[v] = N[u]$, then $D_t(G,x) = D_t(G \setminus e, x) + x^2 D_t(G \setminus N[u], x)$.*

We recall that a leaf is a vertex of degree one and a support vertex is defined as a vertex adjacent to a leaf.

Theorem 2.3. [6] *Let G be a graph and $e = \{u,v\}$ be an edge of G. If u and v are adjacent to the support vertices, then e is an irrelevant edge, i.e., $D_t(G,x) = D_t(G \setminus e, x)$.*

Theorem 2.4. *Let G be a graph and $e = \{u,v\} \in E(G)$ be an edge of G satisfying $N_G[u] = N_G[v]$. If there is a vertex w such as $N_G(w) \subseteq N_G(u)$, then e is an irrelevant edge. That means $D_t(G,x) = D_t(G \setminus e, x)$.*

Proof. Let G be a graph and $e = \{u,v\} \in E(G)$ that $N_G[u] = N_G[v]$. By Theorem 2.2 and by the fact that $G \setminus N_G[u]$ has at least one isolate vertex, w, so $D_t(G \setminus N_G[u]) = 0$ and we have the result. □

The (m,n)-lollipop graph is a special type of graph consisting of a complete graph K_m of order m and a path graph on n vertices, connected with a bridge. See Figure 1.

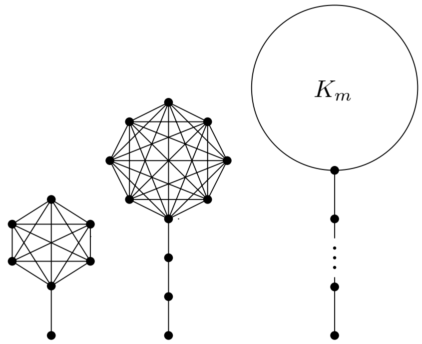


FIGURE 1. The lollipop graphs $L(6,1), L(8,3)$ and $L(m,n)$.

Corollary 2.5. *For every natural number m, the total domination polynomial of $(m,1)$-lollipop graph is equal to*

$$D_t(L(m,1),x) = x(x+1)^m - x.$$

Proof. By Theorem 2.3, all edges of complete graph K_m in $(m,1)$-lollipop graph are irrelevant. So the total domination polynomial of this graph is equal to the total domination polynomial of the star graph $K_{1,m}$ and we have the result. □

Generally, the total domination polynomial of (m,n)-lollipop graphs, i.e., $D_t(L(m,n),x)$ is obtained from the following recursive relation:

$$xD_t(L(m,n-1),x) + x^2[D_t(L(m,n-3),x) + D_t(L(m,n-4),x)],$$

where

$$D_t(L(m,1),x) = x(x+1)^m - x,$$
$$D_t(L(m,2),x) = x^2(x+1)^{m-1}(x+2) - (m-1)x^3 - x^2,$$
$$D_t(L(m,3),x) = x^2(x+1)^m(x+2) - (m-1)x^4 - 2mx^3 - 2x^2,$$
$$D_t(L(m,4),x) = x^2(x+1)^m(x^2+3x+1) - (m-1)x^5 - 2mx^4 - (m+2)x^3 - x^2.$$

Theorem 2.6. *Let G be a graph of order n with a vertex v of degree $\deg(v) = n-1$. Then G is $\mathcal{D}_t$-unique if and only if $G \setminus v$ is $\mathcal{D}_t$-unique.*

Proof. By Theorem 2.1, we have

$$D_t(G,x) = D_t(G \setminus v,x) + D_t(G \odot v,x) - D_t(G \circledcirc v,x)$$

where $D_t(G \odot v, x) = D_t(K_{1,n-1}, x)$ and $D_t(G \circledcirc v, x) = 0$. So we have the result. □

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. The Friendship theorem of Paul Erdős, Alfred Rényi and Vera T. Sós [17], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 4 shows some examples of friendship graphs.

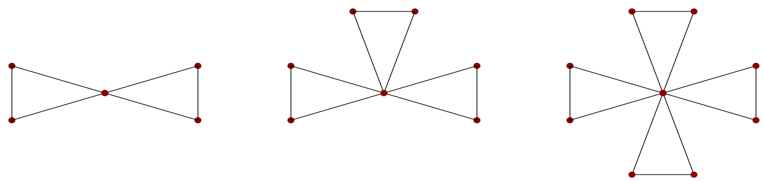


FIGURE 2. Friendship graphs F_2, F_3 and F_4, respectively.

Corollary 2.7. i) *For every $n > 0$, K_n is $\mathcal{D}_t$-unique.*
ii) *The friendship graph F_n is $\mathcal{D}_t$-unique, for every $n \geq 3$.*

Proof. i) The result follows induction and Theorem 2.6.
ii) By Theorem 2.6, since $F_n \setminus v$ is D_t-unique, where v is the center vertex of F_n, so we have the result. □

Theorem 2.8. *For every natural number $n > 2$, $K_{1,n}$ is not $\mathcal{D}_t$-unique and especially $[K_{1,n}] \supseteq \{K_{1,n}, L(n,1), L(n,1) - e, \ldots\}$ where e is any edge of complete graph K_n in lollipop graph that is not adjacent to the pendent edge of this graph.*

Proof. Let v be the center vertex in $K_{1,n}$. We have $D_t(K_{1,n} \setminus v, x) = 0$, so $K_{1,n} \setminus v$ is not $\mathcal{D}_t$-unique and by Theorem 2.6 we have the result. Also, by Theorem 2.5 the second part of theorem is achieved. □

Now, we introduce an infinite family of graphs such that are total dominating equivalent with $G \circ \overline{K_m}$. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. By $G(v_1^{m_1}, v_2^{m_2}, \ldots, v_n^{m_n})$, we mean the graph obtained from G by joining m_i new vertices to each v_i, for $i = 1, \ldots, n$, where $m_1, \ldots, m_n$ are positive integers; this graph is called sunlike. We note that by the new notation, $G \circ K_1$ is equal to $G(v_1^1, v_2^1, \ldots, v_n^1)$.

Theorem 2.9. *Let G be a connected graph of order n. Any graphs in the family*

$$\{G \circ \overline{K_m}, (G \circ \overline{K_m}) \circ \overline{K_m}, ((G \circ \overline{K_m}) \circ \overline{K_m}) \circ \overline{K_m}, \ldots\}$$

is not $\mathcal{D}_t$-unique.

Proof. Actually for every connected graph G of order n,

$$[G \circ \overline{K_m}] \supseteq \{G \circ \overline{K_m}, G(v_1^{m_1}, v_2^{m_2}, \dots, v_n^{m_n})\},$$

where $\sum_{i=1}^{n} m_i = mn$ and for every i, $m_i \geq 1$. □

3. Unimodality of total domination polynomial

In this section, we consider some specific graphs and study the unimodality of their total domination polynomials. We think that the total domination polynomial of a graph is unimodal [23]. First, we study the unimodality of the total domination polynomial of some certain graphs.

3.1. **Unimodality of some specific graphs.** We need the following theorem to state and prove some new results for the unimodality of the total domination polynomial of graphs.

Theorem 3.1. [31] *Let $f(x)$ and $g(x)$ be polynomials with positive coefficients.*

i) *If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.*
ii) *If $f(x)$ is log-concave and $g(x)$ is unimodal, then their product $f(x)g(x)$ is unimodal.*
iii) *If both $f(x)$ and $g(x)$ are symmetric and unimodal, then so is their product $f(x)g(x)$.*

If polynomials $P_i(x)$ for $i = 1, 2, \dots, n$ with positive coefficients are log-concave, then $\prod_{k=1}^{n} P_k(x)$ is log-concave as well. Here, we introduce a family of graphs whose total domination polynomial are unimodal.

An (n, k)-firecracker $F(n, k)$ is a graph obtained by the concatenation of n, k-stars S_k by linking one leaf from each. Also, we generalize the definition of firecracker graphs. An $(k_1, k_2, \dots, k_n)$-firecracker $F(k_1, \dots, k_n)$ is a graph obtained by the concatenation of k_i-stars S_{k_i} by linking one leaf from each (see Figure 3).

Theorem 3.2. [6]

(i) *For every natural numbers n and $k \geq 3$,*

$$D_t(F(n, k), x) = (x(x + 1)^{(k-1)} - x)^n.$$

(ii) $D_t(F(k_1, \dots, k_n), x) = \prod_{i=1}^{n} (x(x + 1)^{(k_i-1)} - x).$

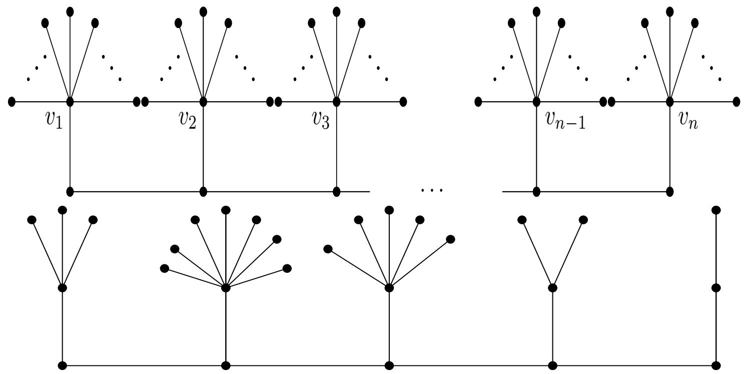


FIGURE 3. The graph $F(n,k)$ and $F(5,9,7,4,3)$.

Theorem 3.3. *For natural numbers n, $k \geq 3$ and $k_i \geq 3$ $(1 \leq i \leq n)$ the total domination polynomial of graphs $L(n,1)$, $F(n,k)$ and $F(k_1,k_2,\ldots,k_n)$ are unimodal.*

Proof. The total domination polynomial of these graphs is equal to the product of the total domination polynomial of some star graphs. For every natural number n , $D_t(K_{1,n},x) = x(x+1)^n - x$ is unimodal and in particular log-concave. So by Theorem 3.1 we have results. □

The following theorem gives many graphs whose the total domination polynomials are unimodal:

Theorem 3.4. *Let G be a graph of order n with r isolated vertices. The total domination polynomial of every graph of the family*

$$\{G \circ \overline{K_m}, (G \circ \overline{K_m}) \circ \overline{K_m}, ((G \circ \overline{K_m}) \circ \overline{K_m}) \circ \overline{K_m}, \cdots\},$$

is unimodal.

Proof. For every graph G of order n with r isolated vertices we have

$$D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{m(n-r)}[(x+1)^n - 1]^r.$$

So by Theorem 3.1 the total domination polynomial of this graph is log-concave and so is unimodal. □

The generalized friendship graph $F_{n,q}$ is a collection of n cycles (all of order q), meeting at a common vertex (see Figure 4). The generalized friendship graph may also be referred to as a flower ([30]). For $q = 3$ the graph $F_{n,q}$ is denoted simply by F_n and is friendship graph as known.

The n-book graph B_n can be constructed by bonding n copies of the cycle graph C_4 along a common edge $\{u,v\}$, see Figure 5. Here we compute the total domination polynomial of n-book graphs.

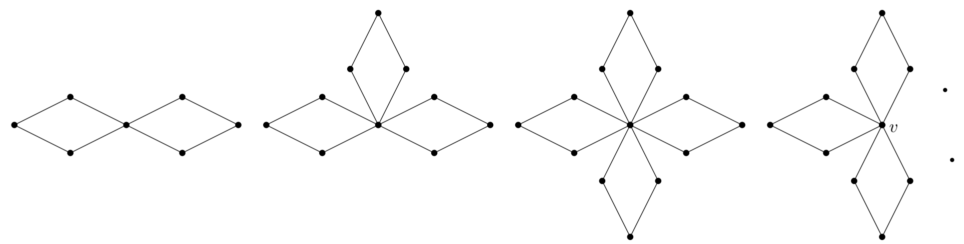


FIGURE 4. Graphs $F_{2,4}$, $F_{3,4}$, $F_{4,4}$ and $F_{n,4}$, respectively.

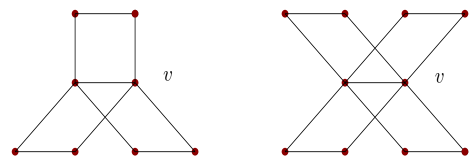


FIGURE 5. The book graphs B_3 and B_4, respectively.

Theorem 3.5. *For each natural number n,*

i) *Total domination polynomial of n-book graph B_n is unimodal.*
ii) *Total domination polynomial of graph $F_{n,4}$ is unimodal.*

Proof. i) We have $D_t(B_n, x) = (x(x+1)^n + x^n)^2$ (see [23]), so

$$D_t(B_n, x) = \big(x^{n+1} + (n+1)x^n + \binom{n}{n-2}x^{n-1} + \ldots + \binom{n}{2}x^3 + nx^2\big)^2.$$

By Theorem 3.1 this polynomial is log-concave and so unimodal.

ii) Since $D_t(F_{n,4}, x) = x^{n+1}(x+2)^n[(x+1)^n + x^{n-1}]$ (see [3]) and

$$\big(2^i\binom{n}{i}\big)^2 \geq 2^{i-1}\binom{n}{i-1}2^{i+1}\binom{n}{i+1} = 2^{2i}\binom{n}{i-1}\binom{n}{i+1},$$

so this polynomial is unimodal. □

Some results about the unimodality of polynomials can be proved by the locations of their roots.

Theorem 3.6. [14] *If a polynomial $p(x)$ with positive coefficients has only real roots, then it is log-concave and unimodal.*

Here, we introduce some family of graphs whose total domination roots are real, and so their total domination polynomial are log-concave and unimodal. The helm graph H_n is obtained from the wheel graph W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel. We consider the generalized helm graph $H_{n,m}$, as the graph is obtained from the wheel graph W_n by attaching m pendent edges at each vertex of the n-cycle of the wheel.

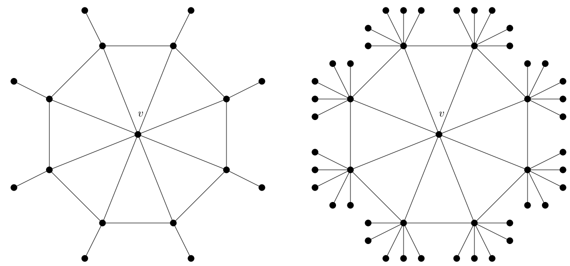


FIGURE 6. Helm graph H_8 and generalized helm graph $H_{8,5}$, respectively.

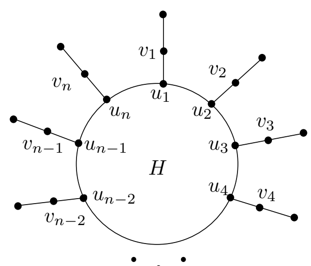


FIGURE 7. The graph $H(3)$.

Theorem 3.7. [3] *For natural numbers m, n,*

$$D_t(H_{n,m}, x) = x^n(x+1)^{mn+1},$$

specially for $m = 1$, $D_t(H_n, x) = x^n(x+1)^{n+1}$.

By the definition, the graph $H(3)$ is obtained by identifying each vertex of H with an end vertex of a P_3 ([7]). See Figure 7.

Theorem 3.8. [6] *For any graph H of order n,*

$$D_t(H(3), x) = x^{2n}(x+2)^n.$$

By Theorems 3.6, 3.7 and 3.8 we have the following corollary:

Corollary 3.9. *The total domination polynomial of graphs H_n, $H_{m,n}$, $H(3)$ and sunlike graphs $G(v_1^{k_1}, v_2^{k_2}, \ldots, v_n^{k_n})$ are unimodal.*

3.2. **Unimodality and minimum degree.** Beaton and Brown in section 3 of [10] have shown that graphs of order n with minimum degree at least $2\log_2(n)$ have unimodal domination polynomial. With the same method, we do it for the total domination polynomial in this subsection. The approach is exactly similar to [10].

For a graph of order n, let $r_i(G)$ proportion of subsets of vertices of G with cardinality i which are total dominating. That is, $r_i(G) = \frac{d_t(G,i)}{\binom{n}{i}}$. As stated in

[10], we have $r_{i+1}(G) \geq r_i(G)$. This allow us to obtain the following lemma. The proof is similar to the proof of Lemma 3.1 in [10].

Lemma 3.10. *Let G be a graph of order n and $k \geq \frac{n}{2}$. If $r_k(G) \geq \frac{n-k}{k+1}$, then $d_t(G, i+1) \leq d_t(G, i)$ for all $i \geq k$. In particular if $k = \lceil \frac{n}{2} \rceil$, then $D_t(G, x)$ is unimodal with mode $\lceil \frac{n}{2} \rceil$.*

The following theorem which is similar to Theorem 3.2 in [10] is useful for the study of the unimodality of the total domination polynomial of most of graphs. The proof of this theorem is almost similar to the proof of Theorem 3.2 in [10], but we prove it here.

Theorem 3.11. *If G is a graph of order n with minimum degree $\delta(G) \geq 2\log_2(n) + 1$, then $D_t(G, x)$ is unimodal with mode at $\lceil \frac{n}{2} \rceil$.*

Proof. Set $\delta = \delta(G)$, $d_i = d_t(G, i)$ and $r_i = r_i(G)$ for all i. Let n_i denote the number of non-total dominating subsets $S \subseteq V(G)$ of cardinality i. Note that $n_i = \binom{n}{i} - d_i$ and so

$$r_i = \frac{d_i}{\binom{n}{i}} = 1 - \frac{n_i}{\binom{n}{i}}.$$

We now show that $n_i \leq n\binom{n-\delta}{i}$. For each vertex $v \in V$, let $n_i(v)$ denote the number of subsets which do not total dominate v. A subset S does not dominate v if and only if it does not contain any vertices in $N(v)$. Therefore $n_i(v)$ simply counts every subset of $V(G)$ with i vertices which omits $N(v)$. So $n_i(v) = \binom{n-\deg(v)}{i}$. Furthermore any non-total dominating set of size i must not total dominate some vertex of G. Therefore

$$n_i \leq \sum_{v\in V} n_i(v) = \sum_{v\in V} \binom{n-\deg(v)}{i} \leq \sum_{v\in V} \binom{n-\delta}{i} = n\binom{n-\delta}{i}.$$

So we have

$$\begin{aligned} r_i &= 1 - \frac{n_i}{\binom{n}{i}} \\ &\geq 1 - \frac{n\binom{n-\delta}{i}}{\binom{n}{i}} \\ &= 1 - \frac{n(n-\delta)!}{i!(n-\delta-i)!} . \frac{i!(n-i)!}{n!} \\ &= 1 - \frac{(n-\delta)!}{(n-1)!} . \frac{(n-i)!}{(n-\delta-i)!} \end{aligned}$$

$$= 1 - \frac{(n-i)(n-i-1)...(n-i-\delta+1)}{(n-1)...(n-\delta+1)}.$$

Note that for any $k \geq 0$, $\frac{n-i-k}{n-k} \geq \frac{n-i-k-1}{n-k-1}$ holds as $i \geq 0$. Therefore

$$\frac{n-i}{n} \geq \frac{n-i-1}{n-1} \geq ... \geq \frac{n-i-\delta+1}{n-\delta+1},$$

and so

$$r_i \geq 1 - (n-i)(\frac{n-i}{n})^{\delta-1}.$$

Now suppose that $f(x,\delta) = 1-(n-x)\left(\frac{n-x}{n}\right)^{\delta-1}$ and $g(x) = \frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$ for $x, \delta \in [0,n]$. Note that $f(x,\delta)$ is an increasing function of both x and δ and $g(x)$ is also a decreasing function of x. By Lemma 3.10, it suffices to show $f(\frac{n}{2}, 2\log_2(n)+1) \geq g(\frac{n}{2})$. Note

$$f(\frac{n}{2}, 2\log_2(n)+1) = 1 - \frac{n}{2}(\frac{1}{2})^{2\log_2(n)} = 1 - \frac{n}{2n^2} = 1 - \frac{1}{2n},$$

and

$$g(\frac{n}{2}) = \frac{\frac{n}{2}}{\frac{n}{2}+1} = \frac{n}{n+2} = 1 - \frac{2}{n+2}.$$

Therefore $f(\frac{n}{2}, 2\log_2(n)+1) \geq g(\frac{n}{2})$ if and only if $\frac{2}{n+2} \geq \frac{1}{2n}$ which holds for all $n \geq 1$. □

Here, similar to [13] we state and prove a result for the unimodality of the total domination polynomial of certain regular graphs. First we need the following theorem:

Theorem 3.12. [23] *Let G be a graph of order n. Then for every $0 \leq i < \frac{n}{2}$ we have $d_t(G,i) \leq d_t(G,i+1)$.*

Theorem 3.13. *Let $G = (V,E)$ be an m-regular graph on $2n$ vertices for some $n \geq 4$ and $m \geq n-1$. Then, $D_t(G,x)$ is unimodal with a mode at n.*

Proof. Note that by by Theorem 3.12, we have

$$d_t(G,1) \leq d_t(G,2) \leq ... \leq d_t(G,n-1) \leq d_t(G,n).$$

By the degree condition, each vertex v has a neighborhood of size at least $n-1$. Every set of vertices that does not total dominate v must be a subset of $V \setminus N(v)$, which has size $2n - |N(v)| \leq 2n-(n-1) = n+1$, and thus there is at most one such set of size $n+1$. Iterating over the vertices, we see that there are at most $2n$ non-total dominating sets of size $n+1$, and furthermore every set of size at least $n+1$ is total dominating. Thus,

$\binom{2n}{n} - 2n \leq d_t(G,n) \leq \binom{2n}{n}$ and $d_t(G,n+r) = \binom{2n}{n+r}$ for $1 \leq r \leq n$. We clearly have $d_t(G,n+1) \geq d_t(G,n+2) \geq ... \geq d_t(G,2n)$. For $n \geq 4$, it is straightforward to check that $\binom{2n}{n} - 2n \geq \binom{2n}{n+1}$. Therefore we can conclude that $D_t(G,x)$ is unimodal with a mode at n. □

Example 3.14. By Theorem 3.13, the total domination polynomial of complete graphs, cube graph Q_3 and Octahedron are unimodal.

We end this paper with the following corollary.

Corollary 3.15. *The total domination polynomial of Cartesian product of* K_n *and* K_2*, i.e.,* $D_t(K_n \square K_2, x)$ *is unimodal.*

Proof. Since $K_n \square K_2$ is an n-regular graph of order $2n$, so we have the result by Theorem 3.13. □

4. Appendix: $\mathcal{D}_t$-equivalence classes of graphs of order ≤ 6

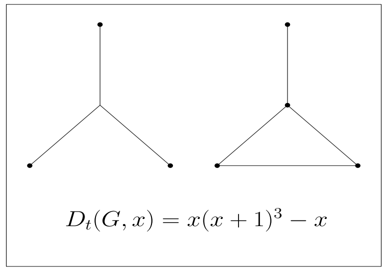


Figure 8. The $\mathcal{D}_t$-equivalence class of connected graphs of order 4.

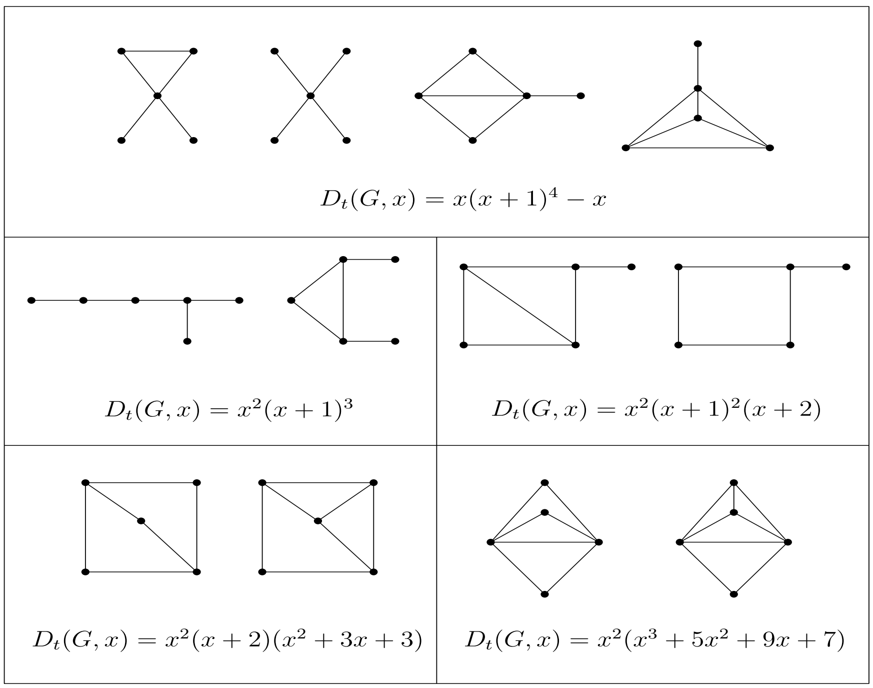
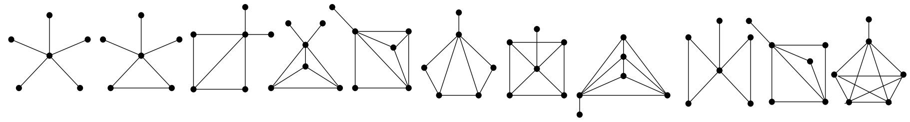
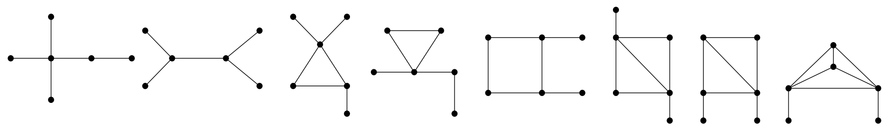


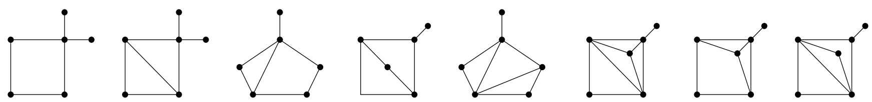
Figure 9. The $\mathcal{D}_t$-equivalence classes of connected graphs of order 5.



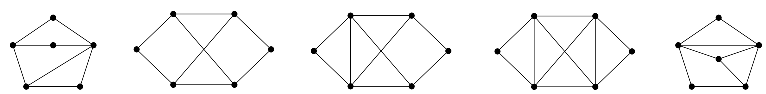
$D_t(G,x) = x(x+1)^5 - x$



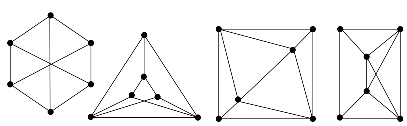
$D_t(G,x) = x^2(x+1)^4$



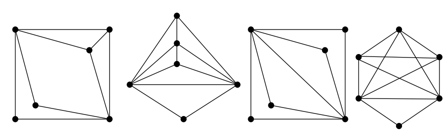
$D_t(G,x) = x^2(x+1)^3(x+2)$



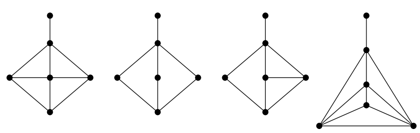
$D_t(G,x) = x^2(x+1)^2(x+2)^2$



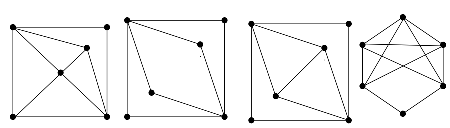
$D_t(G,x) = x^2(x^2+3x+3)^2$



$D_t(G,x) = x^2(x^4+6x^3+14x^2+16x+9)$



$D_t(G,x) = x^2(x+1)^2(x^2+3x+3)$



$D_t(G,x) = x(x+2)^2(x^2+2x+2)$

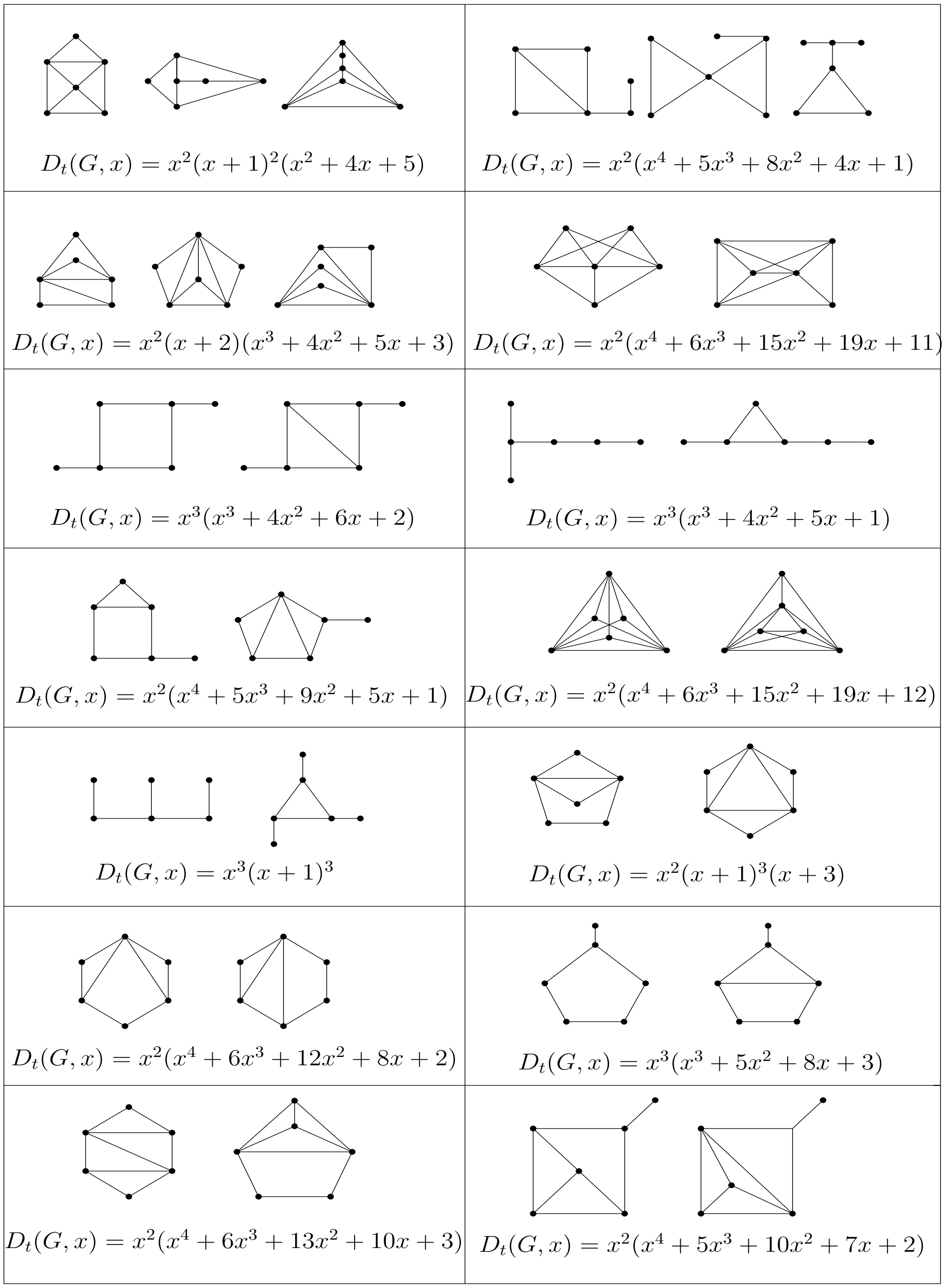


FIGURE 10. The $\mathcal{D}_t$-equivalence classes of connected graphs of order 6.

Acknowledgments

The authors would like to express their gratitude to the referees for their careful reading and helpful comments.

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Journal of Algebraic Systems

MORE ON TOTAL DOMINATION POLYNOMIAL AND

$\mathcal{D}_t$-EQUIVALENCE CLASSES OF SOME GRAPHS

S. ALIKHANI AND N. JAFARI

مطالعه بیشتر چندجمله‌ای احاطه‌گر تام و کلاس‌های $\mathcal{D}_t$ هم‌ارزی برخی گراف‌ها

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فرض کنید $G = (V, E)$ یک گراف ساده از مرتبه n است. مجموعه احاطه‌گر تام G یک زیرمجموعه D از V است به‌طوری که هر راس V در مجاورت برخی از رئوس D است. عدد احاطه‌گر تام G برابر با حداقل اندازه یک مجموعه احاطه‌گر تام در G است و با $\gamma_t(G)$ نشان داده می‌شود. چندجمله‌ای احاطه‌گر تام G چندجمله‌ای $D_t(G, x) = \sum_{i=\gamma_t(G)}^{n} d_t(G, i)x^i$ است، که در آن $d_t(G, i)$ تعداد مجموعه‌های احاطه‌گر تام G با اندازه i است. دو گراف G و H هم‌ارز احاطه‌گری تام یا به سادگی $\mathcal{D}_t$- هم‌ارز هستند، اگر $D_t(G, x) = D_t(H, x)$. کلاس هم ارزی G، که با $[G]$ نشان داده می‌شود، مجموعه همه گراف‌ها $\mathcal{D}_t$-هم‌ارز G است. یک چندجمله‌ای $\sum_{k=0}^{n} a_k x^k$ تک‌مدول نامیده می‌شود، اگر دنباله ضرایب آن تک‌مدول باشد، یعنی عدد $k \in \{0, 1, \ldots, n\}$، موجود باشد به‌گونه‌ای که $a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$. در این مقاله، کلاس‌های $\mathcal{D}_t$-هم‌ارزی برخی گراف‌ها را بررسی می‌کنیم. همچنین تعدادی از خانواده‌ها از گراف‌ها را معرفی می‌کنیم که چندجمله‌ای احاطه‌گر تام آن‌ها تک‌مدول هستند. کلاس‌های $\mathcal{D}_t$-هم‌ارزی گراف‌های از مرتبه ≤ 6 در پیوست ارائه شده‌اند.

کلمات کلیدی: چندجمله‌ای احاطه‌گر تام، کلاس هم‌ارزی، تک‌مدول.