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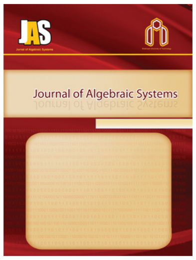

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WEAK IDEMPOTENT NIL-CLEAN RINGS

B. ASMARE, T. ABEBAW AND K. VENKATESWARLU*

Abstract. We introduce the concept of a weak idempotent nil-clean ring as a generalization of a weakly nil-clean ring. We give certain characterizations for weak idempotent nil-clean rings in terms of Jacobson radical and nil-radical. Further, we obtain any weak idempotent nil-clean ring $R \cong R_1 \times R_2$ where R_1 and R_2 are weak idempotent nil-clean rings such that $2 \in J(R_1)$ and $3 \in J(R_2)$.

1. Introduction

Throughout this paper, R stands for associative ring with unity unless and otherwise stated. We denote the set of all idempotents, nilpotents, units, the Jacobson radical, and the prime radicals (nil-radicals) of a ring R by $Id(R)$, $Nil(R)$, $U(R)$, $J(R)$ and $N(R)$ respectively.
We recall the following definitions from [3]. A ring R is called

1. strongly nil-clean if for each $r \in R$, there exists a nilpotent n and an idempotent e such that $r = n + e$ and $ne = en$.
2. nil-clean if every element can be expressed as a sum of a nilpotent and an idempotent.
3. strongly weakly nil-clean if each element $r \in R$ can be represented as either $r = n + e$ or $r = n - e$, $ne = en$ where n is nilpotent and e is idempotent.
4. weakly nil-clean ring if every element can be written as either a sum or a difference of a nilpotent and an idempotent.
5. clean if every element can be written as a sum of a unit and an idempotent.

The following hold: Strongly nil-clean $\Rightarrow$ nil-clean $\Rightarrow$ weakly nil-clean $\Rightarrow$ clean.

It is observed that every element can be represented as a sum of a certain element and an idempotent element in all the above-said rings. It is quite natural to ask whether the representation can be generalized or not. In any

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ring R, if $a^4 = a^2$ then such a is called weak idempotent element. Clearly, every idempotent is weak idempotent but not conversely. For instance, consider the ring of integers modulo 4. Clearly, every element is a weak idempotent element but 2 is not idempotent. In view of these observations, is it possible to replace the idempotent element with a weak idempotent element in the above-said classes of rings? To some extent the answer is affirmative. In this context, we introduce the notion of weak idempotent nil-clean rings which is a subclass of the class of clean rings and a wider class to the class of weakly nil-clean rings.

In this paper, we introduce the notion of weak idempotent nil-clean rings (for short, win-clean rings) and furnish certain examples. Further, we obtain some basic results concerning weak idempotent nil-clean rings. In the next section, we prove $R/Nil(R)$ is a reduced win-clean ring if and only if R is a commutative win-clean ring. Also, we characterize the win-clean ring in Proposition 2.23. The main result of this paper is that every win-clean ring R is isomorphic to a direct product of win-clean rings R_1 and R_2 where $2 \in J(R_1)$ and $3 \in J(R_2)$.

2. Main results

Definition 2.1. Let R be a ring. An element $a \in R$ is called weak idempotent nil-clean if $a = n + w$ for some nilpotent n and some weak idempotent w. R is said to be weak idempotent nil-clean if every element of R is weak idempotent nil-clean.

Remark 2.2. We denote the set of all weak idempotent elements by $wi(R)$ and weak idempotent nil-clean ring by win-clean ring.

Example 2.3. Let $R = M_2(\mathbb{Z}_3)$. Then R is win-clean ring.

Example 2.4. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then R is win-clean ring.

Remark 2.5. If R is a ring and w is a weak idempotent element, then

(1) $w^{2n} = w^2$, and $w^{2n+1} = w^3$.
(2) $Id(R) \cup -Id(R) \subseteq wi(R)$.

We can easily verify that every weakly nil-clean ring is a win-clean ring using remark 2.5 (2) but the converse is not true. For instance, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is win-clean ring but not weakly nil-clean, since $(2, 1)$ cannot be expressed as a sum or a difference of any nilpotent and any idempotent element in $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Theorem 2.6. *Let R be a ring. If $w \in R$ is weak idempotent, then*

(1) w^k *is a weak idempotent element, i.e,* $w^{2k} = w^{4k}$ *where* $k \in \mathbb{N}$.
(2) w^2 *and* $1 - w^2$ *are idempotent elements.*
(3) $2w^2 - 1$ *and* $w - 1 + w^2$ *are units.*
(4) $w^n - w^{n+2}$ *is nilpotent for every* $n \in \mathbb{N}$.
(5) $(1 - w^2)w^2 = 0$
(6) w *is clean.*

Proof. It is straightforward. □

Definition 2.7. Let R be a ring and I be an ideal of R. Then the weak idempotents can be lifted modulo I if there exists $w \in wi(R)$ for a given $a \in R$ with $a^4 - a^2 \in I$ such that $w - a \in I$.

Proposition 2.8. *Let* I *be a nil ideal of a ring* R. *If* $\bar{w}$ *is a weak idempotent element in* R/I, *then* $\bar{w}$ *can be lifted to a weak idempotent in* R.

Proof. Let $\bar{w} \in R/I$ be a weak idempotent and w be any pre-image for $\bar{w}$. Then $\bar{w}^2 = \bar{w}^4$ implies that $w^2 - w^4 \in I$ or $w^2 \equiv w^4 (modI)$ where w^2 and w^4 are pre-images of $\bar{w}^2$ and $\bar{w}^4$, in R/I respectively. Let $z = 1 - w^2$. Then (a) $w^2 z = z w^2$ and (b) $w^2 + z \equiv 1(modI)$.

Now $w^2 z = w^2 - w^4 \in I$. Then $0 = (w^2 z)^k = w^{2k} z^k$ for some positive integer k. Also, w^{2k} is a pre-image of $\bar{w}$, since $w^{2k} \equiv w^2(modI)$. Conditions (a) and (b) are preserved when w and z are replaced by w^{2k} and z^k. Moreover, condition (c) $w^2 z = z w^2 = 0$ is also preserved.

From condition (b), we have $x = 1 - w^2 - z \in I$. Then $(1 - w^2 - z)^m = 0$ for some positive integer m. Thus $1 = 1 - x^m = (1 - x)(1 + x + \cdots + x^{m-1})$ and it follows that $1 - x$ has an inverse $u = 1 + x + \cdots + x^{m-1}$. u commutes with w and z as x commutes with w and z.

Since $x \in I$, $u \equiv 1(modI)$. We can replace w and z with uw^2 and uz, in this case w is again a pre-image for $\bar{w}$ and also conditions (a), (b), and (c) hold true. Further, it is true that (d) $w^2 + z = 1$. By condition (c), we have $w^2 z = 0$, so it gives that $w^2 = w^2(w^2 + z) = w^4 + w^2 z = w^4$. Therefore, $\bar{w}$ lifted to the weak idempotent w in R. □

Proposition 2.9. *The homomorphic image of any win-clean ring is win-clean.*

Proof. It is straightforward. □

Remark 2.10. The converse of Theorem 2.9 is not true. For instance, consider the canonical epimorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(3)$ given by $\alpha(n) = n + (3)$. Then $\mathbb{Z}_3 \cong \mathbb{Z}/(3)$ is a win-clean ring, but $\alpha^{-1}(\mathbb{Z}/(3)) = \mathbb{Z}$ is not a win-clean ring.

Let R be a ring and M a left R-module. Consider the idealization of R and M given by $R(M) = R \oplus M$. For $(r, m), (s, t) \in R(M)$, product and sum defined as follows:

$$(r, m)(s, t) = (rs, rt + sm); (r, m) + (s, t) = (r + s, m + t).$$

Then $R(M)$ is the ring.

Theorem 2.11. *Let R be a ring and M be a left R-module. Then R is win-clean if and only if $R(M)$ is win-clean.*

Proof. Assume that R is win-clean ring and $(r, m) \in R(M)$ where $r \in R$ and $m \in M$. Then $r = n + w$ for $n \in Nil(R)$ and $w \in wi(R)$. Thus $n^k = 0$ for $k \in \mathbb{N}$. So $(n, m)^{k+1} = (n^{k+1}, (k + 1)n^k m) = (0, 0)$ which implies that $(r, m) = (n + w, m) = (n, m) + (w, 0)$ is win-clean expression of (r, m). Hence, $R(M)$ is win-clean. Conversely, $R \cong R(M)/(0 \oplus M)$ is homomorphic image of $R(M)$. So by Theorem 2.9, R is win-clean ring. □

Proposition 2.12. *Let R be a ring. Then weak idempotent elements in $J(R)$ are nilpotents.*

Proof. Let $w \in J(R)$ be a weak idempotent element. Then $w^2 \in J(R)$ and also $1 - w^2$ is an idempotent element. Again, $w^2 \in J(R)$ implies that $1 - w^2 \in U(R)$. So $1 - w^2$ is both idempotent and unit. Thus $1 - w^2 = 1$, since 1 is the only unit and idempotent element. This implies that $w^2 = 0$. Hence w is nilpotent element. □

Proposition 2.13 ([8])**.** *Let R be a ring and $a, b \in R$ such that $ab \neq ba$. Then*

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} + \sum_{k=0}^{n} D_k b^{n-k}$$

where $d_a(x) = ax - xa$ and $D_k = D_k(b, a) = (a + d_b)^n 1 - a^n$, $D_0(b, a) = 0$, $D_{n+1}(b, a) = d_b a^n + (A + d_b)D_n(b, a)$.

Proposition 2.14. *Let R be a win-clean ring, then $J(R) \subseteq Nil(R)$.*

Proof. Let $a \in J(R)$. Then $a = n + w$, where $n \in Nil(R)$ and $w \in wi(R)$. Then $(a - w)^k = 0$ for some $k \in \mathbb{N}$. So $(w - a)^k \in J(R)$. Now

$$(w - a)^k = \textstyle\sum_{k=0}^{n} \binom{n}{k} w^k a^{n-k} + \sum_{k=0}^{n} D_k a^{n-k}$$

implies that $(w - a)^k - [\sum_{k=0}^{n-1} \binom{n}{k} w^k a^{n-k} + \sum_{k=0}^{n} D_k a^{n-k}] = w^k \in J(R) \cap wi(R)$. Since $J(R)$ does not contain units and non-zero idempotents, w must be nilpotent. Now $a - w, w \in Nil(R)$ which in turn implies that $a \in Nil(R)$. Hence $J(R) \subseteq Nil(R)$. □

Corollary 2.15. *If R is a win-clean ring, then $J(R)$ is nil.*

Remark 2.16. A reduced win-clean ring is a ring in which all the elements are weak idempotents.

Proposition 2.17. *Let R be a commutative ring. Then R is win-clean if and only if $R/Nil(R)$ is a reduced win-clean ring.*

Proof. Assume that R is win-clean ring. Let $\bar{x} = x + Nil(R) \in R/Nil(R)$ for some $x \in R$. Now write

$$\bar{x} = (n + w) + Nil(R) = (n + Nil(R)) + (w + Nil(R)) = w + Nil(R)$$

and $w + Nil(R) \in wi(R/Nil(R))$. This implies that $\bar{x}$ is weak idempotent element in $R/Nil(R)$. Since $\bar{x}$ is arbitrary, $R/Nil(R)$ is reduced win-clean ring. Conversely, assume that $R/Nil(R)$ is win-clean ring and let $r \in R$. Since $R/Nil(R)$ is reduced, $Nil(R/Nil(R)) = \{0\}$ and $r + Nil(R) = w + Nil(R)$ for some $w + Nil(R) \in wi(R/Nil(R))$. Then $w^4 - w^2 \in Nil(R)$. By Proposition 2.12, the weak idempotent $w + Nil(R)$ can be lifted to a weak idempotent $w \in wi(R)$ such that $r - w = n$ for some $n \in Nil(R)$, i.e., $r = n + w$. This shows that r is win-clean. Hence R is win-clean ring. □

Corollary 2.18. *Let R be a commutative ring. Then R is win-clean if and only if $R/N(R)$ is win-clean ring.*

Proof. It is obvious. □

Proposition 2.19. *Let I be a nil ideal of a ring R. R is win-clean if and only if R/I is win-clean.*

Proof. ($\Longrightarrow$) It is obvious.

($\Longleftarrow$) Let $r \in R$. Then $\bar{r} = r + I \in R/I$. We can write $\bar{r} = \bar{n} + \bar{w}$ where $\bar{n} \in Nil(R/I)$ and $\bar{w} \in wi(R/I)$ implies that $r + I = (n + w) + I$. The nilpotent $\bar{n}$ in R/I lift to a nilpotent n in R. To see this, $\bar{n}^k = 0$ for $k \geq 1$ in R/I implies that $n^k \in I$. Since I is nil, $(n^k)^m = 0$. So $n^{km} = 0$ for $m \geq 1$. We know that weak idempotents lift modulo any nil ideal, this allows us to assume that w is a weak idempotent in R. Moreover, $r - n - w \in I$. It follows that $r - w = n + d$ where $d \in I$. Since $n^m = 0$ for some $m \in \mathbb{N}$, we have $(n + d)^k \in I$ because I is ideal of R. Thus $(n + d)^{mk} = 0$ for some $m \in \mathbb{N}$ as I is nil ideal. So $n + d$ is nilpotent. Therefore, R is win-clean, as desired. □

Corollary 2.20. *A ring R is win-clean if and only if $R/J(R)$ is win-clean and $J(R)$ is nil.*

Proof. Since $J(R)$ is nil, the proof follows from Proposition 2.19. □

The converse of Proposition 2.14 is not true. Consider example 1.2 in [7]. If we take a simple domain $F = \mathbb{Z}_5$, then $A = M_2(\mathbb{Z}_5)$ is a ring of 2×2 matrices over integer modulo 5, and $B = D_2(\mathbb{Z}_5)$ is a ring of 2×2 diagonal matrices over integer modulo 5 such that $Nil(B) = \begin{pmatrix} 0 & \mathbb{Z}_5 \\ 0 & 0 \end{pmatrix}$. Define $R = B + A[[x]]_x$, where $A[[x]]$ denotes the formal power series ring with an indeterminate x over a ring R. Then $Nil(R) \subsetneq J(R) = Nil(B) + A[[x]]$ and $R/J(R) \cong \mathbb{Z}_5$. But $\mathbb{Z}_5$ is not win-clean and hence $R/J(R)$ is not win-clean. Therefore, By Corollary 2.20, R is not win-clean ring.

Remark 2.21. It is clear that if $x \in R$ a non-zero central nilpotent, then $1 - xr \in U(R)$ for all $r \in R$. Hence $x \in J(R)$, i.e, the non-zero central nilpotents are contained in Jacobson radical, $J(R)$.

Corollary 2.22. *Let R be a win-clean ring such that the weak idempotents are central. Then $C(R)$, the center of R, is a win-clean ring.*

Proposition 2.23. *The following are equivalent for a ring R:*

(1) *R is win-clean.*
(2) *12 is nilpotent and $R/12R$ is win-clean.*
(3) *$R/J(R)$ is win-clean and $J(R)$ is nil.*

Proof. (1) $\implies$ (2). If $12 = 0$, then we are done. Assume that $12 \neq 0$. As R is a ring with 1, $1 + 1 = 2 \in R$ is the least non-unit central element of R. Then there exist a weak idempotent w and a nilpotent n such that $2 = n + w$. Thus $(2-n)^2 = (2-n)^4 \implies 2^2 - 4n + n^2 = 2^4 - 32n + 24n^2 - 8n^3 + n^4$. So $n(-n^3 + 8n^2 - 23n + 28) = 12$. Hence, 12 is nilpotent. Since R is win-clean, $R/12R$ is win-clean by Proposition 2.19.

(2) $\implies$ (1) follows from Proposition 2.19 and (1) $\iff$ (3) obtained immediately from Corollary 2.20. □

Proposition 2.24 ([5])**.** *Let R be a ring, and let I be any nil-ideal of R. Then R is nil-clean if and only if R/I is nil-clean.*

Proposition 2.25. *A ring R is nil-clean if and only if R is win-clean and $2 \in J(R)$.*

Proof. ($\implies$) Suppose R is nil-clean and $r \in R$. Then $r = n + e$ where $n \in Nil(R)$ and $e \in Id(R)$. Thus $e \in wi(R)$. So r is win-clean and hence R is win-clean. Also, $2 = n + e$ implies that $n = 2$. Thus 2 is central nilpotent. This implies that $2 \in J(R)$.

($\Longleftarrow$) Assume R is win-clean. Then $J(R)$ is nil. As $2 \in J(R)$,

$$2 + J(R) = 0 + J(R).$$

We know that a nilpotent modulo nil ideal lifted to nilpotent in R. So we have $2 = 0$, i.e., $char(R/J(R)) = 2$. Thus for all $r \in R$ we have $2\bar{r} = \bar{0}$ and $1 - 2\bar{r} = \bar{1}$. So $R/J(R)$ is Boolean. Hence $R/J(R)$ is nil-clean. By Proposition 2.24, R is nil-clean. □

Proposition 2.26. *A ring R is weakly nil-clean if and only if R is win-clean and $2 \in J(R)$ or $3 \in J(R)$.*

Proof. ($\Longrightarrow$) Obviously R is win-clean. Assume that $2 \notin J(R)$. Then $2 \in U(R)$ and $6^n = 0$ for some positive integer n as 6 is nilpotent element in R ([3], Theorem 2). Thus $2^n 3^n = 0$ implies that $3^n = 0$. Hence $3 \in J(R)$.

($\Longleftarrow$) Assume that R is win-clean. Then $R/J(R)$ is win-clean and $J(R)$ is nil by Corollary 2.20. If $2 \in J(R)$, then by Proposition 2.25, R is nil-clean. So R is weakly nil-clean. Again, if $3 \in J(R)$, then $3 + J(R) = 0 + J(R)$ and also 2 is invertible in R. we can assume $3 = 0$, so that $char(R/J(R)) = 3$. So $\bar{2}$ is unit in $R/J(R)$. Moreover, $3\bar{r} = \bar{0}$, $\bar{1} - 3\bar{r} = \bar{1}$ and $\bar{2} - 3\bar{r} = \bar{2}$ for all $r \in R$. Thus $R/J(R) \cong \mathbb{Z}_3$ and hence $R/J(R)$ is weakly nil-clean. Therefore, R is weakly nil-clean. □

Proposition 2.27. *A finite direct product $R = \prod R_\alpha$ of rings is win-clean ring if and only if each R_α is win-clean ring.*

Proof. It is straightforward. □

Proposition 2.28. *Let R be a ring. Then R is win-clean ring if and only if $R \cong R_1 \times R_2$ where R_1 is win-clean with $2 \in J(R_1)$ and R_2 is 0 or a win-clean ring with $3 \in J(R_2)$.*

Proof. ($\Longrightarrow$) Suppose R is win-clean ring. Then 12 is nilpotent element in R, so that $(12)^n = 0$ for some positive integer n. Then $4^n R \cap 3^n R = 0$ and $4^n R + 3^n R = R$. Thus $R \cong (R/2^{2n}R) \times (R/3^n R)$ by Chinese remainder theorem. By Proposition 2.27, $R_1 = R/2^{2n}R$ and $R_2 = R/3^n R$ are win-clean rings. Thus 2 is central nilpotent in R_1. So $2 \in J(R_1)$. We can assume $R_2 \neq 0$. Then 3 is central nilpotent in R_2 and hence $3 \in J(R_2)$.

($\Longleftarrow$) It is obvious. □

Corollary 2.29. *The following are equivalent for a ring R.*

(1) *R is a win-clean ring with central weak idempotent elements.*

(2) *$R \cong R_1 \times R_2$, where R_1 is win-clean with weak idempotents are central and $J(R_1)$ nil such that $R_1/J(R_1)$ is Boolean, and R_2 is 0 or $R_2/J(R_2) \cong \mathbb{Z}_3$ with $J(R_2)$ nil.*

(3) *R is win-clean ring with central weak idempotent elements, $J(R)$ is nil, and $R/J(R)$ is isomorphic to either a Boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.*

Proof. (1) $\implies$ (2) Using Proposition 2.28, we can write $R \cong R_1 \times R_2$, where R_1 is win-clean ring with central weak idempotents and $2 \in J(R_1)$; and R_2 is 0 or win-clean ring with central weak idempotents and $3 \in J(R_2)$. Thus $char(R_1/J(R_1)) = 2$ which in turn implies that $\bar{x} = -\bar{x}$ for all $\bar{x} \in R_1/J(R_1)$. Hence, $R_1/J(R_1)$ is Boolean. Assume $R_2 \neq 0$. As R_2 is win-clean and $3 \in J(R_2)$, $R_2/J(R_2)$ is win-clean and $char(R_2/J(R_2)) = 3$. Also, 2 is unit in R_2, since $2 \notin J(R_2)$. From this, we conclude that

$$R_2/J(R_2) = \{3R_2, 1 - 3R_2, 2 - 3R_2\},$$

so that every element of $R_2/J(R_2)$ is nilpotent or invertible. Therefore, $R_2/J(R_2) \cong \mathbb{Z}_3$. Furthermore, by Corollary 2.20, $J(R_1)$ and $J(R_2)$ are nil ideals.

(2) $\implies$ (3) and (3) $\implies$ (1) are straightforward. □

Theorem 2.30. *Let R be a reduced commutative ring. The following statements are equivalent.*

(1) *$R = wi(R)$.*
(2) *R is isomorphic to either a Boolean ring B or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$.*
(3) *For all $x \in R$, $x^4 = x^2$.*
(4) *R is win-clean.*

Proof. For a reduced ring R, (1) $\iff$ (3) $\iff$ (4). Thus, it remains to show the equivalence of (1) and (2).

(1) $\implies$ (2) Suppose $R = wi(R)$. If $y \in R$, y^2 is an idempotent. If R is indecomposable, then either $y^2 = 0$ or $y^2 = 1$ for any $y \in R$. This implies that $y = 0$ or $y^2 = 1$ for all $y \in R$. Thus, each nonzero element of R is a unit and hence R is a field. Hence, R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Next, assume R is not indecomposable. Let $R = S \times T$ and $s \in S$, where S and T are coprime ideals of R, that is, $S + T = R$. Then, $(s, 0)$ is not a unit implies that either $(s, 0) = (0, 0)$, or $(s, 0)^2 = (0, 0)$, or $(s, 0)^2 = (s, 0)$, or $(s, 0)^2 = (s, 0)^4$ and $(s, 0)^2 \neq (1, 0)$. If $(s, 0) = (0, 0)$ or $(s, 0)^2 = (0, 0)$, then $(s, 0) = (0, 0)$ since S is reduced. In this case, S is a field. So, S is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. If $(s, 0)^2 = (s, 0)$, then $s \in Id(S) \cup [-Id(S)]$ and

hence $S = Id(S) \cup [-Id(S)]$. By [4, Theorem 1.13], S is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, where B is a Boolean ring. The same holds for T. As a direct product of two Boolean rings is a Boolean ring we get R is isomorphic to a Boolean ring B, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$. If $(s, 0)^2 = (s, 0)^4$ and $(s, 0)^2 \neq (1, 0)$, then let $y = (s, 0)$. Now $R = R(y^2) \oplus R(1 - y^2)$ is the decomposition of R.
Assume $R(y^2)$ is not a Boolean ring. Then, we show that $R(1-y^2)$ is Boolean. Suppose ry^2 is not idempotent. Then, for any $s \in R$, $ry^2 + (-s)(1 - y^2)$ is not idempotent. Thus, $-(ry^2 + (-s)(1 - y^2)) = -ry^2 + s(1 - y^2)$ is idempotent. So, each $s(1 - y^2)$ is idempotent. Thus, $R(1 - y^2)$ is Boolean and also $2R(1 - y^2) = 0$. Hence, for each $y \in R$, either $2y^2 = 0$ or $2(1 - y^2) = 0$.

If $(0 : 2) = \{y \in R \mid 2y^2 = 0\} = R$, then $char(R) = 2$. Hence, $R = wi(R) = Id(R)$ and so R is Boolean. Now assume $(0 : 2) \neq R$. Then, we claim that $(0 : 2)$ is a maximal ideal of R. Suppose there is a maximal ideal M such that $(0 : 2) \subsetneq M$. Let $y^2 \in M - (0 : 2)$. Then, $y^2 \in wi(R) = R$ and $y^2 \notin (0 : 2)$. Thus, $2y^2 \neq 0$ and hence $2(1-y^2) = 0$. So, $1-y^2 \in (0 : 2) \subseteq M$, a contradiction. Hence, $(0 : 2)$ is a maximal ideal. So, $\bar{R} = R/(0 : 2)$ is an indecomposable ring with $\bar{R} = wi(\bar{R})$. By the idea in the first part of this proof, we have that $\bar{R}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Next we show that $2R \cap (0 : 2) = 0$. Assume that $y \in 2R \cap (0 : 2)$. Then, $y = 2s$ and $2y^2 = 0$. But then $y^2 = y^4 = (2s)^4 = 2(2s)^2(2s)^2 = 2y^2 = 0$. If $2R = 0$, then R is Boolean.

Now assume that $2R \neq 0$. If $2R = R$, then $(0 : 2) = 0$ is a maximal ideal of R. Thus, R is a field and hence by the first paragraph of this proof, it is isomorphic to $\mathbb{Z}_3$. If $2R \neq R$, then $R = 2R \oplus (0 : 2)$, where $(0 : 2)$ is a Boolean ring and $2R \cong R/(0 : 2)$ is isomorphic to $\mathbb{Z}_3$ since $2R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ by the first paragraph of this proof and $2R \nsubseteq (0 : 2)$. Therefore, R is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$, where B is a Boolean ring.

(2) $\implies$ (1) It is obvious. □

Recall that a ring is said to be zero dimensional if every prime ideal is maximal ideal.

Corollary 2.31. *Let R be a commutative ring. The following statements hold.*

(1) *A reduced indecomposable ring is win-clean if and only if it is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In particular, any win-clean domain is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

(2) *A win-clean ring is zero-dimensional.*

Proof. (1) Suppose R is a reduced indecomposable win-clean. Then 0 is the only nilpotent, and its idempotents are only 0 and 1. Let $w \in wi(R)$. Then $w^2 \in Id(R)$ implies that $w^2 = 0$ or $w^2 = 1$. If $w^2 = 0$, then w is both weak idempotent and nilpotent. So $w = 0$. If $w^2 = 1$, then w is a unit and weak idempotent. Now we have $R = \{0, 1, w\}$. Since R is closed under $+$, $w + 1 \in R$ which implies that $w + 1 = 0$ or $w + 1 = 1$, or $w + 1 = w$. If $w + 1 = 0$, then $w = -1$. In this case, $R = \{0, 1, -1\}$ which is isomorphic to $\mathbb{Z}_3$. If $w + 1 = 1$ or $w + 1 = w$, then $w = 0$ as $0 \neq 1$. Hence $R = \{0, 1\}$ which is isomorphic to $\mathbb{Z}_2$. The converse is straightforward.

(2) Let R be a win-clean and P a prime ideal of R. Then R/P is an integral domain. By (1), the quotient R/P is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ and hence P is maximal ideal.

□

Proposition 2.32. *Let R be a win-clean ring with central weak idempotent elements and let $a \in R$. If aR contains no non-zero idempotent. Then a is the sum of two nilpotent elements.*

Proof. Suppose aR contains no non-zero idempotent. Choose $w \in wi(R)$ and $n \in Nil(R)$ such that $a = n + w$. Then

$$aw^3 = nw^3 + w^4 = nw^3 + w^2 = (nw + 1)w^2.$$

So $aw^3(nw+1)^{-1} = (nw+1)w^2(nw+1)^{-1}$ in aR. As nw is nilpotent, $nw+1$ is unit and w^2 is idempotent. Thus $(nw + 1)w^2(nw + 1)^{-1}$ is idempotent. Since aR does not contain non-zero idempotent element, we have

$$(nw + 1)w^2(nw + 1)^{-1} = 0$$

which implies $w^2 = 0$ and hence w is nilpotent. Therefore, a is a sum of two nilpotent elements. □

Definition 2.33. Let R be a ring. Then an element x in R is called the square root of idempotent element if there exists an idempotent element e in R such that $x^2 = e$.

Proposition 2.34. *Let R be a win-clean ring with central weak idempotent elements in which, $2 \in U(R)$. Then every element of R can be written as a sum of nilpotent and a square root of idempotent element.*

Proof. Let $a \in R$. Then $a = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$. Let $v = 2w^2 - 1$. Then $v^2 = (2w^2-1)^2 = 4w^4 - 4w^2 + 1 {=} 4w^2 - 4w^2 + 1 = 1$. Thus $vv^{-1} = (2w^2 - 1)(2w^2 - 1)^{-1} = 1$. Now $v = 2w^2 - 1$ implies $w^2 = (v + 1)/2$ and $[(v+1)/2]^2 = (v+1)/2$. Therefore, w is a square root of idempotent. □

Next, we see that a win-clean ring is a subclass of clean rings.

Theorem 2.35. *Every win-clean ring is clean.*

Proof. Let R be a win-clean ring and $a \in R$. Then $a = n + w$ for some nilpotent n and weak idempotent w. So $a = n+w = (n+w-1+w^2)+(1-w^2)$. By Theorem 2.6, $w-1+w^2$ is unit and $1-w^2 \in Id(R)$. To see $n+w-1+w^2$ is unit. Let $u = w - 1 + w^2$. Then $n + w - 1 + w^2 = n + u$. Since n and $(u^{-1}n)$ are nilpotents, we have $n^m = 0$ and $(u^{-1}n)^m = 0$ for some positive integer m. Now

$$\begin{aligned}(n+u)^{-1} =& [u(1+\frac{n}{u})]^{-1} = [1 - \frac{n}{u} + (\frac{n}{u})^2 - (\frac{n}{u})^3 + \cdots + (-\frac{n}{u})^{m-1}]u^{-1} \\ =& [1 - u^{-1}n + (u^{-1}n)^2 - \cdots + (-1)^{m-1}(u^{-1}n)^{m-1}]u^{-1}.\end{aligned}$$

and so

$$\begin{aligned}(n+u)(n+u)^{-1} =& (n+u)[1 - u^{-1}n + \cdots + (-1)^{m-1}(u^{-1}n)^{m-1}]u^{-1} \\ =& n[1 - u^{-1}n + \cdots + (-1)^{m-1}(u^{-1}n)^{m-1}]u^{-1} \\ &+ u[1 - u^{-1}n + (u^{-1}n)^2 - \cdots + (-1)^{m-1}(u^{-1}n)^{m-1}]u^{-1} \\ =& nu^{-1} - (nu^{-1})^2 + (nu^{-1})^3 - \cdots + (-1)^{m-1}(nu^{-1})^m \\ &+ 1 - nu^{-1} + (nu^{-1})^2 - \cdots + (-1)^{m-1}(nu^{-1})^{m-1} \\ =& 1.\end{aligned}$$

Thus a is clean. Therefore, R is clean. □

In general, the converse of Theorem 2.35 does not hold true. For example, integer modulo 5, $\mathbb{Z}_5$, is clean but not win-clean.

Lemma 2.36. *If w is a weak idempotent element in a win-clean ring R and $2 \in J(R)$, then $w \pm w^2$ is nilpotent.*

Proof. Since $2 \in J(R)$, we have $(w{\pm}w^2)^2 = 2(w^2{\pm}w^3) \in J(R)$. As $J(R)$ is nil, there exists some positive integer m such that $2^m = 0$ and also $(w{\pm}w^2)^{2m} = 0$. Hence, $w \pm w^2$ is nilpotent. □

The following proposition sets a condition for which a clean element becomes win-clean.

Proposition 2.37. *Let R be a commutative ring, $2 \in J(R)$ and x be clean in R with clean decomposition $x = u + e$. Then x is win-clean if and only if there exists $w \in wi(R) \cap Nil(R)$ such that $2e - 1 + u$ is nilpotent.*

Proof. ($\Longrightarrow$) Suppose x is win-clean. Then $x = n + f$ for some $n \in Nil(R)$ and $f \in wi(R)$. Now $x = n + f = (n - 1 + f + f^2) + (1 - f^2)$. Since $2 \in J(R)$, $f + f^2$ is nilpotent by Lemma 2.36. Then take $u = n - 1 + f + f^2$ and $e = 1 - f^2$. So

$$2e - 1 + u = 2(1 - f^2) - 1 + (n - 1 + f + f^2) + f^2 = n + f.$$

($\Longleftarrow$) We can rewrite $x = u + e$ as $x = (u + 2e - 1 + w^2) + (1 - e - w^2)$. Since $1 - e - w^2$ is weak idempotent, x is win-clean. □

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