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ON THE NILPOTENT DOT PRODUCT GRAPH OF A COMMUTATIVE RING

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Abstract. Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$, $1 \leq m < \infty$ be an integer and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times). In this paper, we introduce two types of (undirected) graphs, total nilpotent dot product graph denoted by $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and nilpotent dot product graph denoted by $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, in which vertices are from $\mathcal{R}^* = \mathcal{R} \setminus \{(0, 0, ..., 0)\}$ and $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^*$ respectively, where $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = \{w \in \mathcal{R}^* | wz \in \mathcal{N}(\mathcal{R}), \text{for some } z \in \mathcal{R}^*\}$. Two distinct vertices $w = (w_1, w_2, ..., w_m)$ and $z = (z_1, z_2, ..., z_m)$ are said to be adjacent if and only if $w \cdot z \in \mathcal{N}(\mathscr{B})$ (where $w \cdot z = w_1z_1 + \cdots + w_mz_m$, denotes the normal dot product and $\mathcal{N}(\mathscr{B})$ is the set of nilpotent elements of $\mathscr{B}$). We study about connectedness, diameter and girth of the graphs $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Finally, we establish the relationship between $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, $\mathcal{T}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}\mathcal{D}(\mathcal{R})$.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with a non-zero identity, and let $\mathcal{Z}(\mathcal{R})$ and $\mathcal{N}(\mathcal{R})$ be the sets of zero-divisors and nilpotent elements in $\mathcal{R}$, respectively. If H is any non-empty subset of a ring $\mathcal{R}$, then $H^* = H \setminus \{0\}$. A ring $\mathcal{R}$ is said to be reduced, if it has no non-zero nilpotent elements. In recent times, there has been a significant effort to explore the structural properties of a ring in relation to its zero-divisor graph. The concept of the zero-divisor graph for commutative rings was originally introduced by Anderson and Livingston[6]. The zero-divisor graph, denoted as $\Gamma(\mathcal{R})$ for a ring $\mathcal{R}$, is defined as a simple undirected graph with a vertex set $\mathcal{Z}(\mathcal{R})^*$. In this graph, two distinct vertices are adjacent if their product is zero. Several graph structures have been defined on rings and studied by various authors, as referenced in the following works [1, 4, 7, 5, 10, 12, 18, 20, 21, 22, 23, 25]. One can refer [3] for the entire literature on graphs from rings. The concept of a nilpotent graph was introduced by Chen[11]. In which, all the elements of a ring $\mathcal{R}$ are considered as vertices. Two distinct vertices w and z are adjacent if and only if $wz \in \mathcal{N}(\mathcal{R})$. In the mentioned paper, Chen[11] studied vertex colouring of a graph. Motivated by the concept of Chen in 2010 Ai-Hua and Qi-Sheng[16]

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introduced the modified definition of a nilpotent graph denoted by $\Gamma_{\mathcal{N}}(\mathcal{R})$. In which they considered the collection

$$\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = \{w \in \mathcal{R}^* | wz \in \mathcal{N}(\mathcal{R}) \text{ for some } z \in \mathcal{R}^*\}$$

to be a set of vertices of $\Gamma_{\mathcal{N}}(\mathcal{R})$ and two distinct vertices of $\Gamma_{\mathcal{N}}(\mathcal{R})$ are adjacent if $wz \in \mathcal{N}(\mathcal{R})$. They observe that $\Gamma(\mathcal{R})$ is a subgraph of $\Gamma_{\mathcal{N}}(\mathcal{R})$ and studied the basic properties of $\Gamma_{\mathcal{N}}(\mathcal{R})$. Further a lot of work has been done related to the nilpotent graph in [11, 14, 13, 16, 17, 19, 24].

Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$, $1 \leq m \leq \infty$ be an integer, and let $\mathcal{R} = \mathscr{B} \times \cdots \times \mathscr{B}$ (m times). In 2015, Badawi[9] introduced the *total dot product graph* $\mathcal{TD}(\mathcal{R})$ and *zero-divisor dot product graph* $\mathcal{ZD}(\mathcal{R})$ of a ring $\mathcal{R}$ in which vertices are taken from $\mathcal{R}^* = \mathcal{R} \setminus \{(0, 0, ..., 0)\}$ and $\mathcal{Z}(\mathcal{R})^* = \mathcal{Z}(\mathcal{R}) \setminus \{(0, 0, ..., 0)\}$ respectively. Two distinct vertices w and z are adjacent if and only if $w \cdot z = 0$. Motivated by the idea of Badawi and concept of Ai-Hua and Qi-Sheng[16], we introduce the *total nilpotent dot product graph* $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and *nilpotent dot product graph* $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ with vertices from $\mathcal{R}^*$ and $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = \mathcal{Z}_{\mathcal{N}}(\mathcal{R}) \setminus \{(0, 0, ..., 0)\}$ respectively. Two distinct vertices w and z are adjacent if and only if $w \cdot z \in \mathcal{N}(\mathscr{B})$. It can be observed that $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ are extended graphs of $\mathcal{TD}(\mathcal{R})$ and $\mathcal{ZD}(\mathcal{R})$, respectively.

Let G be a simple (undirected) graph with vertex set $V(G)$ and edge set $E(G)$. We say that a graph G is *connected*, if there exists a path between any two distinct vertices of a graph G. The length of the shortest path between two distinct vertices $w, z \in V(G)$ is denoted by $d(w, z)$. If there is no path between w and z, then $d(w, z) = \infty$. The *diameter* of a graph G denoted by $diam(G)$=sup$\{d(w, z)|$ where $w, z \in V(G)\}$. The girth of a graph G denoted by $gr(G)$ is defined as the length of the shortest cycle in G ($gr(G) = \infty$, if G contains no cycle). A *complete graph* is defined as the graph in which every two distinct vertices are adjacent. Recall that G is a *complete bipartite graph* if the vertex set of G can be partitioned into two vertex sets say V_1 and V_2 such that for every $x \in V_1$ is adjacent to every $y \in V_2$ and no distinct vertices in the same set (i.e, either V_1 or V_2) are adjacent. A *complete bipartite graph* in which atmost one vertex has degree greater than one is called a star graph. For more definitions related to the graph one can see [26].

Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$, $1 \leq m \leq \infty$ be an integer, and let $\mathcal{R} = \mathscr{B} \times \cdots \times \mathscr{B}$ (m times). In this paper, we prove that $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is a connected graph with $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 2$ for all $m \geq 1$, provided that $\mathscr{B}$ is a nonreduced ring. Further, we demonstrate that $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected graph and its $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 3$, for all $m \geq 3$. For the graph $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, we show that $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected, $diam(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 3$ and $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 4$ for all $m \geq 1$. Moreover we prove that for $m \geq 2$, if *girth* exists, then $gr(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$. Finally, we establish the relationship between $\mathcal{T}\mathcal{D}(\mathcal{R})$, $\mathcal{Z}\mathcal{D}(\mathcal{R})$, $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$.

2. Basic properties of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$

The purpose of this section is to study about connectedness, diameter, and girth of the graph $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. The following lemma is trivial.

Lemma 2.1. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$ and*

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$$

(m times), where $1 \leq m < \infty$. If $u, v \in \mathcal{R}$ such that $uv \in \mathcal{N}(\mathcal{R})$, then $u \cdot v \in \mathcal{N}(\mathcal{B})$.

Remark 2.2. Converse of the Lemma 2.1 need not be true in general. Let $\mathcal{R} = \mathbb{Z}_6 \times \mathbb{Z}_6$. Then for $a = (1,3), b = (3,1) \in \mathcal{R}^*$, $a \cdot b = 0 \in \mathcal{N}(\mathscr{B})$. However $ab = (3,3) \notin \mathcal{N}(\mathcal{R})$.

Theorem 2.3. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$ and*

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$$

(m times), where $1 \leq m < \infty$. Then $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected and $diam(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 3$. Moreover, if $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ contains a cycle, then

$$gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 4.$$

Proof. By [16, Theorem 2.1], $\Gamma_{\mathcal{N}}(\mathcal{R})$ is connected and $diam(\Gamma_{\mathcal{N}}(\mathcal{R})) \leq 3$. From Lemma 2.1, we can say that $\Gamma_{\mathcal{N}}(\mathcal{R})$ is a subgraph of $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ and therefore $diam(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq diam(\Gamma_{\mathcal{N}}(\mathcal{R}))$. Hence $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected and $diam(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 3$. By assumption, $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ contains a cycle. Suppose that $m \geq 3$. For each $i \in \{1,2,3,...,m\}$, let e_i denote the element of $\mathcal{R}$ whose *i-th* coordinate equals 1 and *j-th* coordinate equals 0 for all $j \in \{1,2,...,m\} \setminus \{i\}$. Note $e_1 - e_2 - e_3 - e_1$ is a cycle of length 3 in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Therefore, $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$ if $m \geq 3$. Assume that $m = 2$. If $\mathscr{B}$ is not reduced, then there exists $c \in \mathscr{B} \setminus \{0\}$ such that $c^2 = 0$. Observe that $(c,0) - (0,c) - (1,0) - (c,0)$ is a cycle of length 3 in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Hence $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$. Suppose that $\mathscr{B}$ is reduced. Then either $\mathscr{B}$ is an integral domain or $\mathscr{B}$ is not an integral domain. If $\mathscr{B}$ is an integral domain, then $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = V_1 \cup V_2$, where $V_1 = \{(a,0) | a \in \mathscr{B}^*\}$ and $V_2 = \{(0,c) | c \in \mathscr{B}^*\}$. It is easily seen that $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$ is a complete bipartite graph with vertex partition $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = V_1 \cup V_2$. Since $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ contains a cycle by assumption, it follows that $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 4$. Assume that $\mathscr{B}$ is a reduced ring but not an integral domain. Then there exist $r, s \in \mathscr{B} \setminus \{0\}$ such that $rs = 0$. Then $r \neq s$ and $(r, 0) - (s, 0) - (0, r) - (r, 0)$ is a cycle of length 3 in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Hence, $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$. Therefore, if $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ contains a cycle, then $gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 4$. □

Theorem 2.4. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$ and*

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$$

(m times), where $2 \leq m < \infty$. Then $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is a star graph if and only if $m = 2$ and $\mathscr{B} \cong \mathbb{Z}_2$.

Proof. First, we have to show that $m = 2$. On contrary suppose that $m \geq 3$ and let $u = (1, 0, 0, ..., 0)$, $v = (0, 1, 0, ..., 0)$ and $w = (0, 0, 1, ..., 0)$ in $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^*$. Then $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ have a cycle, a contradiction. Hence $m = 2$.
It remains to prove that $|\mathscr{B}|= 2$. On contrary suppose that $|\mathscr{B}| > 2$. Then there exists $0 \neq a \in \mathscr{B}$ such that $u = (1, 0)$, $v = (a, 0)$, $w = (0, a)$ and $z = (0, 1)$ are in $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^*$. Clearly $u - w - v - z - u$ forms a cycle in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, a contradiction. Hence $m = 2$ and $\mathscr{B} \cong \mathbb{Z}_2$.

Conversely, if $m = 2$ and $\mathscr{B} \cong \mathbb{Z}_2$, then a simple calculation leads to the desired result. □

Lemma 2.5. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$ and*

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times ... \times \mathscr{B},$$

(m times), where $2 \leq m < \infty$. Then the following hold:

(i) If $w - z$ is a path(edge) of $\mathcal{T}\mathcal{D}(\mathcal{R})$, then $w - z$ is also a path(edge) of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, where $w, z \in \mathcal{R}^*$.

(ii) If $a - b$ is a path(edge) of $\mathcal{Z}\mathcal{D}(\mathcal{R})$, then $a - b$ is also a path(edge) of $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$, where $a, b \in \mathcal{Z}(\mathcal{R})^*$.

Proof. (i) If $w - z$ is an edge of $\mathcal{T}\mathcal{D}(\mathcal{R})$, then $w \cdot z = 0$. This implies that $w \cdot z = 0 \in \mathcal{N}(\mathscr{B})$. Therefore, $w - z$ is also an edge of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Similarly, if $w - z$ is a path in $\mathcal{T}\mathcal{D}(\mathcal{R})$, then $w - z$ is also a path in $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$.

(ii) We can prove it in the similar manner.

□

Remark 2.6. The converse of Lemma 2.5 need not be true in general. Let $\mathcal{R} = \mathbb{Z}_{16} \times \mathbb{Z}_{16}$. Then for $a = (1, 3), b = (3, 1) \in \mathcal{Z}(\mathcal{R})^*$ (or $\mathcal{R}^*$), we have $a \cdot b = 6 \in \mathcal{N}(\mathscr{B})$ but $a \cdot b = 6 \neq 0$. Therefore, $a - b$ is an edge in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ (or $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$) but not in $\mathcal{Z}\mathcal{D}(\mathcal{R})$ (or $\mathcal{T}\mathcal{D}(\mathcal{R})$).

Remark 2.7. Let $\mathscr{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $2 \leq m < \infty$. It is shown in Lemma 2.5 that $\mathcal{T}\mathcal{D}(\mathcal{R})$ (respectively,

$\mathcal{ZD}(\mathcal{R})$) is a subgraph of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ (respectively, $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$). Assume that $\mathscr{B}$ is a reduced ring. Let $w, z \in \mathcal{R}^*$ be such that $w - z$ is an edge of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Then $w \cdot z \in \mathcal{N}(\mathscr{B})$. Since $\mathscr{B}$ a is reduced ring, then $\mathcal{N}(\mathscr{B}) = \{0\}$. Thus, $w - z$ is an edge of $\mathcal{TD}(\mathcal{R})$. Hence, $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is a subgraph of $\mathcal{TD}(\mathcal{R})$ and so, $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \mathcal{TD}(\mathcal{R})$. Similarly, $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is a subgraph of $\mathcal{ZD}(\mathcal{R})$ and so, $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \mathcal{ZD}(\mathcal{R})$. Thus if $\mathscr{B}$ is a reduced ring, then $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \mathcal{TD}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \mathcal{ZD}(\mathcal{R})$.

Remark 2.8. Let $\mathscr{B}$ be a nonreduced ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), $1 \leq m < \infty$. Then by definition $\mathcal{Z}_{\mathcal{N}}(\mathcal{R})^* = \mathcal{R}^*$. Hence $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) = \mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$.

Theorem 2.9. *Let $\mathscr{B}$ be a nonreduced commutative ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), $1 \leq m < \infty$. Then $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is a connected graph and $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 2$.*

Proof. Since $\mathscr{B}$ is a non-reduced ring, then there exists

$$n = (c, c, ..., c) \in \mathcal{N}(\mathcal{R})^*,$$

where $c \in \mathcal{N}(\mathscr{B})^*$. Let $a = (a_1, a_2, ..., a_m) \in \mathcal{R}^*$. Then

$$n \cdot a = na_1 + na_2 + \cdots + na_m \in \mathcal{N}(\mathscr{B}).$$

Therefore, n is adjacent to all the vertices of $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Hence, we obtain that $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected and $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \leq 2$. As $c \in \mathcal{N}(\mathscr{B})^*$, it follows that $1 + c \in \mathscr{B}^\times$ and $1 + c \neq 1$. Let $x = (1, 0, ..., 0)$ and $y = (1 + c, 0, ..., 0)$. Then it is clear that $x \cdot y = 1 + c \notin \mathcal{N}(\mathscr{B})$ and so, x and y are not adjacent in $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$. Hence, $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) \geq 2$ and so, $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 2$. □

Theorem 2.10. *Let $\mathscr{B}$ be a reduced commutative ring with $1 \neq 0$ which is not an integral domain and $\mathcal{R} = \mathscr{B} \times \mathscr{B}$. Then*
(i) $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected and $diam(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$.
(ii) $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ is connected and $diam(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$.
(iii) $gr(\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = gr(\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})) = 3$.

Proof. By hypothesis, $\mathscr{B}$ is a reduced ring but not an integral domain. We know from Remark 2.7 that $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})= \mathcal{TD}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})= \mathcal{ZD}(\mathcal{R})$.
(i) This follows from [9, Theorem 2.3(1)].
(ii) This follows from [9, Theorem 2.3(2)].
(iii) This follows from [9, Theorem 2.3(3)]. □

Theorem 2.11. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$ and*

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$$

(m times), where $3 \leq m < \infty$. Then $\mathcal{T_N D}(\mathcal{R})$ is connected and

$$diam(\mathcal{T_N D}(\mathcal{R})) = 2.$$

Proof. By [9, Theorem 2.4], $\mathcal{TD}(\mathcal{R})$ is connected and $diam(\mathcal{TD}(\mathcal{R})) = 2$. From Lemma 2.5, $\mathcal{TD}(\mathcal{R})$ is a subgraph of $\mathcal{T_N D}(\mathcal{R})$. Therefore, $\mathcal{T_N D}(\mathcal{R})$ is connected and $diam(\mathcal{T_N D}(\mathcal{R})) \leq diam(\mathcal{TD}(\mathcal{R})) = 2$. Now, it remains to demonstrate that $diam(\mathcal{T_N D}(\mathcal{R})) = 2$. For this, let $u, v \in \mathcal{R}^*$ such that $u = (1, 1, 1, 0, ..., 0)$, $v = (1, 0, 0, 0, ..., 0) \in \mathcal{R}^*$. Then $u \cdot v \notin \mathcal{N}(\mathscr{B})$ and hence $d(u, v) > 1$. Also, we have $d(u, v) \leq 2$. We conclude that $d(u, v) = 2$ and hence the result. □

The immidiate consequence of Theorem 2.9, Theorem 2.10(i) and Theorem 2.11 is the following:

Corollary 2.12. *Let $\mathscr{B}$ be a commutative ring with $1 \neq 0$, which is not an integral domain and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $2 \leq m < \infty$. Then $\mathcal{T_N D}(\mathcal{R})$ is connected and $diam(\mathcal{T_N D}(\mathcal{R})) = 2$ or 3.*

Theorem 2.13. *Let $\mathscr{B}$ be a reduced commutative ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \mathscr{B}$. Then the following statements hold:*

(i) If $diam(\mathcal{Z_N D}(\mathcal{R})) = 3$, then $\mathscr{B}$ is an integral domain.

(ii) If $diam(\mathcal{Z_N D}(\mathcal{R})) = 2$, then $\mathscr{B}$ is not an integral domain.

Proof. We know from Theorem 2.3 that $diam(\mathcal{Z_N D}(\mathcal{R})) \leq 3$. Let $w = (1, 1, 0)$ and $z = (0, 1, 1)$. It is clear that $w, z \in \mathcal{Z_N}(\mathcal{R})^*$ with $w \neq z$. Observe that $w \cdot z \notin \mathcal{N}(\mathscr{B})$. Therefore, $d(w, z) \geq 2$ in $\mathcal{Z_N D}(\mathcal{R})$. Hence, $diam(\mathcal{Z_N D}(\mathcal{R})) \geq 2$.

(i) Since $\mathscr{B}$ is an integral domain, then by Remark 2.7, $\mathcal{Z_N D}(\mathcal{R}) = \mathcal{ZD}(\mathcal{R})$. Hence, we obtain from [9, Theorem 2.5(1)] that $diam(\mathcal{Z_N D}(\mathcal{R})) = 3$.

(ii) By assumption $\mathscr{B}$ is not an integal domain. Then from [9, Theorem 2.5(2)], $diam(\mathcal{ZD}(\mathcal{R})) = 2$. Also, from Remark 2.7, $\mathcal{Z_N D}(\mathcal{R}) = \mathcal{ZD}(\mathcal{R})$. Therefore, $diam(\mathcal{Z_N D}(\mathcal{R})) = diam(\mathcal{ZD}(\mathcal{R})) = 2$. Hence the result. □

Using Remark 2.8 and Theorem 2.9, we can prove the following:

Corollary 2.14. *Let $\mathscr{B}$ be a nonreduced commutative ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $1 \leq m < \infty$. Then $\mathcal{Z_N D}(\mathcal{R})$ is connected and $diam(\mathcal{Z_N D}(\mathcal{R})) = 2$.*

3. Relation between $\mathcal{TD}(\mathcal{R})$, $\mathcal{ZD}(\mathcal{R})$, $\mathcal{T_N D}(\mathcal{R})$ and $\mathcal{Z_N D}(\mathcal{R})$

The purpose of this section is to establish the relationship between $\mathcal{T_N D}(\mathcal{R})$, $\mathcal{TD}(\mathcal{R})$, $\mathcal{Z_N D}(\mathcal{R})$ and $\mathcal{ZD}(\mathcal{R})$. Additionally, we aim to demonstrate some corollaries to relate $\mathcal{Z_N D}(\mathcal{R})$ to $\Gamma_{\mathcal{N}}(\mathcal{R})$.

Theorem 3.1. *Let $\mathscr{B}$ be a nonreduced commutative ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $2 \leq m < \infty$. Then the followings hold:*

(i) $\mathcal{TD}(\mathcal{R}) \neq \mathcal{T_N D}(\mathcal{R})$.

(ii) $\mathcal{ZD}(\mathcal{R}) \neq \mathcal{Z_N D}(\mathcal{R})$.

Proof. (i) Since $\mathscr{B}$ is a nonreduced commutative ring, then there exists $0 \neq c \in N(\mathscr{B})$. Let $w = (c, 0, 0, ..., 0)$, $z = (1, 0, 0, ..., 0) \in \mathcal{R}^*$. Then $w \cdot z = c \in N(\mathscr{B})$ but $w \cdot z \neq 0$. Therefore, $w - z$ is adjacent in $\mathcal{T_N D}(\mathcal{R})$ but not in $\mathcal{TD}(\mathcal{R})$. Hence $\mathcal{TD}(\mathcal{R}) \neq \mathcal{T_N D}(\mathcal{R})$.

(ii) We can prove similarly. □

Corollary 3.2. *Let $\mathscr{B}$ be an integral domain and $\mathcal{R} = \mathscr{B} \times \mathscr{B}$. Then the followings hold:*

(i) $\Gamma_{\mathcal{N}}(\mathcal{R}) = \mathcal{Z_N D}(\mathcal{R})$.

(ii) $\mathcal{T_N D}(\mathcal{R})$ *is disconnected.*

Proof. (i) Using Remark 2.7, we have $\mathcal{ZD}(\mathcal{R}) = \mathcal{Z_N D}(\mathcal{R})$. By [9, Theorem 2.1] $\Gamma(\mathcal{R}) = \mathcal{ZD}(\mathcal{R})$. Since $\mathcal{R}$ is a reduced ring, we have $\Gamma(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$. Therefore, $\Gamma_{\mathcal{N}}(\mathcal{R}) = \mathcal{Z_N D}(\mathcal{R})$.

(ii) By Remark 2.7, we have $\mathcal{TD}(\mathcal{R}) = \mathcal{T_N D}(\mathcal{R})$ and using [9, Theorem 2.1], we have $\mathcal{TD}(\mathcal{R})$ is disconnected. Therefore, $\mathcal{T_N D}(\mathcal{R})$ is disconnected. □

Corollary 3.3. *Let $\mathscr{B}$ be a reduced ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $2 \leq m < \infty$. Then $\mathcal{Z_N D}(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$ if and only if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m = 2$ and $\mathscr{B}$ is an integral domain.*

Proof. Since $\mathscr{B}$ is a reduced ring by hypothesis, $\mathcal{ZD}(\mathcal{R}) = \mathcal{Z_N D}(\mathcal{R})$ by Remark 2.7. Since $\mathcal{R}$ is a reduced ring, then it is easy to see that $\Gamma(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$. Thus, $\mathcal{Z_N D}(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$ if and only if $\mathcal{ZD}(\mathcal{R}) = \Gamma(\mathcal{R})$. Hence, we obtain from [9, Theorem 2.2], that $\mathcal{Z_N D}(\mathcal{R}) = \Gamma_{\mathcal{N}}(\mathcal{R})$ if and only if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m = 2$ and $\mathscr{B}$ is an integral domain. □

Corollary 3.4. *Let $\mathscr{B}$ be a nonreduced commutative ring with $1 \neq 0$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$ (m times), where $2 \leq m < \infty$. Then the following hold:*

(i) $\mathcal{Z_N D}(\mathcal{R}) = \mathcal{T_N D}(\mathcal{R})$

(ii) $\mathcal{T_N D}(\mathcal{R}) \neq \mathcal{TD}(\mathcal{R})$.

Proof. (i) It follows from Remark 2.8.

(ii) It follows from Theorem 3.1. □

We end this section by giving an example of a ring by which we can distinguish the graph $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ with $\mathcal{T}\mathcal{D}(\mathcal{R})$ and $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ with $\mathcal{Z}\mathcal{D}(\mathcal{R})$.

Example 3.5. Let $\mathscr{B} \cong \mathbb{Z}_{12}$ and $\mathcal{R} = \mathscr{B} \times \mathscr{B}$. For $(1,3),(3,1) \in \mathcal{R}^*$ and $(1,3)\cdot(3,1) = 6 \neq 0 \in \mathcal{N}(\mathscr{B})$, $(1,3) - (3,1)$ is adjacent in $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ but not in $\mathcal{T}\mathcal{D}(\mathcal{R})$. Hence $\mathcal{T}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) \neq \mathcal{T}\mathcal{D}(\mathcal{R})$. Also $(1,0),(6,0) \in \mathcal{Z}_{\mathcal{N}}(\mathcal{R})$ and $(1,0)\cdot(6,0) = 6 \in \mathcal{N}(\mathscr{B})$ but $(1,0)\cdot(6,0) \neq 0$. Therefore, $(1,0) - (6,0)$ is adjacent in $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R})$ but not in $\mathcal{Z}\mathcal{D}(\mathcal{R})$. Thus $\mathcal{Z}_{\mathcal{N}}\mathcal{D}(\mathcal{R}) \neq \mathcal{Z}\mathcal{D}(\mathcal{R})$.

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ON THE NILPOTENT DOT PRODUCT GRAPH OF A COMMUTATIVE RING

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بررسی گراف ضرب نقطه‌ای پوچ توان حلقه‌ای جابه‌جایی

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فرض کنید $\mathscr{B}$ حلقه‌ی جابه‌جایی با $1 \neq 0$، $1 \leq m < \infty$ یک عدد صحیح و

$$\mathcal{R} = \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B}$$

(m مرتبه) باشد. در این مقاله، دو نوع گراف (غیر جهت‌دار)، گراف ضرب نقطه‌ای پوچ توان کامل که با $\mathcal{T_N D}(\mathcal{R})$ نشان داده می‌شود و گراف ضرب نقطه‌ای پوچ توان که با نماد $\mathcal{Z_N D}(\mathcal{R})$ نمایش داده می‌شود، معرفی می‌کنیم که در آن‌ها به رأس‌ها به ترتیب از $\mathcal{R}^* = \mathcal{R} \setminus \{(0, 0, ..., 0)\}$ و $\mathcal{Z_N}(\mathcal{R})^*$ هستند که $\mathcal{Z_N}(\mathcal{R})^* = \{w \in \mathcal{R}^* | wz \in \mathcal{N}(\mathcal{R}),$ برای برخی $z \in \mathcal{R}^*\}$. دو رأس متمایز $w = (w_1, w_2, ..., w_m)$ و $z = (z_1, z_2, ..., z_m)$ مجاورند اگر و تنها اگر $w \cdot z \in \mathcal{N}(\mathscr{B})$ (که $w \cdot z = w_1 z_1 + \cdots + w_m z_m$ ضرب نقطه‌ای نرمال را نشان می‌دهد و $\mathcal{N}(\mathscr{B})$ مجموعه‌ی عناصر پوچ توان $\mathscr{B}$ می‌باشد.) همچنین، به مطالعه‌ی همبندی، قطر و کمر گراف‌های $\mathcal{T_N D}(\mathcal{R})$ و $\mathcal{Z_N D}(\mathcal{R})$ می‌پردازیم. به علاوه، رابطه‌ی بین $\mathcal{T_N D}(\mathcal{R})$، $\mathcal{Z_N D}(\mathcal{R})$، $\mathcal{TD}(\mathcal{R})$ و $\mathcal{ZD}(\mathcal{R})$ را بیان می‌کنیم.

کلمات کلیدی: گراف ضرب نقطه‌ای، گراف پوچ توان، حلقه‌ی کاهش‌یافته.