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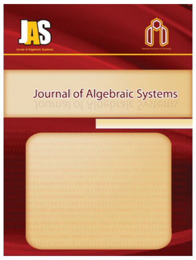

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A SUBCLASS OF BAER IDEALS AND ITS APPLICATIONS

Z. GHARABAGI AND A. TAHERIFAR*

Abstract. An ideal I of a ring R is called a right strongly Baer ideal if $r(I) = r(e)$, where e is an idempotent, and there are right semicentral idempotents e_i $(1 \leq i \leq n)$ with $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$ and each ideal Re_iR is maximal or equals R. In this paper, we provide a topological characterization of this class of ideals in semiprime (resp., semiprimitive) rings. By using these results, we prove that every ideal of a ring R is a right strongly Baer ideal *if and only if* R is a semisimple ring. Next, we give a characterization of right strongly Baer-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings, and semiprime rings. For a semiprimitive commutative ring R, it is shown that $\mathrm{Soc}(R)$ is a right strongly Baer ideal *if and only if* the set of isolated points of $\mathrm{Max}\,(R)$ is dense in it *if and only if* $\mathrm{Soc}_m(R)$ is a right strongly Baer ideal. Finally, we characterize strongly Baer ideals in $C(X)$ (resp., $C(X)_F$).

1. Introduction

Throughout this paper, all rings are assumed to have an identity element. In [4] and [3], Azarpanah provides a topological characterization of essential ideals in the ring of continuous functions, denoted as $C(X)$. It is proven that an ideal I of $C(X)$ is essential *if and only if* $\mathrm{int} \bigcap Z[I] = \emptyset$. This motivates the following question: What kind of ideal I satisfies $\mathrm{int} \bigcap Z[I]$ being a finite subset of X? It is observed that this is equivalent to $r(I) = r(e)$ for some idempotent e of $C(X)$ and the existence of idempotents $e_1, e_2, ..., e_n$ such that $e = e_1 \times e_2 \times ... \times e_n$, where each ideal $C(X)e_i = \langle e_i \rangle$ is either maximal or equals $C(X)$. This motivates the extension of this concept to any associate ring.

In Section 2, we establish that an ideal I of a semiprime (resp., semiprimitive) ring R is a right strongly Baer ideal *if and only if* $\mathrm{int}_S\, V(I)$ $(\mathrm{int}_M\, M(I))$ is a finite subset of $\mathrm{Max}\,(R) \cap I(\mathrm{Spec}\,(R))$ (resp., $\mathrm{Max}\,(R)$), where $I(\mathrm{Spec}\,(R))$ denotes the set of isolated points of the space $\mathrm{Spec}\,(R)$. Furthermore, we prove that every ideal of a ring R is a right strongly Baer ideal *if and only if* R is a semiprime ring and $\mathrm{Spec}\,(R) = \mathrm{Max}\,(R)$ is finite *if and only if* R is

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a semisimple ring *if and only if* R is semiprime and the set of right strongly Baer ideals of R, denoted as $SB(R)$, forms a Boolean algebra when partially ordered by inclusion. We conclude that for any ring R, every ideal of $R/N(R)$ is right strongly Baer *if and only if* $\mathrm{Max}\,(R) = \mathrm{Spec}\,(R)$ is finite.

In Section 3, we characterize right strongly Baer ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings. We show that if $J = M_n(I)$ is a right strongly Baer ideal of $M_n(R)$, then I is a right strongly Baer ideal of R. By using these results, we obtain some well-known results about the semisimplicity of 2-by-2 generalized triangular matrix rings, full, and upper triangular matrix rings.

Section 4 focuses on commutative reduced (resp., semiprimitive) rings. For a semiprimitive ring R, it is demonstrated that $\mathrm{Soc}(R)$ is strongly Baer *if and only if* $\mathrm{Soc}_{\mathsf{max}}(R)$ is strongly Baer *if and only if* the set of isolated points of $\mathrm{Max}\,(R)$ is dense in it. We also show that whenever R is a reduced ring, every intersection of essential minimal prime ideals of R is a strongly Baer ideal *if and only if* the set of isolated points of $\mathrm{Min}\,(R)$ is dense in it.

In Section 5, we investigate strongly Baer ideals in $C(X)$ (resp., $C(X)_F$). We prove that an ideal I of $C(X)$ (resp., $C(X)_F$) is strongly Baer *if and only if* $\mathrm{int} \bigcap Z[I]$ (resp., $Fr \bigcap Z[I]$) is a finite subset of X. Furthermore, we demonstrate that the ideal $C_K(X)$ is a strongly Baer ideal *if and only if* the set of points of X with compact neighborhoods, denoted as X_L, is dense in it.

For any subset S of R, $l(S)$ and $r(S)$ denote the left and right annihilators of S in R, respectively. The ring of n-by-n (upper triangular) matrices over R is denoted by $\mathbf{M_n}(\mathbf{R})$ ($\mathbf{T_n}(\mathbf{R})$). An idempotent e of a ring R is called *left semicentral* (resp., *right semicentral*) if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. It can be easily checked that an idempotent e of R is left semicentral if and only if eR is an ideal (resp., $1-e$ is right semicentral if and only if Re is an ideal). See [5] and [7] for a more detailed account of semicentral idempotents. For a left (right) ideal I of a ring R, if $l(I) = l(e)$ (resp., $r(I) = r(e)$) with an idempotent e, then e is left (right) semicentral, since $l(e)$ (resp., $r(e)$) is an ideal. We use $S_l(R)$ ($S_r(R)$) to denote the set of left (right) semicentral idempotents of R. An ideal I of R is a *Baer ideal* if $r(I) = eR$ for some idempotent e of R, see [21]. It is well known that every ideal of R is a Baer ideal *if and only if* R is a quasi-Baer ring.

For $a \in R$, let $supp(a) = P \in \mathrm{Spec}\,(R) : a \notin P$. Shin [19, Lemms 3.1] proved that for any R, $supp(a) : a \in R$ forms a basis of open sets in the Zariski topology on $\mathrm{Spec}\,(R)$. We use $V(I)$ ($V(a)$) to denote the set of all

$P \in \operatorname{Spec}(R)$ such that $I \subseteq P$ ($a \in P$). Note that $V(I) = \cap_{a\in I} V(a)$ and $V(a) = \operatorname{Spec}(R) \setminus supp(a)$. $Max(R)$ is the set of all maximal ideals of R. For $a \in R$, let $M(a) = M \in Max(R) : a \in M$. It is easy to see that for any ring R, the set $D(a) : a \in R$ (where $D(a) = Max(R) \setminus M(a)$) forms a basis of open sets in the Zariski topology on $Max(R)$. We say R is a semiprimitive ring if $J(R) = 0$, where $J(R)$ is the intersection of all maximal right ideals of R. In the sequel, we denote $\operatorname{int}_S V(I)$ (resp., $\operatorname{int}_M M(I)$) as $\operatorname{int}_{\operatorname{Spec}(R)} V(I)$ (resp., $\operatorname{int}_{\operatorname{Max}(R)} M(I)$).

Recall that for any ring R with identity, the socle of R, denoted as $Soc(R)$, is the sum of all simple right ideals of R, and it is also the intersection of all essential right ideals of R, see [17]. Similarly, in [12], $Soc_m(R)$ is used to denote the intersection of all essential maximal ideals of a commutative ring R. We denote the socle of $C(X)$ by $C_F(X)$; it is the set of all functions which vanish everywhere except on a finite number of points of X.

2. Preliminary results and examples

Definition 2.1. An ideal I of a ring R is called a right strongly Baer ideal if $r(I) = r(e)$, where e is an idempotent, and there are right semisentral idempotents e_i $(1 \leq i \leq n)$ with $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$ and each ideal Re_iR is maximal or equals R.

Since $e_1, e_2, ..., e_n \in S_r(R)$, we have

$$Re_1R \cap Re_2R \cap ... \cap Re_nR = Re_1 \times e_2 \times ... \times e_n.$$

Thus I is a right strongly Baer ideal if $r(I) = r(e)$, where e is idempotent, and there are right semisentral idempotents e_i $(1 \leq i \leq n)$ with

$$Re = Re_1 \times e_2 \times ... \times e_n$$

and each ideal Re_iR is maximal or equals R.

Example 2.2. (1) Trivially every left dense ideal (i.e., the ideal which its right annihilator is zero) is a right strongly Baer ideal.

(2) If M is a left maximal ideal of R which is not left essential, then MR is a right strongly Baer ideal. For, if M is not a left essential ideal, then there is a non-zero left ideal I of R such that $I \cap M = 0$. As $I + M = R$, we have $M = Re$ for some idempotent e of R. We have $MR = R$ or $MR = M$. If $MR = R$, then $r(MR) = r(R) = r(1)$. If $M = MR$ (i.e., M is an ideal of R), then $r(MR) = r(M) = r(e)$ and $ReR = M$.

(3) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then every non-zero ideal of R is a right strongly Baer ideal. For, the only non-zero ideals of R are

$I_1 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $I_3 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, and we have $r_R(I_1) = 0$, $r_R(I_2) = r_R(I_3) = I_1$, where I_2 is a maximal ideal generated by idempotent $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. But we can see that the zero-ideal is not a right strongly Baer ideal. In this ring, we have I_1, I_3 are right strongly Baer ideals which are not essential as right ideals and I_2 is not essential as a left ideal. For, consider $J = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Then J is a right ideal of R and $I_3 \cap J = I_1 \cap J = 0$. Also, put $K = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$. Then K is a left ideal and $I_2 \cap K = 0$.

Lemma 2.3. [6, Lemma 4.2] Let R be a semiprime ring.

(1) For any $a \in R$ and any ideal I of R, $supp(a) \cap supp(I) = supp(Ia)$.
(2) If I and J are two ideals of R, then $r(I) \subseteq r(J)$ if and only if $\operatorname{int}_S V(I) \subseteq \operatorname{int}_S V(J)$
(3) $A \subseteq \operatorname{Spec}(R)$ is a clopen subset if and only if there exists a central idempotent $e \in R$ such that $A = V(e)$.

Similar to the above lemma we have the following result.

Lemma 2.4. Let R be a semiprimitive ring.

(1) For any $a \in R$ and any ideal I of R, $D(a) \cap D(I) = D(Ia)$.
(2) If I and J are two ideals of R, then $r(I) \subseteq r(J)$ if and only if $\operatorname{int}_M M(I) \subseteq \operatorname{int}_M M(J)$

For a subset A of $\operatorname{Spec}(R)$ (resp., $\operatorname{Max}(R)$), put $O_A = \{a \in R : A \subseteq V(a)\}$ (resp., $\{a \in R : A \subseteq M(a)\}$) . Then it is easy to see that $O_A = \bigcap_{P \in A} P$ (resp., $\bigcap_{M \in A} M$) and $V(O_A) = cl_S A$ (resp., $M(O_A) = cl_M A$), where $cl_S A$ (resp., $cl_M A$) is the closure of A in the space $\operatorname{Spec}(R)$ (resp., $\operatorname{Max}(R)$). It is easy to see that for $A, B \subseteq \operatorname{Spec}(R)$ (resp., $\operatorname{Max}(R)$), $O_A = O_B$ if and only if $\operatorname{cl}_S A$ (resp., $\operatorname{cl}_M(A)) = \operatorname{cl}_S(B)$ (resp., $\operatorname{cl}_M(B)$).

Lemma 2.5. The following statements hold.

(1) A maximal ideal M of a semiprime ring R is generated by a right semicentral idempotent if and only if M is an isolated point in the space $\operatorname{Spec}(R)$.
(2) A maximal ideal M of a semiprimitive ring R is generated by a right semicentral idempotent if and only if M is an isolated point in the space $\operatorname{Max}(R)$.

Proof. (1) We assume $M = ReR$ for some right semicentral idempotent e of R. Then $eR(1-e) = 0$. This implies $supp(1-e) = \{M\}$. Thus $\{M\}$ is open in $\mathrm{Spec}\,(R)$. Conversely, the set $\{M\}$ is open in $\mathrm{Spec}\,(R)$. Thus $\mathrm{Spec}\,(R) \setminus \{M\}$ is closed in $\mathrm{Spec}\,(R)$. Therefore

$$I = \bigcap_{P \in \mathrm{Spec}\,(R) \setminus \{M\}} P = O_{\mathrm{Spec}\,(R) \setminus \{M\}}$$

is a non-zero ideal of R. In fact, whenever $O_{\mathrm{Spec}\,(R) \setminus \{M\}} = 0$, then

$$O_{\mathrm{Spec}\,(R) \setminus \{M\}} = O_{\mathrm{Spec}\,(R)}.$$

By comments before lemma, this shows $\mathrm{Spec}\,(R) \setminus \{M\} = \mathrm{Spec}\,(R)$, which is a contradiction. By semiprime hypothesis, $I \cap M = 0$ and by maximality of M, $I + M = R$. Thus $M = eR$ for some idempotent e of R. Since M is an ideal of R, e is a right semicentral idempotent.

(2) This follows from (1). □

Similar to the commutative case we have the following result.

Lemma 2.6. The following statements hold.

1. (1) An ideal I of a semiprime ring R is an essential right ideal if and only if $\mathrm{int}_S\, V(I) = \emptyset$.
2. (2) An ideal I of a semiprimitive ring R is an essential right ideal if and only if $\mathrm{int}_M\, M(I) = \emptyset$.

Proof. (1) First assume I is an essential right ideal and $P \in int_S V(I)$. Then there is a non-zero element $a \in R$ such that

$$P \in supp(a) = supp(RaR) \subseteq V(I).$$

This implies

$$supp(RaR \cap I) = supp(RaR) \cap supp(I) = \emptyset,$$

i.e., $RaR \cap I = 0$. This is a contradiction. Next, suppose J is a non-zero right ideal of R such that $I \cap J = 0$. Thus $V(I) \cup V(J) = V(I \cap J) = \mathrm{Spec}\,(R)$. This says $\mathrm{Spec}\,(R) \setminus V(J)$ is a non-empty open set contained in $V(I)$, which is a contradiction.

(2) The proof is similar to the (1). □

Lemma 2.7. The following statements hold.

1. (1) An ideal I of a semiprime ring R is a right strongly Baer ideal if and only if $\mathrm{int}_S\, V(I)$ is a finite subset of $\mathrm{Max}\,(R) \cap I(\mathrm{Spec}\,(R))$.
2. (2) An ideal I of a semiprimitive ring R is a right strongly Baer ideal if and only if $\mathrm{int}_M\, M(I)$ is a finite subset of $\mathrm{Max}\,(R)$.

Proof. (1) Let I be a right strongly Baer ideal of R. Then there exists an idempotent $e \in R$ such that $r(I) = r(e)$ and $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$, where each ideal Re_iR $(1 \leq i \leq n)$ is maximal or each $e_i = 1$. Now let for each $1 \leq i \leq n$, $M_i = Re_iR$ is maximal. By Lemmas 2.3 and 2.5 we have;

$$\operatorname{int}_S V(I) = \operatorname{int}_S V(ReR) = \operatorname{int}_S(V(Re_1R) \cup ... \cup V(Re_nR)) = \{M_1, ..., M_n\}.$$

Hence $\operatorname{int}_S V(I)$ is a finite subset of $\operatorname{Max}(R) \cap I(\operatorname{Spec}(R))$. If each $e_i = 1$ $(1 \leq i \leq n)$, then $\operatorname{int}_S V(I) = \emptyset$.
Conversely, suppose that $\operatorname{int}_S V(I) = \{P_1, ..., P_n\}$ is a finite subset of $\operatorname{Max}(R) \cap I(\operatorname{Spec}(R))$. Then for each $1 \leq i \leq n$, the ideal P_i is a maximal ideal and each P_i is an isolated point of $\operatorname{Spec}(R)$. By Lemma 2.5, for each $1 \leq i \leq n$ there is a right semicentral idempotent f_i such that $P_i = Rf_iR$. This implies that;

$$\operatorname{int}_S V(I) = \operatorname{int}_S(V(Rf_1R) \cup ... \cup V(Rf_nR)) = \operatorname{int}_S V(Rf_1R \cap ... \cap Rf_nR).$$

Now consider $e = f_1 \cdot f_2 \cdot ... \cdot f_n$. Then by [22, Lemma 2.3], e is a right semicentral idempotent and we can see that

$$ReR = Rf_1R \cap Rf_2R \cap ... \cap Rf_nR.$$

By Lemma 2.3, we have $r(I) = r(ReR) = r(e)$ and each ideal Rf_iR $(1 \leq i \leq n)$ is maximal. If $\operatorname{int}_S V(I) = \emptyset$, then $r(I) = r(1)$. Therefore I is a right strongly Baer ideal.

(2) Let I be a right strongly Baer ideal. Then $r(I) = r(ReR)$, where $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$ and each ideal Re_iR $(i = 1, ..., n)$ is maximal or each $e_i = 1 (i = 1, ..., n)$. Now let for each $1 \leq i \leq n$, $Re_iR = M_i$, where M_i is a maximal ideal. Then by Lemma 2.4, we have

$$\begin{aligned}\operatorname{int}_M M(I) &= int_M M(ReR)\\ &= \operatorname{int}_M(M(Re_1R) \cup ... \cup M(Re_{nR})))\\ &= \{M_1, ..., M_n\}.\end{aligned}$$

Hence $\operatorname{int}_M M(I)$ is finite. If each $e_i = 1$ $(1 \leq i \leq n)$, then $\operatorname{int}_M M(I) = \emptyset$.
Conversely, suppose that $\operatorname{int}_M M(I) = \{M_1, ...M_n\}$ is a finite subset of $\operatorname{Max}(R)$. Then for each $1 \leq i \leq n$, the point M_i is an isolated point of $\operatorname{Max}(R)$, so by Lemma 2.5, for each $1 \leq i \leq n$ there is a right semicentral idempotent e_i such that $M_i = Re_iR$. Thus

$$\begin{aligned}\operatorname{int}_M M(I) &= \operatorname{int}_M(M(Re_1R) \cup ... \cup M(Re_nR))\\ &= \operatorname{int}_M M(Re_1R \cap Re_2R \cap ... \cap Re_nR).\end{aligned}$$

Now put $e = e_1 \cdot \ldots \cdot e_n$. Then we have $ReR = Re_1R \cap Re_2R \cap \ldots \cap Re_nR$ and $r(I) = r(ReR)$, by Lemma 2.4. If $\mathrm{int}_M\, M(I) = \emptyset$, then $r(I) = 0 = r(1)$. So we are done. □

Lemma 2.8. A ring R is semiprime if and only if for any two ideals I, J of R, $r(IJ) = r(I \cap J)$.

Proof. First, assume R is semiprime and I, J are two ideals of R. We must prove that $r(IJ) = r(I \cap J)$. Evidently $r(I \cap J) \subseteq r(IJ)$. Now suppose that $x \in r(IJ)$ and $a \in I \cap J$. Then $RxR \subseteq r(IJ)$, $RaRaR \subseteq IJ$ and we have $(RaRxR)^2 = RaRxRaRxR \subseteq RaR.RaR.RxR = 0$. By hypothesis, $RaRxR = 0$. Thus $ax = 0$, i.e., $r(IJ) \subseteq r(I \cap J)$. So we are done. Conversely, suppose I is an ideal of R and $I^2 = 0$. Then by hypothesis,

$$r(I) = r(I \cap I) = r(I^2) = R.$$

This implies $I = 0$, i.e., R is semiprime. □

Proposition 2.9. Let R be a semiprime ring.

(1) The intersection of two right strongly Baer ideals of R is a right strongly Baer ideal.
(2) The sum of a right strongly Baer ideal and any other ideal of R is a right strongly Baer ideal.
(3) Every ideal of R which is an essential right ideal is a right strongly Baer ideal.
(4) For every right maximal ideal M of R, RM is a right strongly Baer ideal.

Proof. (1) Let I and J be two right strongly Baer ideals of R. Then there are two idempotents $e, f \in R$ such that $r(I) = r(e)$, $r(J) = r(f)$ and e, f satisfy in our definition. Then we have $r(I) + r(J) = r(e) + r(f) = r(fe)$. As $e, f \in S_r(R)$, we claim that $r(fe) = r(IJ)$ and hence by lemma 2.8, $r(I \cap J) = r(fe)$, where, $RefR = ReR \cap RfR$. This says $I \cap J$ is a right strongly Baer ideal. To prove our claim, let $x \in r(fe)$. Then $fex = 0$. This implies $ex \in r(f) = r(J)$. Thus $Jex = 0$ and so $eJex = 0$, i.e., $eJx = 0$. This says $Jx \in r(e) = r(I)$. Therefore $IJx = 0$, i.e., $x \in r(IJ)$. Now assume $x \in r(IJ)$. Then $IJx = 0$. Thus $Jx \subseteq r(I) = r(e)$. Hence $eJx = 0$. This shows $(JRexR)^2 = JRexRJRexR \subseteq JeJxR = 0$. By semiprime hypothesis, $JRexR = 0$. Thus $Jex = 0$, i.e., $ex \in r(J) = r(f)$. Hence $fex = 0$. This says $x \in r(fe)$, so $r(fe) = r(IJ)$.

(2) Let I be a right strongly Baer ideal and J be an ideal in R. Then Lemma 2.7 implies $\mathrm{int}_S V(I)$ is finite, so we have $\mathrm{int}_S V(I+J) = int_S V(I) \cap int_S V(J)$ is finite. Hence by Lemma 2.7, $I + J$ is a right strongly Baer ideal.

(3) This follows from Lemmas 2.6 and 2.7.

(4) If M is essential as the right ideal, then by part (3) we are done. Otherwise, $M = eR$, for some idempotent e. We may have $RM = R$ or $RM = M = eR$ (i.e., $e \in S_l(R)$). When $RM = R$, it is a right strongly Baer ideal. For the second case, by semiprime hypothesis, e is a central idempotent and hence $M = Re$. Thus $r(M) = r(e)$ and $M = ReR$. □

Put $SB(R) = \{I : I \text{ is a right strongly Baer ideal of } R\}$. Then by Proposition 2.9, whenever R is a semiprime ring, $SB(R)$ partially ordered by inclusion is a complete sub-lattice of the lattice of ideals with $I \vee J = I + J$ and $I \wedge J = I \cap J$.

Let L be a lattice with a least element 0 and a greatest element 1. A *complemented* of the element $a \in L$ is an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. If each element in a lattice L has a complement, then L is said to be complemented. A *Boolean algebra* is a complemented distributive lattice, see [20]. A ring R is called *semisimple* if R is a direct sum of minimal right ideals. It is well-known that R is semisimple if and only if every right ideal of R is a direct summand of R.

Theorem 2.10. For any ring R the following statements are equivalent.

(1) Every ideal of R is a right strongly Baer ideal.
(2) Every Baer ideal of R is a right strongly Baer ideal.
(3) The zero ideal is a right strongly Baer ideal.
(4) R is a semiprime ring and $\mathrm{Spec}\,(R) = \mathrm{Max}\,(R)$ is finite.
(5) R is a semisimple ring.
(6) R is a semiprime ring and $SB(R)$ is a Boolean algebra.

Proof. (1)⇒(2) Trivial.

(2)⇒(3) The zero-ideal is a Baer ideal and hence is a right strongly Baer ideal, by hypothesis.

(3)⇒(4) The zero ideal is right strongly Baer. Thus $R = r(0) = r(e)$, where $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$ and each ideal Re_iR $(1 \leq i \leq n)$ is maximal or equals R. There is a $1 \leq j \leq n$ such that Re_jR is maximal. Thus $Re_1R \cap Re_2R \cap ... \cap Re_nR = 0$. This equality shows

$$\mathrm{Spec}\ R = \mathrm{Max}\,(R) = \{Re_1R, Re_2R, ..., Re_nR\}.$$

Thus $N(R) = \bigcap_{i=1}^{n} Re_iR = 0$, i.e., R is semiprime.

(4)⇒(5) Trivially R is a finite product of fields. Hence R is a semisimple ring.

(5)⇒(6) The ring R is semisimple, so is a semiprime ring and there are division rings $D_1, D_2, ..., D_n$ such that $R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times ... \times M_{n_k}(D_k)$. Thus every ideal of R is generated by an idempotent and it is a finite intersection of maximal ideals and any of these maximal ideals is generated by an idempotent. Hence every ideal of R is right strongly Baer. Now let I be an ideal of R. Then $I = eR$ for some idempotent e of R. Put $J = (1-e)R$. Then J is a right strongly Baer ideal, $I \wedge J = 0$ and $I \vee J = R$, i.e., J is a complement of I. It is enough to show $SB(R)$ is a distributive lattice. To see this, first, we note that since R is semiprime and every ideal of R is a right annihilator ideal, it is a semiprime ideal. Next, let I, J and K be three ideals of R. Always we have $I \cap J + I \cap K \subseteq I \cap (J+K)$. Suppose $x \in I \cap (J+K)$. Then $x^2 \in I \cap K + I \cap J$. As $I \cap J$ and $I \cap K$ are two right strongly Baer ideals, $I \cap J + I \cap K$ is a semiprime ideal, so $x \in I \cap J + I \cap K$. Thus $I \cap (J+K) = I \cap J + I \cap K$.

(6)⇒(1) As in a distributive lattice complements are unique, and R is a right strongly Baer ideal, so the complement of R which is zero ideal is a right strongly Baer ideal. Thus $r(0) = r(e)$, where $ReR = Re_1R \cap Re_2R \cap ... \cap Re_nR$ and each ideal Re_iR $(1 \leq i \leq n)$ is maximal or equals R. This implies $Re_1R \cap Re_2R \cap ... \cap Re_nR = 0$ and each ideal Re_iR is maximal. At least one of them is maximal. This equality shows that $\text{Spec}\,(R) = \text{Max}\,(R)$ is finite. By Lemma 2.7, every ideal of R is a right strongly Baer ideal. □

For the proof of (5)⇒(6) of the above theorem we can give an alternative proof. As R is semisimple, every ideal of R is a right annihilator ideal. Also, every ideal of R is generated by an idempotent and it is a finite intersection of maximal ideals and any of these maximal ideals is generated by an idempotent. Hence every ideal of R is right strongly Baer. Thus $SB(R) = rAnn(id(R))$. Now, Theorem 1.1 of [10] implies $SB(R)$ is a Boolean algebra.

Corollary 2.11. Every ideal of $R/N(R)$ is right strongly Baer if and only if $\text{Spec}\,(R) = \text{Max}\,(R)$ is finite.

Proof. (1) If every ideal of $R/N(R)$ is strongly Baer, then by Theorem 2.10, we have $\text{Spec}\,(R/N(R)) = \text{Max}\,(R/N(R))$ is finite and hence $\text{Spec}\,(R)$ is finite. Now let $P \in \text{Spec}\,(R)$. Then $P/N(R) \in \text{Max}\,(R/N(R)$. So we must have $P \in \text{Max}\,(R)$, i.e., $\text{Spec}\,(R) = \text{Max}\,(R)$ is finite. Conversely, the finiteness of Spec (R) = Max (R), implies Spec $(R/N(R))$ = Max $(R/N(R))$ is finite. Thus by Theorem 2.10, we are done. □

Lemma 2.12. The following statements hold.

(1) If I is an ideal of R containing $J(R)$, then
$$\operatorname{int}_M M(\overline{I}) = \{\overline{M} : M \in \operatorname{int}_M M(I)\},$$
where $\overline{I} = I/J(R)$.

(2) If I is an ideal of R containing $N(R)$, then
$$\operatorname{int}_S V(\overline{I}) = \{\overline{P} : P \in \operatorname{int}_S V(I)\},$$
where $\overline{I} = I/N(R)$.

Proof. (1) It is easy to see that $aI \subseteq M$ if and only if $\overline{a}\overline{I} \subseteq \overline{M}$. Thus $D(aI) = \emptyset$ if and only if $D(\overline{a}\overline{I}) = \emptyset$. Hence we have $\overline{M} \in \operatorname{int}_M M(\overline{I})$ if and only if $\overline{M} \in D(\overline{a}) \subseteq M(\overline{I})$ for some $\overline{a} \in R/J(R)$ if and only if $D(\overline{a}\overline{I}) = D(\overline{aI}) = \emptyset$, and $M \in D(a)$ for some $a \in R$. This is equivalent to the $D(aI) = \emptyset$ for some $a \in R$ and $M \in D(a)$, i.e., $M \in D(a) \subseteq M(I)$, for some $a \in R$. So we are done.

(2) The proof is similar to the proof of (1). □

Lemma 2.12 implies the next result.

Corollary 2.13. Let I be an ideal of a semiprime ring R containing $J(R)$. Then I is right strongly Baer in R if and only if $\overline{I}$ is right strongly Baer in $R/J(R)$.

The above result together with Theorem 2.10 imply the next result.

Corollary 2.14. For a semiprime ring R the following statements are equivalent.

(1) Every ideal of $R/J(R)$ is right strongly Baer.
(2) Max (R) is finite.
(3) $J(R)$ is a right strongly Baer ideal.

3. Right strongly Baer ideals in extension rings

In this section T will denote a 2-by-2 generalized (or formal) triangular matrix ring $\begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where R and S are rings and M is an (S, R)-bimodule. Whenever N is an (S, R)-submodule of M (briefly, ${}_SN_R \leq_S M_R$), $Ann_R N = \{r \in R : Nr = 0\}$ and $Ann_S N = \{s : sN = 0\}$, see [16] . In this section we use the results of Birkenmeier, Kim, and Park in [8] and characterize right strongly Bear-ideals of 2-by-2 generalized triangular matrix rings.

Also, we characterize right strongly Baer-ideals in full and upper triangular matrix rings. By using these results, we obtain the well-known results about T and $M_n(R)$, as a semisimple ring. We use the notation $[a_{ij}]$ for the square matrix whose (i,j)th position is a_{ij}.

Theorem 3.1. An ideal J of $\mathbf{M_n}(\mathbf{R})$ is a right strongly Baer-ideal if and only if $J = \mathbf{M_n}(\mathbf{I})$, for some right strongly Baer-ideal I of R.

Proof. Let J be a right strongly Baer-ideal of $\mathbf{M_n}(\mathbf{R})$. By [15, Theorem 3.1], $J = \mathbf{M_n}(\mathbf{I})$, for some ideal I of R. We claim that I is a right strongly Baer-ideal. By hypothesis, there exists an idempotent $E \in M_n(R)$ such that $r(J) = r(E)$ and

$$M_n(R)EM_n(R) = M_n(R)E_1M_n(R) \cap M_n(R)E_2M_n(R) \cap ... \cap M_n(R)EM_n(R),$$

where for each $1 \leq i \leq n$, $E_i \in S_r(M_n(R))$ and each ideal $M_n(R)E_iM_n(R)$ is maximal or it is $M_n(R)$. By [6, Lemma 3.1], $r_{M_n(R)}(J) = \mathbf{M_n}(\mathbf{r_R}(\mathbf{I}))$. If for each $1 \leq i \leq n$, the ideal $M_n(R)E_iM_n(R)$ equals $M_n(R)$, then $r_{M_n(R)}(J) = r_{M_n(R)}(M_n(R)) = 0$, and hence $r_R(I) = 0 = r(R)$, so we are done. We assume the next case. Then there is $l \in \mathbb{N}$ such that for each $1 \leq i \leq l$, there is a maximal ideal H_i in R such that $M_n(R)E_iM_n(R) = M_n(H_i)$. By [22, Theorem 3.3], in each matrix E_k $(1 \leq k \leq l)$, $(e_k)_{ij} = (e_k)_{ij}(e_k)_{11}$, where $(e_k)_{ij}$ is the (i,j)-position in the matrix E_k. Thus, it is easy to see that, for each $1 \leq k \leq l$, $H_k = R(e_k)_{11}R$. We claim that $r_R(I) = r_R(e_{11})$ and $Re_{11}R = H_1 \cap H_2 \cap ... \cap H_l$, where e_{11} is the $(1,1)$-th position in E. Let $x \in r_R(I)$. Then $A \in M_n(r_R(I)) = r_{M_n(R)}(J) = r_{M_n(R)}(E)$, where $a_{11} = x$ and zero elsewhere. This implies $EA = 0$, and hence $e_{11}x = 0$, i.e., $x \in r_R(e_{11})$. Now let $z \in r_R(e_{11})$. By [22, Theorem 3.3], in matrix E we have $e_{ij} = e_{ij}e_{11}$, for each $1 \leq i,j \leq n$. Then $EB = 0$, where $b_{11} = z$ and zero elsewhere. This shows $B \in r_{M_n(R)}(J) = M_n(r_R(I))$. Thus $z \in r_R(I)$. For the proof of other our claim, we know that $C \in M_n(R)EM_n(R)$, where $c_{11} = e_{11}$ and zero elsewhere. Thus $C \in M_n(H_i)$, for each $1 \leq i \leq l$. This implies $e_{11} \in H_i$, for each $1 \leq i \leq l$. Thus $Re_{11}R \subseteq H_1 \cap H_2 \cap ... \cap H_l$. For the converse of the inclusion, consider $x \in H_1 \cap H_2 \cap ... \cap H_l$. Then

$$A \in M_n(R)E_1M_n(R) \cap M_n(R)E_2M_n(R) \cap ... \cap M_n(R)E_lM_n(R),$$

where $a_{11} = x$ and zero elsewhere. Thus $A \in M_n(R)EM_n(R)$. This implies $x \in Re_{11}R$, so $H_1 \cap H_2 \cap ... \cap H_l \subseteq Re_{11}R$.

Now suppose that I is a strongly right Baer ideal of R. Then $r_R(I) = 0$ or $r_R(I) = r_R(e)$, for some idempotent e of R and there are right semicentral idempotents $e_1, e_2, ..., e_l$ of R such that $ReR = Re_1R \cap Re_2R \cap ... \cap Re_lR$

and each ideal Re_iR is maximal in R. Put $J = M_n(I)$. Then by [6, Lemma 3.1], $r_{M_n(R)}(J) = \mathbf{M_n}(\mathbf{r_R}(\mathbf{I}))$. If $r_R(I) = 0$, then $r_{M_n(R)}(J) = 0 = r_{M_n(R)}(1)$. Assume I satisfies in the next case. We have

$$r_{M_n(R)}(J) = M_n(r_R(I)) = M_n(r_R(e)).$$

We claim that $M_n(r_R(e)) = r_{M_n(R)}(E)$, and

$$\begin{aligned} M_n(R)EM_n(R) &= M_n(R)E_1M_n(R) \cap M_n(R)E_2M_n(R) \cap ... \cap M_n(R)E_lM_n(R) \\ &= M_n(Re_1R) \cap M_n(Re_2R) \cap ... \cap M_n(Re_lR), \end{aligned}$$

where for each $1 \le k \le l$ in matrix E_k, the (i,i)-th position equals e_k and elsewhere is zero and in matrix E for each $1 \le i \le n$, $e_{ii} = e$ and $e_{ij} = 0$ for all $j \neq i$ $(1 \le j \le n)$. To see this, first assume $A = [a_{ij}] \in M_n(r_R(e))$. Then for each $1 \le i, j \le n$, $a_{ij} \in r_R(e)$. Thus $EA = 0$, i.e., $A \in r_{M_n(R)}(E)$. Now let $B = [b_{ij}] \in r_{M_n(R)}(E)$. Then $eb_{ij} = 0$ for each $1 \le i, j \le n$. This says $b_{ij} \in r_R(e)$ for each $1 \le i, j \le n$, i.e., $B \in M_n(r_R(e))$. Trivially, the equality

$$\begin{aligned} &M_n(R)E_1M_n(R) \cap M_n(R)E_2M_n(R) \cap ... \cap M_n(R)E_lM_n(R) \\ &= M_n(Re_1R) \cap M_n(Re_2R) \cap ... \cap M_n(Re_lR) \end{aligned}$$

holds. We have $E \in M_n(Re_1R) \cap M_n(Re_2R) \cap ... \cap M_n(Re_lR)$. Thus

$$M_n(R)EM_n(R) \subseteq M_n(Re_1R) \cap M_n(Re_2R) \cap ... \cap M_n(Re_lR).$$

Now let $A = [a_{kj}] \in M_n(Re_1R) \cap M_n(Re_2R) \cap ... \cap M_n(Re_lR)$. Then $a_{kj} \in \bigcap_{i=1}^{l} Re_iR = ReR$, for each $1 \le k, j \le n$. Thus $A \in M_n(R)EM_n(R)$. On the other hand, since for each $1 \le k \le l$, $e_k \in S_r(R)$, hence $E_k \in S_r(M_n(R))$. So this completes the proof. □

From Theorems 2.10 and 3.1, we conclude the following well-known result.

Corollary 3.2. A ring R is semisimple if and only if $M_n(R)$ is semisimple.

For every $I \trianglelefteq \mathbf{T_n}(\mathbf{R})$, there are ideals J_{ik} of R, $1 \le i, k \le n$ such that

$$I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}, J_{ik} \subseteq J_{ik+1}$$

and $J_{i+1k} \subseteq J_{ik}$, see Part 1 of Theorem 3.2 in [21]. Trivially, if I is a maximal ideal of $T_n(R)$, then all $J_{ij} = R$ $(1 \le i, j \le n)$ except one of J_{ii} which is a maximal ideal. Now, we want to give a characterization of strongly Baer ideals in an upper triangular matrices ring.

Theorem 3.3. If an ideal I of $\mathbf{T_n}(\mathbf{R})$ is a right strongly Baer ideal, then each J_{1k} $(1 \leq k \leq n)$ is a right strongly Baer-ideal of R.

Proof. Let I be a right strongly Baer ideal of $T_n(R)$. By comments before of the theorem,

$$I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}, J_{ik} \subseteq J_{ik+1} \quad \text{and } J_{i+1k} \subseteq J_{ik}.$$

By hypothesis, $r_{Tn(R)}(I) = 0$ or there is an $E \in S_r(T_n(R))$ with $r_{T_n(R)}(I) = r_{T_n(R)}(E)$, and there are $E_1, E_2, ..., E_l \in S_r(T_n(R))$ such that;

$$Tn(R)ET_n(R) = T_n(R)E_1T_n(R) \cap T_n(R)E_2T_n(R) \cap ... \cap T_n(R)E_lT_n(R),^{(1)}$$

and for each $1 \leq i \leq l$, the ideal $T_n(R)E_iT_n(R)$ is maximal. We have,

$$r_{T_n(R)}(I) = \begin{pmatrix} r_R(J_{11}) & r_R(J_{11}) & . & . & . & r_R(J_{11}) \\ 0 & r_R(J_{12}) & . & . & . & r_R(J_{12}) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_R(J_{1n}) \end{pmatrix}.$$

Thus for each $1 \leq j \leq n$, $r_R(J_{1j}) = r_R(e_{jj})$, where e_{jj} is the (j, j)-position in the matrix E. Let

$$E_k = \begin{pmatrix} (e_k)_{11} & (e_k)_{12} & . & . & . & (e_k)_{1n} \\ 0 & (e_k)_{22} & . & . & . & (e_k)_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & (e_k)_{nn} \end{pmatrix},$$

for each $1 \leq k \leq l$. By [22, Theorem 3.3], for each $1 \leq k \leq l$, we have,

$$T_n(R)E_kT_n(R) = \begin{pmatrix} R(e_k)_{11}R & R(e_k)_{22}R & . & . & . & R(e_k)_{nn}R \\ 0 & R(e_k)_{22}R & . & . & . & R(e_k)_{nn}R \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & R(e_k)_{nn}R \end{pmatrix}.$$

As for each $1 \leq k \leq l$, $T_n(R)E_kT_n(R)$ is a maximal ideal, so all $R(e_k)_{ij}R$ ($1 \leq k \leq l$, $1 \leq i, j \leq n$ and $i \geq j$) equal R except one of $(e_k)_{ii}$ which is a maximal ideal. Thus, by the Equality (1), for each $1 \leq k \leq l$ we have $Re_{kk}R = R$ or it is a finite intersection of maximal ideals which any of them is generated by a right semicentral idempotent of R. So we are done. □

The converse of the above result is not true. If we consider the field $\mathbb{R}$, then the zero-ideal is a right strongly Baer ideal. However, the zero-ideal in $T_2(\mathbb{R})$ is not a right strongly Baer ideal. Since $T_2(\mathbb{R})$ is not semisimple.

We are including the following lemma for completeness since it is used in the next result. Its proof is easy and get from Proposition 1.17 in [15].

Lemma 3.4. An ideal $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ is a maximal ideal if and only if

(i) $N = M$.

(ii) $I = S$ and L is a maximal ideal of R
or $L = R$ and I is a maximal ideal of S.

As $e \in S_r(R)$ if and only if $1 - e \in S_l(R)$, from Lemma 2.3 in [8], we have the following result.

Lemma 3.5. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$ be an idempotent element of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then $e \in S_r(T)$ if and only if

(i) $e_1 \in S_r(S)$;

(ii) $e_2 \in S_r(R)$;

(iii) $ke_2 = k$; and

(iv) $e_1me_2 = e_1m$, for all $m \in M$.

Lemma 3.6. [8, Lemma 3.1]. Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then

$$r(J) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}$$

and

$$l(J) = \begin{pmatrix} l_S(I) \cap Ann_S(N) & l_M(L) \\ 0 & l_R(L) \end{pmatrix}.$$

The next result gives a characterization of right strongly Baer ideals in a 2-by-2 generalized triangular matrix ring.

Theorem 3.7. Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

Then J is a right strongly Baer-ideal of T if and only if
(i) I is a right strongly Baer-ideal of S;
(ii) $r_M(I) = (r_S(I))M$; and
(iii) $r_R(L) \cap Ann_R(N) = r_R(a)$, for some $a^2 = a \in R$ and there are $a_1, a_2, ..., a_n \in S_r(R)$ such that each ideal Ra_iR is a maximal ideal of R or equals R, $RaR = Ra_1R \cap Ra_2R \cap ... \cap Ra_nR$ and $Ma_i = M$

Proof. Let J be a right strongly Baer-ideal of T. Then there exists an idempotent $e \in T$ such that $r_T(J) = r_T(e)$ and there are $e_1, e_2, ..., e_n \in S_r(T)$ with

$$TeT = Te_1T \cap Te_2T \cap ... \cap Te_nT, \quad (1)$$

and each ideal Te_iT is maximal or equals T. By Lemma 3.5, $e = \begin{pmatrix} e_{11} & k \\ 0 & e_{22} \end{pmatrix}$, and $e_i = \begin{pmatrix} (e_i)_{11} & k_i \\ 0 & (e_i)_{22} \end{pmatrix}$, where $e_{11}, (e_i)_{11} \in S_r(S), e_{22}, (e_i)_{22} \in S_r(R)$ and $k, k_i \in M$, for each $1 \leq i \leq n$. By Part 4 of Lemma 2.3 in [8], we can see that

$$r_T(e) = \begin{pmatrix} r_S(e_{11}) & r_S(e_{11})M \\ 0 & r_R(e_{22}) \end{pmatrix}.$$

Thus by the equality $r_T(J) = r_T(e)$, we have $r_S(I) = r_S(e_{11})$, $r_M(I) = r_S(I)M$ and $r_R(L) \cap Ann_R(N) = r_R(e_{22})$. On the other hand, by Lemma 3.5, for each $1 \leq i \leq n$, $k_i = k_i(e_i)_{22}$. Thus

$$Sk_i = Sk_i(e_i)_{22} \subseteq M(e_i)_{22}.$$

For each $1 \leq i \leq n$, this implies;

$$Te_iT = Te_i = \begin{pmatrix} S(e_i)_{11} & Sk_i + M(e_i)_{22} \\ 0 & R(e_i)_{22} \end{pmatrix} = \begin{pmatrix} S(e_i)_{11} & M(e_i)_{22} \\ 0 & R(e_i)_{22} \end{pmatrix}.$$

This together with equality (1) shows that $Se_{11}S = Se_{11} = \bigcap_{i=1}^{n} S(e_i)_{11}$ and $Re_{22}R = Re_{22} = \bigcap_{i=1}^{n} R(e_i)_{22}$. By Lemma 3.4, the maximality of each Te_iT ($1 \leq i \leq n$) implies each ideal $S(e_i)_{11}$ (resp., $R(e_i)_{22}$) is maximal or equals S (resp., R) and $M(e_i)_{22} = M$. So we are done.

Conversely, by hypothesis, there are $e \in S_r(S)$ and $a^2 = a \in R$ such that $r_S(I) = r_S(e)$ and $r_R(L) \cap Ann_R(N) = r_R(a)$. Also, there are $e_i \in S_r(S)$ and $a_j \in S_r(R)$ ($1 \leq i \leq n$, $1 \leq j \leq k$) such that $SeS = Se = Se_1 \cap Se_2 \cap ... \cap Se_n$ and $Ra = Ra_1 \cap Ra_2 \cap ... \cap Ra_k$ and $Ma_j = M$. Since $Ann_R(N) \trianglelefteq R$, hence $a \in S_r(R)$. By (ii), $r_M(I) = (r_S(I))M = r_S(e)M$. Now let $E = \begin{pmatrix} e & 0 \\ 0 & a \end{pmatrix}$. Then we can see that;

$$r_S(E) = \begin{pmatrix} r_S(e) & r_S(e)M \\ 0 & r_R(a) \end{pmatrix} = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}.$$

By this equality and Lemma 3.5, $r_T(J) = r_T(E)$. Now for each $1 \leq i \leq n$ and $1 \leq j \leq k$, put $E_i = \begin{pmatrix} e_i & 0 \\ 0 & 1 \end{pmatrix}$ and $A_j = \begin{pmatrix} 1 & 0 \\ 0 & a_j \end{pmatrix}$. Then by hypothesis ($Ma_j = M$) and Lemma 3.5, for each $1 \leq i \leq n$ and $1 \leq j \leq k$, $E_i, A_j \in S_r(T)$ and we have;

$$TE_i = \begin{pmatrix} Se_i & M \\ 0 & R \end{pmatrix}, TA_j = \begin{pmatrix} S & Ma_j \\ 0 & Ra_j \end{pmatrix}.$$

The Lemma 3.4 shows that TE_i and TA_j are maximal ideals of T, for each $1 \leq i \leq n$ and $1 \leq j \leq k$. On the other hand, it is easily seen that $TET = ET = (\bigcap_{i=1}^{n} E_iT) \cap (\bigcap_{j=1}^{k} A_jT)$. This completes the proof. □

From Theorems 2.10 and 3.7 we have the following result.

Corollary 3.8. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where M is an (S, R)-bimmodule. Then T is semisimple if and only if

(i) S is semisimple;

(ii) For every ideal I of S, $r_S(I) = r_M(I)S$;

(iii) For every ideal L of R and every submodule N of M,

$$r_R(L) \cap Ann_R(N) = r_R(a),$$

for some $a^2 = a \in R$ and there are $a_1, a_2, ..., a_n \in S_r(R)$ such that each ideal Ra_iR is a maximal ideal of R or equals R, $RaR = Ra_1R \cap Ra_2 \cap ... \cap Ra_nR$ and $Ma_i = M$.

4. Soc(R) (resp., Soc$_{\mathsf{max}}(R)$) as strongly Baer ideal in commutative rings

In this section all rings are commutative. The intersection of all essential maximal ideals of a commutative ring R is denoted by $\mathrm{Soc}_{\mathsf{max}}(R)$, see [12] and [24]. Now in the next results, we give a point-wise characterization of $\mathrm{Soc}(R)$ (resp., $\mathrm{Soc}_{\mathsf{max}}(R)$), whenever R is a commutative semiprimitive ring.

Theorem 4.1. Let R be a commutative semiprimitive ring.

(1) $\mathrm{Soc}(R) = \{a \in R : \mathrm{Max}\,(R) \setminus M(a)$ is a finite subset of $I(\mathrm{Max}\,(R))\}$.
(2) $\mathrm{Soc}_{\mathsf{max}}(R) = \{a \in R : \forall b \in R, M(1-ab)$ is a finite subset of $I(\mathrm{Max}\,(R))\}$.

Proof. (1) See Theorem 2.3 in [18].

(2) Let $a \in \mathrm{Soc}_{\mathsf{max}}(R)$. For each $b \in R$, we have $M(1-ab) \cap M(a) = \emptyset$. Thus $M(1-ab) \subseteq \mathrm{Max}\,(R) \setminus M(a) \subseteq I(\mathrm{Max}\,(R))$, by Lemma 3.2 in [12]. As $\mathrm{Max}\,(R)$ is compact, hence $M(1-ab)$ is a compact subset of $I(\mathrm{Max}\,(R))$. Thus it must be finite. Now, assume $M(1-ab)$ is a finite subset of $I(\mathrm{Max}\,(R))$ for each $b \in R$, and M is an essential maximal ideal with $a \notin M$. Then $M + Ra = R$. So there exists $c \in R$ such that $1 - ca \in M$. This shows $M \in M(1-ca)$. Thus M must be an isolated point of $\mathrm{Max}\,(R)$, which is a contradiction, by Lemma 2.6. This shows $a \in \mathrm{Soc}_{\mathsf{max}}(R)$. □

Proposition 4.2. Let R be a commutative semiprimitive ring. The following statements are equivalent.

(1) $\mathrm{Soc}(R)$ is an essential ideal.
(2) $\mathrm{Soc}(R)$ is a strongly Baer ideal.
(3) The set of isolated points of $\mathrm{Max}\,(R)$ is dense in it.

Proof. The equivalency of (1) and (3) is proved in Proposition 3.3 of [12]. It is enough to prove the equivalency of (2) and (3).

(2)⇒(3) First we claim that $M(\mathrm{Soc}(R)) = \mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$. To see this, assume M is a maximal ideal containing $\mathrm{Soc}(R)$ and $\{M\}$ is an isolated point of $\mathrm{Max}\,(R)$. Then by Lemma 2.5, $M = Re$, for some idempotent $e \in R$. This implies $R(1-e)$ is a minimal ideal of R and hence

$$R(1-e) \subseteq \mathrm{Soc}(R) \subseteq M,$$

a contradiction. Next, let $M \in \mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$. Then M is an essential ideal and hence M containing $\mathrm{Soc}(R)$. The hypothesis and Lemma 2.7 imply $\mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$ is finite. As $\mathrm{Max}\,(R)$ is a T_1-space, the finiteness of $\mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R)))$ implies each $x \in \mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R)))$ is an isolated point, i.e., $x \in I(\mathrm{Max}\,(R))$, a contradiction. Thus we must have $\mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))) = \emptyset$, i.e., $I(\mathrm{Max}\,(R))$ is dense in $\mathrm{Max}\,(R)$.

(3)⇒(2) As $M(\mathrm{Soc}(R)) = \mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$, the hypothesis implies that $\mathrm{int}_M M(\mathrm{Soc}(R)) = \mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))) = \emptyset$. So by Lemma 2.7, $\mathrm{Soc}(R)$ is a strongly Baer ideal. □

Proposition 4.3. Let R be a commutative semiprimitive ring. The following statements are equivalent.

(1) $\mathrm{Soc}_{\mathsf{max}}(R)$ is an essential ideal.
(2) $\mathrm{Soc}_{\mathsf{max}}(R)$ is a strongly Baer ideal.
(3) The set of isolated points of $\mathrm{Max}\,(R)$ is dense in it.

Proof. (1)⇒(2) This follows from Proposition 2.9.

(2)⇒(3) We know that $\mathrm{Soc}_{\mathsf{max}}(R)$ is the intersection of all essential maximal ideals. By Lemma 2.5, every essential maximal ideal is a non-isolated point of $\mathrm{Max}\,(R)$. Thus $\mathrm{Soc}_{\mathsf{max}}(R) = O_{\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))}$. The Lemma 2.7 implies that $\mathrm{int}_M M(\mathrm{Soc}_{\mathsf{max}}(R)) = \mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$ is finite. As $\mathrm{Max}\,(R)$ is a T_1-space, the finiteness of $\mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R)))$ implies each $x \in \mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R)))$ is an isolated point, i.e., $x \in I(\mathrm{Max}\,(R))$, a contradiction. Thus we must have $\mathrm{int}_M(\mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))) = \emptyset$, i.e., $I(\mathrm{Max}\,(R))$ is dense in $\mathrm{Max}\,(R)$.

(3)⇒(1) As $M(\mathrm{Soc}_{\mathsf{max}}(R)) = \mathrm{Max}\,(R) \setminus I(\mathrm{Max}\,(R))$. The hypothesis implies $\mathrm{int}_M M(\mathrm{Soc}_{\mathsf{max}}(R)) = \emptyset$. So by Lemma 2.6, $\mathrm{Soc}_{\mathsf{max}}(R)$ is a essential ideal. □

As a subspace of $\mathrm{Spec}\,(R)$, the space of minimal prime ideals, is denoted by $\mathrm{Min}\,(R)$. Thus the set $\{D(a) : a \in R\}$ is a base for open sets in this space, where $D(a) = \mathrm{Min}\,(R) \setminus m(a)$ and $m(a) = \{P \in \mathrm{Min}\,(R) : a \in P\}$. For an open subset A of $\mathrm{Min}\,(R)$, O_A is the intersection of all minimal prime ideals in A. For a subset H of $\mathrm{Min}\,(R)$, we denote by $\mathrm{int}_{\mathcal{M}} H$, the interior of H in $\mathrm{Min}\,(R)$.

Proposition 4.4. Let R be a commutative reduced ring.

(1) Every intersection of essential minimal prime ideals is an essential ideal.
(2) Every intersection of essential minimal prime ideals is a strongly Baer ideal.
(3) The set of isolated points of $\mathrm{Min}\,(R)$ is dense in it.

Proof. (1)⇒(2) This follows from Proposition 2.9.

(2)⇒(3) By hypothesis and Lemma 3.2 in [23], $O_{\mathrm{Min}\,(R)\setminus I(\mathrm{Min}\,(R))}$ is a strongly Baer ideal. Hence

$$\mathrm{int}_{\mathcal{M}}\, m(O_{\mathrm{Min}\,(R)\setminus I(\mathrm{Min}\,(R))}) = \mathrm{int}_{\mathcal{M}}(\mathrm{Min}\,(R) \setminus I(\mathrm{Min}\,(R)))$$

is a finite subset of $\mathrm{Min}\,(R)$. As $\mathrm{Min}\,(R)$ is a Hausdorff space, every point of $\mathrm{int}_{\mathcal{M}}(\mathrm{Min}\,(R) \setminus I(\mathrm{Min}\,(R)))$ (since it is finite) is an isolated point of $\mathrm{Min}\,(R)$, which is a contradiction. Thus $\mathrm{int}_{\mathcal{M}}(\mathrm{Min}\,(R) \setminus I(\mathrm{Min}\,(R)))$ is an empty set. This shows $I(\mathrm{Min}\,(R))$ is dense in $\mathrm{Min}\,(R)$.

(3)⇒(1) See Proposition 3.3 in [23]. □

5. STRONGLY BAER IDEALS IN $C(X)$ AND $C(X)_F$

In this section, we investigate strongly Baer ideals in $C(X)$ (resp., $C(X)_F$). We denote by $C(X)$ (resp., $C(X)_F$), the ring of all real-valued continuous functions on a completely regular Hausdorff space X (resp., the ring of functions which have at most a finite number of non-continuous points). We note that $C(X)$ (resp., $C(X)_F$) is a reduced ring. For any $f \in C(X)$,

$$Z(f) = \{x \in X : f(x) = 0\}$$

is called a zero-set. For $f \in C(X)$, the ideal generated by f is denoted by $< f >$. A maximal ideal in $C(X)$ is of the for M^p, where $p \in \beta X$. If $p \in X$, it is denoted by M_p. For $A \subset \beta X$, M^A (resp., O^A) is all $f \in C(X)$, with $A \subseteq \mathrm{cl}_{\beta X} Z(f)$ (resp., $A \subseteq \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X} Z(f)$). For more details about $C(X)$ the reader ie referred to [13]. We need the following well-known results in the sequel.

Lemma 5.1. For $p \in X$, the following statements hold.

(1) The ideal M_p is a principal ideal of $C(X)$ if and only if p is an isolated point of X.
(2) If $p \in I(X)$, then $(\chi_{\{p\}}) = M_{X\setminus\{p\}}$ and $(\chi_{X\setminus\{p\}}) = M_p$.

We recall that $Z[I] = \{Z(f) : f \in I\}$.

Lemma 5.2. For ideals I and J of $C(X)$, $r(I) \subseteq r(J)$ if and only if

$$\mathrm{int} \bigcap Z[I] \subseteq \mathrm{int} \bigcap Z[J].$$

Below we use the above lemma and give a topological characterization of strongly Baer ideals in $C(X)$.

Lemma 5.3. An ideal I of $C(X)$ is strongly Baer if and only if $\mathrm{int} \cap Z[I]$ is finite (hence is a finite subset of isolated points of X).

Proof. Let I be a strongly Baer ideal. Then $r(I) = r(e)$, where

$$< e >=< e_1 > \cap < e_2 > \cap ... \cap < e_n >$$

and each ideal $< e_i >$ $(1 \leq i \leq n)$ is maximal or each $e_i = 1$. Now, lemma 5.1 implies that there is $p_i \in I(X)$ such that $e_i = \chi_{X \setminus \{p_i\}}$ for each $1 \leq i \leq n$. Thus we have

$$\text{int} \cap Z[I] = intZ(e) = int(Z(e_1) \cup ... \cup Z(e_n)) = \{p_1, ..., p_n\}.$$

Conversely, suppose that $\text{int} \cap Z[I] = \{p_1, ..., p_n\}$. Then for each $1 \leq i \leq n$, $\{p_i\} = Z(\chi_{X \setminus \{p_i\}})$. For each $1 \leq i \leq n$, put $e_i = \chi_{X \setminus \{p_i\}}$. Then

$$\text{int} \cap Z[I] = intZ(e_1 \cdot ... \cdot e_n).$$

Therefore $r(I) = r(e)$, where $e = e_1 \cdot ... \cdot e_n$. It is clear that if $\text{int} \cap Z[I] = \emptyset$, then $r(I) = 0 = r(1)$. □

Corollary 5.4. Every ideal of $C(X)$ is strongly Baer if and only if X is finite.

Example 5.5. (1) It is easy to see that for every prime ideal P of $C(X)$, we have $\text{int} \cap Z[P]$ is $\emptyset$ or one point, so every prime ideal of $C(X)$ is a strongly Baer ideal.

(2) For each $p \in \beta X$, the ideal $O^p = \{f \in C(X) : p \in int_{\beta X} cl_{\beta X} Z(f)\}$ is a strongly Baer ideal. So if we take $p \in I(X)$, then O_p is a strongly Baer ideal which is not essential.

(3) Let A be an infinite proper clopen subset of X (e.g., X be an infinite disconnected space). Then M_A is a Bear ideal which is not a strongly Baer ideal.

As many famous ideals of $C(X)$ are of the form M^A or O^A (e.g., $C_K(X) = O^{\beta X \setminus X}$, $C_F(X) = M^{\beta X \setminus I(X)}$, the intersection of all free maximal ideals and the intersection of all essential maximal ideals), so this is important to know when is O^A (M^A) a Baer (resp., strongly Baer) ideal?

Lemma 5.6. Let A be a closed subset of βX. Then the following are equivalent.

(1) The ideal O^A is a Baer-ideal.
(2) The ideal M^A is a Baer-ideal.
(3) $\text{int}(A \cap X)$ is a closed subset of X.

Proof. (1)⇔(3) By Lemma 1.6 in [9], $\bigcap_{f \in O^A} cl_{\beta X} Z(f) = A$. Hence

$$\text{int}(\bigcap_{f \in O^A} Z(f)) = int(A \cap X).$$

Hence by Proposition 4.5 in [21], O^A is a Baer ideal if and only if $\operatorname{int}(A \cap X)$ is a closed subset of X.

(2)⇔(3) This is similar to the (1)⇔(3). □

Lemma 5.7. Let A be a closed subset of βX. Then the following are equivalent.

(1) The ideal O^A is a strongly Baer ideal.
(2) The ideal M^A is a strongly Baer ideal.
(3) $\operatorname{int}(A \cap X)$ is a finite subset of X.

Proof. (1)⇔(3) By Lemma 1.6 in [9], $\bigcap_{f \in O^A} cl_{\beta X} Z(f) = A$. Hence

$$\operatorname{int}(\bigcap_{f \in O^A} Z(f)) = int(A \cap X).$$

By Lemma 5.3, O^A is a strongly Baer ideal if and only if $\operatorname{int}(A \cap X)$ is a finite subset of X.

(2)⇔(3) This is similar to the (1)⇔(3). □

The Lemma 5.6 and [23, Theorem 3.6] imply the following result.

Proposition 5.8. The following statements are equivalent.

(1) $Soc_m(C(X))$ is a Baer-ideal.
(2) $cl_X I(X)$ is an open subset of X.
(3) $C_F(X)$ is a Baer-ideal.

As $I(\mathrm{Max}\,(C(X))) = I(\beta X) = I(X)$. Proposition 4.3 implies the following result. Also, this could obtain from Lemma 5.7.

Lemma 5.9. The following statements are equivalent.

(1) $Soc_m(C(X))$ is a strongly Baer ideal.
(2) The set of isolated points of X is dense in it.
(3) $C_F(X)$ is a strongly Baer ideal.

We denote by X_L the set of all points of X with compact nhods. In fact, we have X is locally compact if and only if $X = X_L$. It also is well known that $X_L = int_{\beta X} X$. Recall from [13] that $C_K(X)$ is the set of all $f \in C(X)$ with $\mathrm{cl}_X(X \setminus Z(f))$ is a compact subset of X. It is clear that $C_K(X)$ is an ideal of $C(X)$. It is also well known that the intersection of all free maximal ideals of $C(X)$ is $M^{\beta X \setminus X}$.

Proposition 5.10. The following statements are equivalent.

(1) $C_K(X)$ is strongly Baer.
(2) X_L is dense in X.

(3) The intersection of all free maximal ideals is a strongly Baer ideal.

Proof. (1)⇔(2) It is well known that $C_K(X) = O^{\beta X \setminus X}$, by 7.E in [13]. So we have $M(C_K(X)) = cl_{\mathrm{Max}\,(C(X))}(\beta X \setminus X) = cl_{\beta X}(\beta X \setminus X)$. Now Lemma 2.7 implies $C_K(X)$ is strongly Baer if and only if $\mathrm{int}_{\beta X}\, cl_{\beta X}(\beta X \setminus X)$ is a finite subset of $I(\beta X)$. If we have $p \in I(\beta X) = I(X)$ and $p \in cl_{\beta X}(\beta X \setminus X)$, then we must have $(\beta X \setminus X) \cap I(X) \neq \emptyset$, which is a contradiction. Therefore $\mathrm{int}_{\beta X}\, cl_{\beta X}(\beta X \setminus X) = \emptyset$. This is equivalent to $cl_{\beta X}(X_L) = \beta X$, i.e., $cl_X(X_L) = X$.

(2)⇔(3) The proof is similar to (1)⇔(2). □

Let $T' = \{f : f|D \in C(D)$, for some dense subset D of $X\}$. It is well known that $T'(X)$ is a regular ring, see [1]. As we recalled in the first of the section for a topological space X, we have
$C(X)_F = \{f \in \mathbb{R}^X : f$ has at most a finite number of discontinuous points$\}$.
It is manifest that $C(X)_F$ is a sub-ring of $T'(X)$ containing $C(X)$. For terminology and notations, the reader is referred to [11].

Lemma 5.11. [11, Lemma 4.11] For ideals I, J of $C(X)_F$ (resp., $T'(X)$), $r(I) \subseteq r(J)$ if and only if $\bigcap Z[I] \subseteq \bigcap Z[J]$.

For any subset A of a space X the *boundary* of A is denoted by FrA and equals to the $cl_X(A) \cap cl_X(X \setminus A)$. In the next result, we give a characterization of idempotents in $C(X)_F$ (resp., $T'(X)$).

Lemma 5.12. An element $e \in C(X)_F$ (resp., $T'(X)$) is idempotent if and only if $e = \chi_A$ for some subset A of X with $FrA = Fr(X \setminus A)$ is finite (resp., closed and nowhere dense).

Proof. Suppose that e is an idempotent of $C(X)_F$ (resp., $T'(X)$). Clearly $e = \chi_A$ for some subset A of X. It is sufficient to show that $FrA = disc(f)$ (i.e., the set of points of X which f is discontinuous on them). Let $x \in disc(f)$ such that $x \notin FrA$. Then there is a neighborhood U of x such that $U \subseteq A$ or $U \subseteq X \setminus A$. Let V be a neighborhood of $e(x)$, clearly $e(U) \subseteq V$, so f is continuous at x, a contradiction. This implies that $disc(f) \subseteq FrA$. Now let $x \in FrA \setminus disc(f)$. Then every neighborhood U of x has both points of A and of $X \setminus A$. It is easy to find a neighborhood V of $e(x)$ such that $e(U) \nsubseteq V$, which is a contradiction. So $x \in disc(f)$. □

The next result characterizes the class of Baer ideals and strongly Baer ideals in $C(X)_F$ (resp., $T'(X)$).

Proposition 5.13. The following statements hold.

(1) An ideal I of $C(X)_F$ (resp.,$T^{'}(X)$) is a Baer ideal if and only if $Fr\bigcap Z(I)$ is a finite (resp., closed and nowhere dense) subset of X.
(2) An ideal I of $C(X)_F$ (resp., $T^{'}(X)$) is a strongly Baer ideal if and only if $\bigcap Z[I]$ is a finite (resp., closed and nowhere dense) subset of X.

Proof. (1) Let I be a Baer-ideal of $C(X)_F$ (resp., $T^{'}(X)$). Then there exists an idempotent $e \in C(X)_F$ (resp., $T^{'}(X)$) such that $r(I) = eR = r(1-e)$. By Lemma 5.11, $\bigcap Z[I] = Z(1-e)$. This and Lemma 5.12 show that $Fr\bigcap Z[I]$ is a finite (resp., closed nowhere dense) subset of X. Conversely, suppose $Fr\bigcap Z[I]$ is finite. put $e = \chi_{X\setminus\bigcap Z[I]}$. Then by Lemma 5.12, e is an idempotent of $C(X)_F$ and we have $Z(e) = \bigcap Z[I]$. By Lemma 5.11,

$$r(I) = r(e) = (1-e)R.$$

(2) Assume I is a strongly Baer ideal of $C(X)_F$. Then $r(I) = r(e_1 \cdot ... \cdot e_n)$ and each ideal $< e_i >$ $(1 \leq i \leq n)$ is maximal or each $e_i = 1$. Let for each $1 \leq i \leq n$, the ideal $< e_i >$ is maximal, so the ideal $< 1 - e_i >$ is minimal. This and Proposition 4.13 of [11] show there exists $\alpha_i \in X$ such that $< 1 - e_i >=< \chi_{\alpha_i} >$. Thus by Lemma 5.11,

$$\bigcap Z[I] = Z(e_1) \cup Z(e_2) \cup ... \cup Z(e_n) = \{\alpha_1, \alpha_2, ..., \alpha_n\}.$$

If for each $1 \leq i \leq n$. $e_i = 1$, then Lemma 5.11 implies $\bigcap Z[I] = \emptyset$. Conversely, suppose that $\bigcap Z[I] = \{p_1, ..., p_n\}$. Then for each $1 \leq i \leq n$, $\{p_i\} = Z(\chi_{X\setminus\{p_i\}}) =$. For each $1 \leq i \leq n$, put $e_i = \chi_{X\setminus\{p_i\}}$. Proposition 4.13 of [11], shows each ideal $< \chi_{p_i} >$ is minimal and hence each ideal $< e_i >$ is maximal and $\bigcap Z[I] = Z(e_1 \cdot ... \cdot e_n)$. Therefore $r(I) = Ann(e)$, where $e = e_1 \cdot ... \cdot e_n$. It is clear that if $\bigcap Z[I] = \emptyset$, then $r(I) = r(1)$.

□

Corollary 5.14. Every ideal of $C(X)_F$ (resp., $T^{'}(X)$) is a strongly Baer ideal if and only if X is finite.

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