Journal of Algebraic Systems, Vol. 14(No. 1): (2026), pp 1-28.
https://doi.org/10.22044/JAS.2024.13268.1736

TRUE-FALSE STRUCTURES AND ITS APPLICATION IN GROUPS AND BCK/BCI-ALGEBRAS

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Abstract. By utilizing the concept of interval-valued fuzzy sets, a novel structure called True-False structures is introduced. Its application in groups and BCK/BCI-algebras is discussed. The introduction of (limited) $T\&F$-subgroups and (limited) $T\&F$-subalgebras is carried out, along with an investigation of various associated properties. Characterizations of (limited) $T\&F$-subgroups and (limited) $T\&F$-subalgebras are provided, and it is demonstrated that the intersection of two $T\&F$-subgroups (respectively, $T\&F$-subalgebras) also forms a $T\&F$-subgroup (respectively, $T\&F$-subalgebra). Additionally, the union and intersection of two $T\&F$-subgroups (respectively, $T\&F$-subalgebras) are explored.

1. Introduction

In the original fuzzy set introduced by Zadeh [23], the membership degree is represented by a single function known as the membership function. Zadeh later extended this concept by introducing interval-valued fuzzy sets [24], where the membership degree is expressed as an interval rather than a single value. As a further generalization of fuzzy sets, Atanassove [2] introduced the concept of intuitionistic fuzzy sets. In this framework, two functions are used- the membership function and the non-membership function. The membership function represents the degree of truth, while the non-membership function represents the degree of falsity or uncertainty. Building upon these concepts, Jun et al. [14] introduced the concept of cubic sets, which combines the ideas of fuzzy sets and interval-valued fuzzy sets. Cubic sets provide a more flexible and expressive way to represent uncertainty and ambiguity in a set. Overall, these extensions and generalizations of fuzzy sets have expanded the scope and applicability of fuzzy logic in various fields. BCK/BCI-algebras are important classes of logical algebras that were introduced by Iseki in 1966 [8, 9, 16]. Since then, there has been a significant amount of research on the theory of BCK/BCI-algebras [15]. Fuzzy mathematics is the study

Published online: 25 October 2025
MSC(2020): Primary: 03G25; Secondary: 06F35, 06B99.
Keywords: True-False structure; (Limited) $T\&F$-Subgroup; (Limited) $T\&F$-Subalgebra.
Received: 16 June 2023, Accepted: 12 January 2024.
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of fuzzy subsets and their applications in mathematical contexts. Fuzzy algebra is a fundamental branch of fuzzy mathematics. In 1971, Rosenfeld [21] introduced the concept of fuzzy subgroups. In 1991, Xi [22] applied the idea of fuzzy sets to BCK-algebras and studied their properties. Since then, fuzzy BCI/BCK-algebras have been extensively investigated by several researchers [3]. Interval-valued fuzzy subgroups, based on Rosenfeld's nature, were defined by Biswas [3], who also studied some elementary properties. Interval-valued fuzzy set theory has been extensively studied in the context of groups and BCK/BCI-algebras [1, 4, 5, 6, 10, 12, 18, 19, 17, 20]. Overall, the combination of fuzzy mathematics and BCK/BCI-algebras, along with the introduction of interval-valued fuzzy sets, has led to a rich body of research in this area.

This paper introduces True-False structures, a new type of structure constructed using (interval-valued) fuzzy sets. The application of these structures to groups and BCK/BCI-algebras is explored. The definitions of (limited) $T\&F$-groups and (limited) $T\&F$-subalgebras are provided and their properties are studied. Characterizations of these structures are investigated and various associated properties are examined. It is proven that the intersection of two $T\&F$-groups is also a $T\&F$-group, and similarly, the intersection of two $T\&F$-subalgebras is also a $T\&F$-subalgebra. However, examples are presented to show that the union of two $T\&F$-groups (respectively, $T\&F$-subalgebras) may not necessarily be a $T\&F$-group (respectively, $T\&F$-subalgebra). By studying True-False structures in the context of groups and BCK/BCI-algebras, this paper aims to contribute to the understanding and application of fuzzy set theory in these areas.

2. Preliminaries

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to I$ where $I = [0, 1]$. Denote by I^X the collection of all fuzzy sets in a set X [23]. Define a relation $\leq$ on I^X as follows:

$$(\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}\ ,\ (\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\},$$

respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by

$$(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x)).$$

For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\Big(\bigvee_{i\in\Lambda} \lambda_i\Big)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\}\ ,\ \Big(\bigwedge_{i\in\Lambda} \lambda_i\Big)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\},$$

respectively, for all $x \in X$.

By an *interval number* we mean a closed subinterval $\tilde{a} := [a^-, a^+]$ of $[0,1]$, where $0 \le a^- \le a^+ \le 1$. Denote by $int([0,1])$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $int([0,1])$. We also define the symbols "$\succcurlyeq$", "$\preccurlyeq$", "$=$" in case of two elements in $int([0,1])$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \tag{2.1}$$

$$\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \tag{2.2}$$

$$\tilde{a}_1 \succcurlyeq \tilde{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+, \tag{2.3}$$

and similarly we may have $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succcurlyeq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$ (resp. $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$). Let $\tilde{a}_i \in int([0,1])$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in int([0,1])$, its *complement*, denoted by $\tilde{a}^c$, is defined to be the interval number $\tilde{a}^c = [1 - a^+, 1 - a^-]$.

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X [24]. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^X$, and for all $x \in X$,we define

$$A \subseteq B \Leftrightarrow A(x) \preceq B(x),\ A = B \Leftrightarrow A(x) = B(x).$$

The complement A^c of $A \in [I]^X$ is defined as follows: $A^c(x) = A(x)^c$ for all $x \in X$, that is, $A^c(x) = [1 - A^+(x), 1 - A^-(x)]$. For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets in X where Λ is an index set, the *union* $G = \bigcup_{i\in\Lambda} A_i$ and the *intersection* $F = \bigcap_{i\in\Lambda} A_i$ are defined as follows:

$$G(x) = \left(\bigcup_{i\in\Lambda} A_i\right)(x) = \operatorname{rsup}_{i\in\Lambda} A_i(x),\ \ F(x) = \left(\bigcap_{i\in\Lambda} A_i\right)(x) = \operatorname{rinf}_{i\in\Lambda} A_i(x)$$

for all $x \in X$, respectively. For a point $p \in X$ and for $\tilde{a} = [a^-, a^+] \in [I]$ with $a^+ > 0$, the IVF set which takes the value $\tilde{a}$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by $\tilde{a}_p$. The set of all IVF points in X is denoted by $IVFP(X)$. For any $\tilde{a} \in [I]$ and $x \in X$, the IVF point $\tilde{a}_x$ is said to *belong* to an IVF set A in X, denoted by $\tilde{a}_x \widetilde{\in} A$, if $A(x) \succeq \tilde{a}$. It can be easily shown that $A = \cup\{\tilde{a}_x \mid \tilde{a}_x \widetilde{\in} A\}$.

An intuitionistic fuzzy set (briefly IFS) A in X (see [2]) is defined by $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ such that $0 \leq \mu(x) + \nu(x) \leqslant 1$, where μ and ν define the degree of membership and the degree of non-membership of the element $x \in X$, respectively. Obviously, each ordinary fuzzy set may be written as $A = \{(x, \mu_A(x), 1 - \mu_A(x)) | x \in X\}$.

3. True-False structures

Definition 3.1. Let U be a universal set. A *True-False structure* (briefly, *T&F-structure*) over U is defined to be a pair $(U, \mathcal{A})$ where $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ and it is the following function:

$$\begin{aligned} \mathcal{A} : U &\to [0,1] \times int[0,1] \times [0,1] \times int[0,1], \\ a &\mapsto (\varphi_A(a), \widetilde{\varphi}_A(a), \partial_A(a), \widetilde{\partial}_A(a)). \end{aligned} \tag{3.1}$$

Definition 3.2. A $T\&F$-structure $(U, \mathcal{A})$ with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ is said to be *limited* if $\varphi_A(a) + \partial_A(a) \leq 1$ and $\sup \widetilde{\varphi}_A(a) + \sup \widetilde{\partial}_A(a) \leq 1$ for all $a \in U$.

Given a (limited) $T\&F$-structure $(U, \mathcal{A})$ with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, consider the sets

$$\begin{aligned} &U(\varphi_A; \alpha) := \{a \in U \mid \varphi_A(a) \geq \alpha\}\ ,\ U(\widetilde{\varphi}_A; \tilde{t}) := \{a \in U \mid \widetilde{\varphi}_A(a) \succcurlyeq \tilde{t}\}, \\ &L(\partial_A; \beta) := \{a \in U \mid \partial_A(a) \leq \beta\}\ ,\ L(\widetilde{\partial}_A; \tilde{s}) := \{a \in U \mid \widetilde{\partial}_A(a) \preccurlyeq \tilde{s}\}, \\ &\mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s}) := U(\varphi_A; \alpha) \cap U(\widetilde{\varphi}_A; \tilde{t}) \cap L(\partial_A; \beta) \cap L(\widetilde{\partial}_A; \tilde{s}) \end{aligned}$$

where $\alpha, \beta \in [0,1]$ and $\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0,1])$. We say that $\mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s})$ is a *$T\&F$-level set* of $\mathcal{A}$ over U.

Definition 3.3. If $(U, \mathcal{A})$ and $(U, \mathcal{B})$ are $T\&F$-structures with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$, then the *union* of $(U, \mathcal{A})$ and $(U, \mathcal{B})$ is a $T\&F$-structure $(U, \mathcal{A} \Cup \mathcal{B})$ with

$$\mathcal{A} \Cup \mathcal{B} := (\varphi_{A\vee B}, \widetilde{\varphi}_{A\cup B}, \partial_{A\wedge B}, \widetilde{\partial}_{A\cap B})$$

where

$$\varphi_{A\vee B}(a) = \max\{\varphi_A(a), \varphi_B(a)\},\ \widetilde{\varphi}_{A\cup B}(a) = \mathrm{rmax}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_B(a)\},$$

$$\partial_{A\wedge B}(a) = \min\{\partial_A(a), \partial_B(a)\} \text{ and } \widetilde{\partial}_{A\cap B}(a) = \mathrm{rmin}\{\widetilde{\partial}_A(a), \widetilde{\partial}_B(a)\}.$$

Definition 3.4. If $(U, \mathcal{A})$ and $(U, \mathcal{B})$ are $T\&F$-structures with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$, then the *intersection* of $(U, \mathcal{A})$ and $(U, \mathcal{B})$ is a $T\&F$-structure $(U, \mathcal{A} \Cap \mathcal{B})$ with $\mathcal{A} \Cap \mathcal{B} := (\varphi_{A\wedge B}, \widetilde{\varphi}_{A\cap B}, \partial_{A\vee B}, \widetilde{\partial}_{A\cup B})$ where

$$\varphi_{A\wedge B}(a) = \min\{\varphi_A(a), \varphi_B(a)\},\ \widetilde{\varphi}_{A\cap B}(a) = \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_B(a)\},$$

$$\partial_{A\vee B}(a) = \max\{\partial_A(a), \partial_B(a)\} \text{ and } \widetilde{\partial}_{A\cup B}(a) = \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_B(a)\}.$$

Proposition 3.5. *If $(U, \mathcal{A})$ and $(U, \mathcal{B})$ are limited $T\&F$-structures with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$, then so are the union and intersection of $(U, \mathcal{A})$ and $(U, \mathcal{B})$.*

Proof. It is straightforward. □

Definition 3.6. Let $f : X \to Y$ be a mapping from a set X to a set Y. If $(Y, \mathcal{B})$ is a $T\&F$-structure with $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$, then the *inverse image* of $(Y, \mathcal{B})$ under f is defined to be a $T\&F$-structure $(X, f^{-1}(\mathcal{B}))$ with $f^{-1}(\mathcal{B}) := (f^{-1}(\varphi_B),\, f^{-1}(\widetilde{\varphi}_B),\, f^{-1}(\partial_B),\, f^{-1}(\widetilde{\partial}_B))$ where

$$f^{-1}(\varphi_B)(a) = \varphi_B(f(a)),\ f^{-1}(\widetilde{\varphi}_B)(a) = \widetilde{\varphi}_B(f(a)),\ f^{-1}(\partial_B)(a) = \partial_B(f(a))$$

and $f^{-1}(\widetilde{\partial}_B)(a) = \widetilde{\partial}_B(f(a))$.

Definition 3.7. Let $f : X \to Y$ be a mapping from a set X to a set Y. If $(X, \mathcal{A})$ is a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$, then the *image* of $(X, \mathcal{A})$ under f is defined to be a $T\&F$-structure $(Y, f(\mathcal{A}))$ with $f(\mathcal{A}) := (f(\varphi_A),\, f(\widetilde{\varphi}_A),\, f(\partial_A),\, f(\widetilde{\partial}_A))$ in which

$$f(\varphi_A) : Y \to [0,1],\ b \mapsto \begin{cases} \sup\limits_{a\in f^{-1}(b)} \varphi_A(a) & \text{if } f^{-1}(b) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(\widetilde{\varphi}_A) : Y \to int[0,1],\ b \mapsto \begin{cases} \mathrm{rsup}\limits_{a\in f^{-1}(b)} \widetilde{\varphi}_A(a) & \text{if } f^{-1}(b) \neq \emptyset, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$f(\partial_A) : Y \to [0,1],\ b \mapsto \begin{cases} \inf\limits_{a\in f^{-1}(b)} \partial_A(a) & \text{if } f^{-1}(b) \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$f(\widetilde{\partial}_A) : Y \to int[0,1],\ b \mapsto \begin{cases} \operatorname{rinf}\limits_{a\in f^{-1}(b)} \widetilde{\partial}_A(a) & \text{if } f^{-1}(b) \neq \emptyset, \\ [1,1] & \text{otherwise,} \end{cases}$$

4. Applications in groups

In this section, we define limited $T\&F$-structure and consider some property of (limited) $T\&F$-structure on group theory. Let $G := (G, e)$ be a group unless otherwise stated.

Definition 4.1. A $T\&F$-structure $(G, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ is called a *$T\&F$-subgroup* of G if the following assertions are valid.

$$(\forall a, b \in G) \begin{pmatrix} \varphi_A(ab) \geq \min\{\varphi_A(a), \varphi_A(b)\} \\ \widetilde{\varphi}_A(ab) \succcurlyeq \operatorname{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\} \\ \partial_A(ab) \leq \max\{\partial_A(a), \partial_A(b)\} \\ \widetilde{\partial}_A(ab) \preccurlyeq \operatorname{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\} \end{pmatrix}, \tag{4.1}$$

and

$$(\forall a \in G) \begin{pmatrix} \varphi_A(a^{-1}) \geq \varphi_A(a),\ \widetilde{\varphi}_A(a^{-1}) \succcurlyeq \widetilde{\varphi}_A(a) \\ \partial_A(a^{-1}) \leq \partial_A(a),\ \widetilde{\partial}_A(a^{-1}) \preccurlyeq \widetilde{\partial}_A(a) \end{pmatrix}. \tag{4.2}$$

If a $T\&F$-subgroup is limited, then we say that it is a *limited $T\&F$-subgroup*.

Example 4.2. Let $G = \{e, a, b, ab\}$ be the Klein's four group where $a^2 = e = b^2$ and $ab = ba$. Define a $T\&F$-structure $(G, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ given as follows:

$$\varphi_A = \begin{pmatrix} e & a & b & ab \\ 0.78 & 0.56 & 0.47 & 0.47 \end{pmatrix},$$

$$\widetilde{\varphi}_A = \begin{pmatrix} e & a & b & ab \\ [0.5, 0.8] & [0.4, 0.6] & [0.1, 0.5] & [0.1, 0.5] \end{pmatrix},$$

$$\partial_A = \begin{pmatrix} e & a & b & ab \\ 0.22 & 0.33 & 0.63 & 0.63 \end{pmatrix},$$

$$\widetilde{\partial}_A = \begin{pmatrix} e & a & b & ab \\ [0.2, 0.5] & [0.3, 0.6] & [0.5, 0.8] & [0.5, 0.8] \end{pmatrix}.$$

It is routine to verify that $(G, \mathcal{A})$ is a $T\&F$-subgroup of G. But it is not limited since $\widetilde{\varphi}_A^+(e) + \widetilde{\partial}_A(e) > 1$.

Example 4.3. Let $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the cyclic group with the group operation "+" which is addition modulo eight. Its Cayley table is given by Table 1.

TABLE 1. Cayley table for the group operation "+"

+	0	2	4	6	1	3	5	7
0	0	2	4	6	1	3	5	7
2	2	4	6	0	3	5	7	1
4	4	6	0	2	5	7	1	3
6	6	0	2	4	7	1	3	5
1	1	3	5	7	2	4	6	0
3	3	5	7	1	4	6	0	2
5	5	7	1	3	6	0	2	4
7	7	1	3	5	0	2	4	6

Define a $T\&F$-structure $(\mathbb{Z}_8, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ given by Table 2. It is routine to verify that $(\mathbb{Z}_8, \mathcal{A})$ is a limited $T\&F$-subgroup of $\mathbb{Z}_8$.

TABLE 2. Tabular representation of $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$

$\mathbb{Z}_8$	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
0	0.6	$[0.4, 0.7]$	0.34	$[0.1, 0.25]$
1	0.2	$[0.1, 0.4]$	0.64	$[0.3, 0.45]$
2	0.4	$[0.2, 0.5]$	0.54	$[0.2, 0.35]$
3	0.2	$[0.1, 0.4]$	0.64	$[0.3, 0.45]$
4	0.5	$[0.3, 0.6]$	0.44	$[0.1, 0.25]$
5	0.2	$[0.1, 0.4]$	0.64	$[0.3, 0.45]$
6	0.4	$[0.2, 0.5]$	0.54	$[0.2, 0.35]$
7	0.2	$[0.1, 0.4]$	0.64	$[0.3, 0.45]$

Lemma 4.4. *If $(G, \mathcal{A})$ is a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$, then $\varphi_A(a^{-1}) = \varphi_A(a)$, $\widetilde{\varphi}_A(a^{-1}) = \widetilde{\varphi}_A(a)$, $\partial_A(a^{-1}) = \partial_A(a)$ and $\widetilde{\partial}_A(a^{-1}) = \widetilde{\partial}_A(a)$ for all $a \in G$.*

Proof. For any $a \in G$, we have

$$\begin{aligned} \varphi_A(a) &= \varphi_A((a^{-1})^{-1}) \geq \varphi_A(a^{-1}) \geq \varphi_A(a), \\ \widetilde{\varphi}_A(a) &= \widetilde{\varphi}_A((a^{-1})^{-1}) \succcurlyeq \widetilde{\varphi}_A(a^{-1}) \succcurlyeq \widetilde{\varphi}_A(a), \\ \partial_A(a) &= \partial_A((a^{-1})^{-1}) \leq \partial_A(a^{-1}) \leq \partial_A(a), \\ \widetilde{\partial}_A(a) &= \widetilde{\partial}_A((a^{-1})^{-1}) \preccurlyeq \widetilde{\partial}_A(a^{-1}) \preccurlyeq \widetilde{\partial}_A(a). \end{aligned}$$

Therefore, $\varphi_A(a^{-1}) = \varphi_A(a)$, $\widetilde{\varphi}_A(a^{-1}) = \widetilde{\varphi}_A(a)$, $\partial_A(a^{-1}) = \partial_A(a)$ and $\widetilde{\partial}_A(a^{-1}) = \widetilde{\partial}_A(a)$ for all $a \in G$. □

Proposition 4.5. *If $(G, \mathcal{A})$ is a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, then $\varphi_A(e) \geq \varphi_A(a)$, $\widetilde{\varphi}_A(e) \succcurlyeq \widetilde{\varphi}_A(a)$, $\partial_A(e) \leq \partial_A(a)$ and $\widetilde{\partial}_A(e) \preccurlyeq \widetilde{\partial}_A(a)$ for all $a \in G$, where e is the identity element of G.*

Proof. Using (4.1) and Lemma 4.4 induces

$$\begin{aligned} \varphi_A(e) &= \varphi_A(aa^{-1}) \geq \min\{\varphi_A(a), \varphi_A(a^{-1})\} = \varphi_A(a), \\ \widetilde{\varphi}_A(e) &= \widetilde{\varphi}_A(aa^{-1}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(a^{-1})\} = \varphi_A(a), \\ \partial_A(e) &= \partial_A(aa^{-1}) \leq \max\{\partial_A(a), \partial_A(a^{-1})\} = \partial_A(a), \\ \widetilde{\partial}_A(e) &= \widetilde{\partial}_A(aa^{-1}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(a^{-1})\} = \widetilde{\partial}_A(a) \end{aligned}$$

for all $a \in G$. □

Proposition 4.6. *Let $(G, \mathcal{A})$ be a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. If $\varphi_A(ab^{-1}) = \varphi_A(e)$, $\widetilde{\varphi}_A(ab^{-1}) = \widetilde{\varphi}_A(e)$, $\partial_A(ab^{-1}) = \partial_A(e)$ and $\widetilde{\partial}_A(ab^{-1}) = \widetilde{\partial}_A(e)$ for all $a, b \in G$, then $\varphi_A(a) = \varphi_A(b)$, $\widetilde{\varphi}_A(a) = \widetilde{\varphi}_A(b)$, $\partial_A(a) = \partial_A(b)$ and $\widetilde{\partial}_A(a) = \widetilde{\partial}_A(b)$.*

Proof. Let $a, b \in G$ such that $\varphi_A(ab^{-1}) = \varphi_A(e)$, $\widetilde{\varphi}_A(ab^{-1}) = \widetilde{\varphi}_A(e)$, $\partial_A(ab^{-1}) = \partial_A(e)$ and $\widetilde{\partial}_A(ab^{-1}) = \widetilde{\partial}_A(e)$. Using Proposition 4.5, we get

$$\begin{aligned} \varphi_A(a) &= \varphi_A((ab^{-1})b) \geq \min\{\varphi_A(e), \varphi_A(b)\} = \varphi_A(b), \\ \widetilde{\varphi}_A(a) &= \widetilde{\varphi}_A((ab^{-1})b) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(e), \widetilde{\varphi}_A(b)\} = \widetilde{\varphi}_A(b), \\ \partial_A(a) &= \partial_A((ab^{-1})b) \leq \max\{\partial_A(e), \partial_A(b)\} = \partial_A(b), \\ \widetilde{\partial}_A(a) &= \widetilde{\partial}_A((ab^{-1})b) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(e), \widetilde{\partial}_A(b)\} = \widetilde{\partial}_A(b). \end{aligned}$$

Similarly, we have $\varphi_A(a) \leq \varphi_A(b)$, $\widetilde{\varphi}_A(a) \preccurlyeq \widetilde{\varphi}_A(b)$, $\partial_A(a) \geq \partial_A(b)$ and $\widetilde{\partial}_A(a) \succcurlyeq \widetilde{\partial}_A(b)$. This completes the proof. □

We consider a characterization of $T\&F$-subgroup.

Theorem 4.7. *A $T\&F$-structure $(G, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ is a $T\&F$-subgroup of G if and only if the following assertions are valid.*

$$(\forall a, b \in G) \left(\begin{array}{l} \varphi_A(ab^{-1}) \geq \min\{\varphi_A(a), \varphi_A(b)\} \\ \widetilde{\varphi}_A(ab^{-1}) \succcurlyeq \operatorname{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\} \\ \partial_A(ab^{-1}) \leq \max\{\partial_A(a), \partial_A(b)\} \\ \widetilde{\partial}_A(ab^{-1}) \preccurlyeq \operatorname{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\} \end{array} \right). \tag{4.3}$$

Proof. Assume that $(G, \mathcal{A})$ is a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$. For any $a, b \in G$, we have

$$\begin{aligned} &\varphi_A(ab^{-1}) \geq \min\{\varphi_A(a), \varphi_A(b^{-1})\} = \min\{\varphi_A(a), \varphi_A(b)\}, \\ &\widetilde{\varphi}_A(ab^{-1}) \succcurlyeq \operatorname{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b^{-1})\} = \operatorname{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\}, \\ &\partial_A(ab^{-1}) \leq \max\{\partial_A(a), \partial_A(b^{-1})\} = \max\{\partial_A(a), \partial_A(b)\}, \\ &\widetilde{\partial}_A(ab^{-1}) \preccurlyeq \operatorname{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b^{-1})\} = \operatorname{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\} \end{aligned}$$

by Lemma 4.4.

Conversely, suppose that a $T\&F$-structure $(G, \mathcal{A})$ with

$$\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$$

satisfies the condition (4.3). If we take $b := a$ in (4.3), then

$$\begin{aligned} &\varphi_A(e) = \varphi_A(aa^{-1}) \geq \min\{\varphi_A(a), \varphi_A(a)\} = \varphi_A(a), \\ &\widetilde{\varphi}_A(e) = \widetilde{\varphi}_A(aa^{-1}) \succcurlyeq \operatorname{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(a)\} = \widetilde{\varphi}_A(a), \\ &\partial_A(e) = \partial_A(aa^{-1}) \leq \max\{\partial_A(a), \partial_A(a)\} = \partial_A(a), \\ &\widetilde{\partial}_A(e) = \widetilde{\partial}_A(aa^{-1}) \preccurlyeq \operatorname{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(a)\} = \widetilde{\partial}_A(a). \end{aligned}$$

It follows from (4.3) that

$$\begin{aligned} &\varphi_A(b^{-1}) = \varphi_A(eb^{-1}) \geq \min\{\varphi_A(e), \varphi_A(b)\} = \varphi_A(b), \\ &\widetilde{\varphi}_A(b^{-1}) = \widetilde{\varphi}_A(eb^{-1}) \succcurlyeq \operatorname{rmin}\{\widetilde{\varphi}_A(e), \widetilde{\varphi}_A(b)\} = \widetilde{\varphi}_A(b), \\ &\partial_A(b^{-1}) = \partial_A(eb^{-1}) \leq \max\{\partial_A(e), \partial_A(b)\} = \partial_A(b), \\ &\widetilde{\partial}_A(b^{-1}) = \widetilde{\partial}_A(eb^{-1}) \preccurlyeq \operatorname{rmax}\{\widetilde{\partial}_A(e), \widetilde{\partial}_A(b)\} = \widetilde{\partial}_A(b). \end{aligned}$$

Hence

$$\begin{aligned}
&\varphi_A(ab) = \varphi_A(a(b^{-1})^{-1}) \geq \min\{\varphi_A(a), \varphi_A(b^{-1})\} \geq \min\{\varphi_A(a), \varphi_A(b)\},\\
&\widetilde{\varphi}_A(ab) = \widetilde{\varphi}_A(a(b^{-1})^{-1}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b^{-1})\} \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\},\\
&\partial_A(ab) = \partial_A(a(b^{-1})^{-1}) \leq \max\{\partial_A(a), \partial_A(b^{-1})\} \leq \max\{\partial_A(a), \partial_A(b)\},\\
&\widetilde{\partial}_A(ab) = \widetilde{\partial}_A(a(b^{-1})^{-1}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b^{-1})\} \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\}.
\end{aligned}$$

Therefore $(G, \mathcal{A})$ is a $T\&F$-subgroup of G. □

Theorem 4.8. *If $(G, \mathcal{A})$ is a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, then*

$$S := \{a \in G \mid \varphi_A(a) = \varphi_A(e), \widetilde{\varphi}_A(a) = \widetilde{\varphi}_A(e), \partial_A(a) = \partial_A(e), \widetilde{\partial}_A(a) = \widetilde{\partial}_A(e)\},$$

is a subgroup of G.

Proof. Let $a, b \in S$. Then $\varphi_A(b) = \varphi_A(a) = \varphi_A(e)$, $\widetilde{\varphi}_A(b) = \widetilde{\varphi}_A(a) = \widetilde{\varphi}_A(e)$, $\partial_A(b) = \partial_A(a) = \partial_A(e)$ and $\widetilde{\partial}_A(b) = \widetilde{\partial}_A(a) = \widetilde{\partial}_A(e)$. It follows from Theorem 4.7 that

$$\begin{aligned}
&\varphi_A(ab^{-1}) \geq \min\{\varphi_A(a), \varphi_A(b)\} = \varphi_A(e),\\
&\widetilde{\varphi}_A(ab^{-1}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\} = \widetilde{\varphi}_A(e),\\
&\partial_A(ab^{-1}) \leq \max\{\partial_A(a), \partial_A(b)\} = \partial_A(e),\\
&\widetilde{\partial}_A(ab^{-1}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\} = \widetilde{\partial}_A(e),
\end{aligned}$$

which implies from Proposition 4.5 that $\varphi_A(ab^{-1}) = \varphi_A(e)$, $\widetilde{\varphi}_A(ab^{-1}) = \widetilde{\varphi}_A(e)$, $\partial_A(ab^{-1}) = \partial_A(e)$ and $\widetilde{\partial}_A(ab^{-1}) = \widetilde{\partial}_A(e)$. Hence $ab^{-1} \in S$, and so S is a subgroup of G. □

Theorem 4.9. *Let $(G, \mathcal{A})$ be a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. If $\mathcal{A}$ is a $T\&F$-subgroup of G, then the $T\&F$-level set $\mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s})$ of $\mathcal{A}$ over G is a subgroup of G for all $\alpha, \beta \in [0, 1]$ and $\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0, 1])$.*

Proof. Assume that $(G, \mathcal{A})$ is a $T\&F$-subgroup with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. Let $a, b \in \mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s})$ for all $\alpha, \beta \in [0, 1]$ and

$$\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0, 1]).$$

Then $\varphi_A(a) \geq \alpha$, $\varphi_A(b) \geq \alpha$, $\widetilde{\varphi}_A(a) \succcurlyeq \tilde{t}$, $\widetilde{\varphi}_A(b) \succcurlyeq \tilde{t}$, $\partial_A(a) \leq \beta$, $\partial_A(b) \leq \beta$, $\widetilde{\partial}_A(a) \preccurlyeq \tilde{s}$ and $\widetilde{\partial}_A(b) \preccurlyeq \tilde{s}$. It follows from Theorem 4.7 that

$$\begin{aligned}
&\varphi_A(ab^{-1}) \geq \min\{\varphi_A(a), \varphi_A(b)\} \geq \alpha,\\
&\widetilde{\varphi}_A(ab^{-1}) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\} \geq \tilde{t},\\
&\partial_A(ab^{-1}) \leq \max\{\partial_A(a), \partial_A(b)\} \leq \beta,\\
&\widetilde{\partial}_A(ab^{-1}) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\} \leq \tilde{s}.
\end{aligned}$$

Hence $ab^{-1} \in \mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s})$, and so $\mathcal{L}_{\mathcal{A}}(\alpha, \tilde{t}, \beta, \tilde{s})$ is a subgroup of G. □

Theorem 4.10. *Let $(G, \mathcal{A})$ be a T&F-structure with*

$$\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A).$$

If $U(\varphi_A; \alpha)$, $U(\widetilde{\varphi}_A; \tilde{t})$, $L(\partial_A; \beta)$ and $L(\widetilde{\partial}_A; \tilde{s})$ are subgroups of G for all $\alpha, \beta \in [0, 1]$ and $\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0, 1])$, then $(G, \mathcal{A})$ is a T&F-group of G.

Proof. The proof is straightforward. □

Theorem 4.11. *The intersection of two T&F-subgroups is also a T&F-subgroup.*

Proof. Let $(U, \mathcal{A})$ and $(U, \mathcal{B})$ are limited T&F-subgroups with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$. Then

$$\begin{aligned}
\varphi_{A\wedge B}(ab^{-1}) &= \min\{\varphi_A(ab^{-1}), \varphi_B(ab^{-1})\}\\
&\geq \min\{\min\{\varphi_A(a), \varphi_A(b)\}, \min\{\varphi_B(a), \varphi_A(b)\}\}\\
&= \min\{\min\{\varphi_A(a), \varphi_B(a)\}, \min\{\varphi_A(b), \varphi_B(b)\}\}\\
&= \min\{\varphi_{A\wedge B}(a), \varphi_{A\wedge B}(b)\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\varphi}_{A\cap B}(ab^{-1}) &= \mathrm{rmin}\{\widetilde{\varphi}_A(ab^{-1}), \widetilde{\varphi}_B(ab^{-1})\}\\
&\succcurlyeq \mathrm{rmin}\{\mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_A(b)\}, \mathrm{rmin}\{\widetilde{\varphi}_B(a), \widetilde{\varphi}_A(b)\}\}\\
&= \mathrm{rmin}\{\mathrm{rmin}\{\widetilde{\varphi}_A(a), \widetilde{\varphi}_B(a)\}, \mathrm{rmin}\{\widetilde{\varphi}_A(b), \widetilde{\varphi}_B(b)\}\}\\
&= \mathrm{rmin}\{\widetilde{\varphi}_{A\cap B}(a), \widetilde{\varphi}_{A\cap B}(b)\},
\end{aligned}$$

$$\begin{aligned}
\partial_{A\vee B}(ab^{-1}) &= \max\{\partial_A(ab^{-1}), \partial_B(ab^{-1})\}\\
&\leq \max\{\max\{\partial_A(a), \partial_A(b)\}, \max\{\partial_B(a), \partial_B(b)\}\}\\
&= \max\{\max\{\partial_A(a), \partial_B(a)\}, \max\{\partial_A(b), \partial_B(b)\}\}\\
&= \max\{\partial_{A\vee B}(a), \partial_{A\vee B}(b)\},
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\partial}_{A\cup B}(ab^{-1}) &= \text{rmax}\{\widetilde{\partial}_A(ab^{-1}), \widetilde{\partial}_B(ab^{-1})\}\\ &\preccurlyeq \text{rmax}\{\text{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_A(b)\}, \text{rmax}\{\widetilde{\partial}_B(a), \widetilde{\partial}_B(b)\}\}\\ &= \text{rmax}\{\text{rmax}\{\widetilde{\partial}_A(a), \widetilde{\partial}_B(a)\}, \text{rmax}\{\widetilde{\partial}_A(b), \widetilde{\partial}_B(b)\}\}\\ &= \text{rmax}\{\widetilde{\partial}_{A\cup B}(a), \widetilde{\partial}_{A\cup B}(b)\}.\end{aligned}$$

for all $a, b \in G$. It follows from Theorem 4.7 that $(U, \mathcal{A} \Cap \mathcal{B})$ is a $T\&F$-subgroup of G. □

The following example shows that the union of two $T\&F$-subgroups may not be a $T\&F$-subgroup in general.

Example 4.12. Consider the symmetric group

$$S_3 = \{(), (12), (23), (13), (123), (132)\}$$

with Table 3.

Table 3. Cayley table for S_3

	()	(12)	(23)	(13)	(123)	(132)
()	()	(12)	(23)	(13)	(123)	(132)
(12)	(12)	()	(123)	(132)	(23)	(13)
(23)	(23)	(132)	()	(123)	(13)	(12)
(13)	(13)	(123)	(132)	()	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	()
(132)	(132)	(23)	(13)	(12)	()	(123)

Let $(S_3, \mathcal{A})$ and $(S_3, \mathcal{B})$ be $T\&F$-structures with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B, \widetilde{\varphi}_B, \partial_B, \widetilde{\partial}_B)$ given by Tables 4 and 5 respectively.
It is routine to verify that $(S_3, \mathcal{A})$ and $(S_3, \mathcal{B})$ are $T\&F$-subgroups of S_3. The union $(S_3, \mathcal{A} \Cup \mathcal{B})$ of $(S_3, \mathcal{A})$ and $(S_3, \mathcal{B})$ is given by Table 6, and it is not a $T\&F$-subgroup of S_3 since

$$\varphi_{A\vee B}((12)(13)) = 0.4 < 0.5 = \min\{\varphi_{A\vee B}((12)), \varphi_{A\vee B}((13))\}$$

and/or $\partial_{A\wedge B}((23)(12)) = 0.48 > 0.44 = \max\{\partial_{A\wedge B}((23)), \partial_{A\wedge B}((12))\}$.

5. Applications in BCK/BCI-algebras

In this section, we apply the notion of $T\&F$-structure to a BCK/BCI-algebra. For the convenience of readers, we first display the basic concept of BCK/BCI-algebras and its fuzzy version although it is well known.

Table 4. Tabular representation of $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$

S_3	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
()	0.6	$[0.4, 0.7]$	0.34	$[0.1, 0.25]$
(12)	0.5	$[0.1, 0.4]$	0.54	$[0.3, 0.45]$
(23)	0.4	$[0.1, 0.4]$	0.44	$[0.3, 0.45]$
(13)	0.4	$[0.3, 0.6]$	0.54	$[0.3, 0.45]$
(123)	0.4	$[0.1, 0.4]$	0.54	$[0.2, 0.35]$
(132)	0.4	$[0.1, 0.4]$	0.54	$[0.2, 0.35]$

Table 5. Tabular representation of $\mathcal{B} := (\varphi_B, \widetilde{\varphi}_B, \partial_B, \widetilde{\partial}_B)$

S_3	$\varphi_B(x)$	$\widetilde{\varphi}_B(x)$	$\partial_B(x)$	$\widetilde{\partial}_B(x)$
()	0.7	$[0.5, 0.7]$	0.28	$[0.2, 0.36]$
(12)	0.3	$[0.3, 0.4]$	0.38	$[0.4, 0.57]$
(23)	0.3	$[0.3, 0.4]$	0.48	$[0.3, 0.45]$
(13)	0.6	$[0.3, 0.4]$	0.48	$[0.4, 0.57]$
(123)	0.3	$[0.4, 0.6]$	0.48	$[0.4, 0.57]$
(132)	0.3	$[0.4, 0.6]$	0.48	$[0.4, 0.57]$

Table 6. Tabular representation of the union $\mathcal{A} \Cup \mathcal{B}$

X	$\varphi_{A\vee B}(x)$	$\widetilde{\varphi}_{A\cup B}(x)$	$\partial_{A\wedge B}(x)$	$\widetilde{\partial}_{A\cap B}(x)$
()	0.7	$[0.5, 0.7]$	0.28	$[0.1, 0.25]$
(12)	0.5	$[0.3, 0.4]$	0.38	$[0.3, 0.45]$
(23)	0.4	$[0.3, 0.4]$	0.44	$[0.3, 0.45]$
(13)	0.6	$[0.3, 0.6]$	0.48	$[0.3, 0.45]$
(123)	0.4	$[0.4, 0.6]$	0.48	$[0.2, 0.35]$
(132)	0.4	$[0.4, 0.6]$	0.48	$[0.2, 0.35]$

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)$ $(x * x = 0)$,
(IV) $(\forall x, y \in X)$ $(x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.*

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{5.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\ z * y \leq z * x)\,, \tag{5.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{5.3}$$

where $x \leq y$ if and only if $x * y = 0$.

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [7, 16] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X (see [11]) if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

An intuitionistic fuzzy set (μ, ν) in a BCK/BCI-algebra X is called an *intuitionistic fuzzy subalgebra* of X (see [13]) if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ and $\nu(x * y) \leq \max\{\nu(x), \nu(y)\}$ for all $x, y \in X$.

An interval-valued fuzzy set $\tilde{\mu}$ in a BCK/BCI-algebra X is called an *interval-valued fuzzy subalgebra* of X (see [10]) if $\tilde{\mu}(x * y) \succeq \mathrm{rmin}\{\tilde{\mu}(x), \tilde{\mu}(y)\}$ for all $x, y \in X$.

An interval-valued intuitionistic fuzzy subset $(\tilde{\mu}, \tilde{\nu})$ in a BCK/BCI-algebra X is called an *interval-valued intuitionistic fuzzy subalgebra* of X (see [5]) if $\tilde{\mu}(x * y) \succeq \mathrm{rmin}\{\tilde{\mu}(x), \tilde{\mu}(y)\}$ and $\tilde{\nu}(x * y) \preceq \mathrm{rmax}\{\tilde{\nu}(x), \tilde{\nu}(y)\}$ for all $x, y \in X$.

In what follows, let X be a BCK/BCI-algebra unless otherwise stated.

Definition 5.1. A $T\&F$-structure $(X, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ is called a *$T\&F$-subalgebra* of X if the following assertions are valid.

$$(\forall x, y \in X) \begin{pmatrix} \varphi_A(x * y) \geq \min\{\varphi_A(x), \varphi_A(y)\} \\ \widetilde{\varphi}_A(x * y) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} \\ \partial_A(x * y) \leq \max\{\partial_A(x), \partial_A(y)\} \\ \widetilde{\partial}_A(x * y) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\} \end{pmatrix}. \tag{5.4}$$

If a $T\&F$-subalgebra is limited, then we say that it is a *limited $T\&F$-subalgebra.*

Example 5.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ given in the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Let $(X, \mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ given by the next table:

x	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
0	0.63	$[0.4, 0.7]$	0.24	$[0.1, 0.25]$
1	0.26	$[0.4, 0.6]$	0.45	$[0.4, 0.56]$
2	0.48	$[0.3, 0.5]$	0.34	$[0.2, 0.33]$
3	0.57	$[0.2, 0.4]$	0.37	$[0.3, 0.37]$

It is routine to verify that $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X. But it is not limited since $\sup \widetilde{\varphi}_A(1) + \sup \widetilde{\partial}_A(1) = 0.6 + 0.56 = 1.16 > 1$.

Example 5.3. Consider a BCI-algebra $(\mathbb{Z}, -, 0)$ and let $(\mathbb{Z}, \mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ given as follows:

$$\varphi_A : \mathbb{Z} \to [0, 1],\ x \mapsto \begin{cases} 0.6 & \text{if } x \in 4\mathbb{Z}, \\ 0.4 & \text{if } x \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \\ 0.3 & \text{otherwise,} \end{cases}$$

$$\widetilde{\varphi}_A : \mathbb{Z} \to int[0, 1],\ x \mapsto \begin{cases} [0.6, 0.8] & \text{if } x \in 6\mathbb{Z}, \\ [0.4, 0.5] & \text{if } x \in 3\mathbb{Z} \setminus 6\mathbb{Z}, \\ [0.2, 0.3] & \text{otherwise,} \end{cases}$$

$$\partial_A : \mathbb{Z} \to [0, 1],\ x \mapsto \begin{cases} 0.2 & \text{if } x \in 8\mathbb{Z}, \\ 0.4 & \text{if } x \in 4\mathbb{Z} \setminus 8\mathbb{Z}, \\ 0.5 & \text{otherwise.} \end{cases}$$

$$\widetilde{\partial}_A : \mathbb{Z} \to int[0, 1],\ x \mapsto \begin{cases} [0.2, 0.3] & \text{if } x \in 10\mathbb{Z}, \\ [0.4, 0.5] & \text{if } x \in 5\mathbb{Z} \setminus 10\mathbb{Z}, \\ [0.6, 0.8] & \text{otherwise,} \end{cases}$$

It is routine to verify that $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is a $T\&F$-subalgebra of $(\mathbb{Z}, -, 0)$.

Proposition 5.4. *If* $(X, \mathcal{A})$ *is a T&F-subalgebra of* X *with*

$$\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A),$$

then $\varphi_A(x) \leq \varphi_A(0)$, $\widetilde{\varphi}_A(x) \preccurlyeq \widetilde{\varphi}_A(0)$ $\partial_A(x) \geq \partial_A(0)$, *and* $\widetilde{\partial}_A(x) \succcurlyeq \widetilde{\partial}_A(0)$ *for all* $x \in X$.

Proof. It is straightforward by taking $x = y$ in (5.4). □

Proposition 5.5. *Given a T&F-subalgebra* $(X, \mathcal{A})$ *with*

$$\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A),$$

the following are equivalent.

(1) $(\forall x \in X) \begin{pmatrix} \varphi_A(x) = \varphi_A(0),\ \widetilde{\varphi}_A(x) = \widetilde{\varphi}_A(0) \\ \partial_A(x) = \partial_A(0),\ \widetilde{\partial}_A(x) = \widetilde{\partial}_A(0) \end{pmatrix}.$

(2) $(\forall x, y \in X) \begin{pmatrix} \varphi_A(x * y) \geq \varphi_A(y)\},\ \widetilde{\varphi}_A(x * y) \succcurlyeq \widetilde{\varphi}_A(y) \\ \partial_A(x * y) \leq \partial_A(y),\ \widetilde{\partial}_A(x * y) \preccurlyeq \widetilde{\partial}_A(y) \end{pmatrix}.$

Proof. Assume that (1) is valid. Using (5.4), we obtain

$$\begin{aligned}
&\varphi_A(y) = \min\{\varphi_A(0), \varphi_A(y)\} = \min\{\varphi_A(x), \varphi_A(y)\} \leq \varphi_A(x * y), \\
&\widetilde{\varphi}_A(y) = \mathrm{rmin}\{\widetilde{\varphi}_A(0), \widetilde{\varphi}_A(y)\} = \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} \preccurlyeq \widetilde{\varphi}_A(x * y), \\
&\partial_A(y) = \max\{\partial_A(0), \partial_A(y)\} = \max\{\partial_A(x), \partial_A(y)\} \geq \partial_A(x * y), \\
&\widetilde{\partial}_A(y) = \mathrm{rmax}\{\widetilde{\partial}_A(0), \widetilde{\partial}_A(y)\} = \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\} \succcurlyeq \widetilde{\partial}_A(x * y)
\end{aligned}$$

for all $x, y \in X$.

Suppose that (2) is true. Since $x * 0 = 0$ for all $x \in X$, it follows that

$$\begin{aligned}
&\varphi_A(0) \leq \varphi_A(x * 0) = \varphi_A(x), \\
&\widetilde{\varphi}_A(0) \preccurlyeq \widetilde{\varphi}_A(x * 0) = \widetilde{\varphi}_A(x), \\
&\partial_A(0) \geq \partial_A(x * 0) = \partial_A(x)
\end{aligned}$$

and $\widetilde{\partial}_A(0) \succcurlyeq \widetilde{\partial}_A(x * 0) = \widetilde{\partial}_A(x)$. Combining this and Proposition 5.4 induces (1). □

Proposition 5.6. *Let* $(X, \mathcal{A})$ *be a T&F-subalgebra of* X *with* $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$. *If there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} \varphi_A(x_n) = 1$, $\lim_{n\to\infty} \widetilde{\varphi}_A(x_n) = [1, 1]$, $\lim_{n\to\infty} \partial_A(x_n) = 0$, *and* $\lim_{n\to\infty} \widetilde{\partial}_A(x_n) = [0, 0]$, *then* $\varphi_A(0) = 1$, $\widetilde{\varphi}_A(0) = [1, 1]$, $\partial_A(0) = 0$ *and* $\widetilde{\partial}_A(0) = [0, 0]$.

Proof. Using Proposition 5.4, we know that $\varphi_A(0) \geq \varphi_A(x_n)$, $\widetilde{\varphi}_A(0) \succeq \widetilde{\varphi}_A(x_n)$, $\partial_A(0) \leq \partial_A(x_n)$ and $\widetilde{\partial}_A(0) \preceq \widetilde{\partial}_A(x_n)$ for every positive integer n. Note that

$$1 \geq \varphi_A(0) \geq \lim_{n\to\infty} \varphi_A(x_n) = 1 \ , \ [1,1] \succeq \widetilde{\varphi}_A(0) \succeq \lim_{n\to\infty} \widetilde{\varphi}_A(x_n) = [1,1],$$

$$0 \leq \partial_A(0) \leq \lim_{n\to\infty} \partial_A(x_n) = 0 \ , \ [0,0] \preceq \widetilde{\partial}_A(0) \preceq \lim_{n\to\infty} \widetilde{\partial}_A(x_n) = [0,0].$$

Therefore $\varphi_A(0) = 1$, $\widetilde{\varphi}_A(0) = [1,1]$, $\partial_A(0) = 0$ and $\widetilde{\partial}_A(0) = [0,0]$. □

Proposition 5.7. *Every T&F-subalgebra $(X, \mathcal{A})$ of a BCI-algebra X with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ satisfies the following assertion:*

$$(\forall x, y \in X) \left(\begin{array}{l} \varphi_A(x * (0 * y)) \geq \min\{\varphi_A(x), \varphi_A(y)\}, \\ \widetilde{\varphi}_A(x * (0 * y)) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} \\ \partial_A(x * (0 * y)) \leq \max\{\partial_A(x), \partial_A(y)\}, \\ \widetilde{\partial}_A(x * (0 * y)) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\} \end{array} \right). \tag{5.5}$$

Proof. Using (5.4) and Proposition 5.4, we obtain

$$\begin{aligned} \varphi_A(x * (0 * y)) &\geq \min\{\varphi_A(x), \varphi_A(0 * y)\} \\ &\geq \min\{\varphi_A(x), \min\{\varphi_A(0), \varphi_A(y)\}\} \\ &= \min\{\varphi_A(x), \varphi_A(y)\}, \end{aligned}$$

$$\begin{aligned} \widetilde{\varphi}_A(x * (0 * y)) &\succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(0 * y)\} \\ &\succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(x), \mathrm{rmin}\{\widetilde{\varphi}_A(0), \widetilde{\varphi}_A(y)\}\} \\ &= \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}, \end{aligned}$$

$$\begin{aligned} \partial_A(x * (0 * y)) &\leq \max\{\partial_A(x), \partial_A(0 * y)\} \\ &\leq \max\{\partial_A(x), \max\{\partial_A(0), \partial_A(y)\}\} \\ &= \max\{\partial_A(x), \partial_A(y)\}, \end{aligned}$$

$$\begin{aligned} \widetilde{\partial}_A(x * (0 * y)) &\preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(0 * y)\} \\ &\preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x), \mathrm{rmax}\{\widetilde{\partial}_A(0), \widetilde{\partial}_A(y)\}\} \\ &= \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\} \end{aligned}$$

for all $x, y \in X$. This completes the proof. □

Theorem 5.8. *Every subalgebra of X can be realized as a T&F-subalgebra of X.*

Proof. Let K be a subalgebra of X and let $(X, \mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ which is defined by

$$\varphi_A(x) = \begin{cases} t & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{\varphi}_A(x) = \begin{cases} [\gamma_1, \gamma_2] & \text{if } x \in K, \\ [0, 0] & \text{otherwise,} \end{cases}$$
$$\partial_A(x) = \begin{cases} s & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \quad \widetilde{\partial}_A(x) = \begin{cases} [\delta_1, \delta_2] & \text{if } x \in K, \\ [1, 1] & \text{otherwise,} \end{cases} \tag{5.6}$$

where $t \in (0, 1]$, $s \in [0, 1)$, $\gamma_1, \gamma_2 \in (0, 1]$ with $\gamma_1 < \gamma_2$ and $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 < \delta_2$. It is clear that $U(\varphi_A; t) = K$, $U(\widetilde{\varphi}_A; [\gamma_1, \gamma_2]) = K$, $L(\partial_A; s) = K$ and $L(\widetilde{\partial}_A; [\delta_1, \delta_2]) = K$. Let $x, y \in X$. If $x, y \in K$, then $x * y \in K$ and so

$$\begin{aligned}
&\varphi_A(x * y) = t = \min\{\varphi_A(x), \varphi_A(y)\} \\
&\widetilde{\varphi}_A(x * y) = [\gamma_1, \gamma_2] = \text{rmin}\{[\gamma_1, \gamma_2], [\gamma_1, \gamma_2]\} = \text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}, \\
&\partial_A(x * y) = s = \max\{\partial_A(x), \partial_A(y)\}, \\
&\widetilde{\partial}_A(x * y) = [\delta_1, \delta_2] = \text{rmax}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = \text{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\}.
\end{aligned}$$

If any one of x and y is contained in K, say $x \in K$, then $\varphi_A(x) = t$, $\widetilde{\varphi}_A(x) = [\gamma_1, \gamma_2]$, $\partial_A(x) = s$, $\widetilde{\partial}_A(x) = [\delta_1, \delta_2]$, $\varphi_A(y) = 0$, $\widetilde{\varphi}_A(y) = [0, 0]$, $\partial_A(y) = 1$ and $\widetilde{\partial}_A(y) = [1, 1]$. Hence

$$\begin{aligned}
&\varphi_A(x * y) \geq 0 = \min\{t, 0\} = \min\{\varphi_A(x), \varphi_A(y)\} \\
&\widetilde{\varphi}_A(x * y) \succeq [0, 0] = \text{rmin}\{[\gamma_1, \gamma_2], [0, 0]\} = \text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}, \\
&\partial_A(x * y) \leq 1 = \max\{s, 1\} = \max\{\partial_A(x), \partial_A(y)\}, \\
&\widetilde{\partial}_A(x * y) \preceq [1, 1] = \text{rmax}\{[\delta_1, \delta_2], [1, 1]\} = \text{rmax}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\},
\end{aligned}$$

If $x, y \notin K$, then $\varphi_A(x) = 0 = \varphi_A(y)$, $\widetilde{\varphi}_A(x) = [0, 0] = \widetilde{\varphi}_A(y)$, $\partial_A(x) = 1 = \partial_A(y)$ and $\widetilde{\partial}_A(x) = [1, 1] = \widetilde{\partial}_A(y)$. It follows that

$$\begin{aligned}
&\varphi_A(x * y) \geq 0 = \min\{0, 0\} = \min\{\varphi_A(x), \varphi_A(y)\} \\
&\widetilde{\varphi}_A(x * y) \succeq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}, \\
&\partial_A(x * y) \leq 1 = \max\{1, 1\} = \max\{\partial_A(x), \partial_A(y)\}, \\
&\widetilde{\partial}_A(x * y) \preceq [1, 1] = \text{rmax}\{[1, 1], [1, 1]\} = \text{rmax}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}.
\end{aligned}$$

Therefore $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X. □

Theorem 5.9. *For any non-empty subset K of X, let $(X, \mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ which is given in (5.6). If $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X, then K is a subalgebra of X.*

Proof. Let $x, y \in K$. Then $\varphi_A(x) = t = \varphi_A(y)$, $\widetilde{\varphi}_A(x) = [\gamma_1, \gamma_2] = \widetilde{\varphi}_A(y)$, $\partial_A(x) = s = \partial_A(y)$ and $\widetilde{\partial}_A(x) = [\delta_1, \delta_2] = \widetilde{\varphi}_A(y)$. Thus

$$\begin{aligned}
&\varphi_A(x * y) \geq \min\{\varphi_A(x), \varphi_A(y)\} = t,\\
&\widetilde{\varphi}_A(x * y) \succeq \text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} = [\gamma_1, \gamma_2],\\
&\partial_A(x * y) \leq \max\{\partial_A(x), \partial_A(y)\} = s,\\
&\widetilde{\partial}_A(x * y) \preceq \text{rmax}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} = [\delta_1, \delta_2].
\end{aligned}$$

and therefore $x * y \in K$. Hence K is a subalgebra of X. □

Theorem 5.10. *If $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are (limited) $T\&F$-subalgebras with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\ \widetilde{\varphi}_B,\ \partial_B,\ \widetilde{\partial}_B)$ respectively, then their intersection is also a (limited) $T\&F$-subalgebra.*

Proof. Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are (limited) $T\&F$-subalgebras with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B,\ \widetilde{\varphi}_B,\ \partial_B,\ \widetilde{\partial}_B)$ respectively. For any $x, y \in X$, we get

$$\begin{aligned}
\varphi_{A\wedge B}(x * y) &= \min\{\varphi_A(x * y), \varphi_B(x * y)\}\\
&\geq \min\{\min\{\varphi_A(x), \varphi_A(y)\}, \min\{\varphi_B(x), \varphi_A(y)\}\\
&= \min\{\min\{\varphi_A(x), \varphi_B(x)\}, \min\{\varphi_A(y), \varphi_B(y)\}\\
&= \min\{\varphi_{A\wedge B}(x), \varphi_{A\wedge B}(y)\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\varphi}_{A\cap B}(x * y) &= \text{rmin}\{\widetilde{\varphi}_A(x * y), \widetilde{\varphi}_B(x * y)\}\\
&\succcurlyeq \text{rmin}\{\text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}, \text{rmin}\{\widetilde{\varphi}_B(x), \widetilde{\varphi}_A(y)\}\\
&= \text{rmin}\{\text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_B(x)\}, \text{rmin}\{\widetilde{\varphi}_A(y), \widetilde{\varphi}_B(y)\}\\
&= \text{rmin}\{\widetilde{\varphi}_{A\cap B}(x), \widetilde{\varphi}_{A\cap B}(y)\},
\end{aligned}$$

$$\begin{aligned}
\partial_{A\vee B}(x * y) &= \max\{\partial_A(x * y), \partial_B(x * y)\}\\
&\leq \max\{\max\{\partial_A(x), \partial_A(y)\}, \max\{\partial_B(x), \partial_B(y)\}\\
&= \max\{\max\{\partial_A(x), \partial_B(x)\}, \max\{\partial_A(y), \partial_B(y)\}\\
&= \max\{\partial_{A\vee B}(x), \partial_{A\vee B}(y)\},
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\partial}_{A\cup B}(x*y) &= \text{rmax}\{\widetilde{\partial}_A(x*y), \widetilde{\partial}_B(x*y)\} \\ &\preccurlyeq \text{rmax}\{\text{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\}, \text{rmax}\{\widetilde{\partial}_B(x), \widetilde{\partial}_B(y)\} \\ &= \text{rmax}\{\text{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_B(x)\}, \text{rmax}\{\widetilde{\partial}_A(y), \widetilde{\partial}_B(y)\} \\ &= \text{rmax}\{\widetilde{\partial}_{A\cup B}(x), \widetilde{\partial}_{A\cup B}(y)\}.\end{aligned}$$

Therefore $(X, \mathcal{A} \Cap \mathcal{B})$ is a $T\&F$-subalgebra of X. It is clear that if $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are limited, then so is $(X, \mathcal{A} \Cap \mathcal{B})$. □

The following example shows that the union of two $T\&F$-subalgebras may not be a $T\&F$-subalgebra in general.

Example 5.11. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ given in the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	1	1	0	1
4	4	4	4	4	1

Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be $T\&F$-structures with $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ and $\mathcal{B} := (\varphi_B, \widetilde{\varphi}_B, \partial_B, \widetilde{\partial}_B)$ given by Tables 7 and 8, respectively.

TABLE 7. Tabular representation of $\mathcal{A} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$

X	$\varphi_A(x)$	$\widetilde{\varphi}_A(x)$	$\partial_A(x)$	$\widetilde{\partial}_A(x)$
0	0.7	$[0.5, 0.7]$	0.3	$[0.2, 0.25]$
1	0.5	$[0.3, 0.5]$	0.5	$[0.4, 0.56]$
2	0.6	$[0.3, 0.4]$	0.4	$[0.4, 0.56]$
3	0.5	$[0.2, 0.3]$	0.5	$[0.2, 0.25]$
4	0.3	$[0.1, 0.2]$	0.7	$[0.6, 0.89]$

It is routine to verify that $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are $T\&F$-subalgebras of X. The union $(X, \mathcal{A} \Cup \mathcal{B})$ of $(X, \mathcal{A})$ and $(X, \mathcal{B})$ is given by Table 9, and it is not a $T\&F$-subalgebras of X since

$$\varphi_{A\vee B}(2*3) = \varphi_{A\vee B}(1) = 0.5 < 0.6 = \min\{\varphi_{A\vee B}(2), \varphi_{A\vee B}(3)\}$$

TABLE 8. Tabular representation of $\mathcal{B} := (\varphi_B,\, \widetilde{\varphi}_B,\, \partial_B,\, \widetilde{\partial}_B)$

X	$\varphi_B(x)$	$\widetilde{\varphi}_B(x)$	$\partial_B(x)$	$\widetilde{\partial}_B(x)$
0	0.8	$[0.6, 0.8]$	0.2	$[0.14, 0.28]$
1	0.4	$[0.5, 0.7]$	0.6	$[0.35, 0.45]$
2	0.4	$[0.4, 0.6]$	0.6	$[0.24, 0.38]$
3	0.8	$[0.5, 0.6]$	0.2	$[0.35, 0.45]$
4	0.3	$[0.2, 0.3]$	0.7	$[0.53, 0.65]$

and/or

$$\begin{aligned}\widetilde{\partial}_{A\cap B}(3*2) = \widetilde{\partial}_{A\cap B}(1) &= [0.35, 0.45] \\ &\succ [0.24, 0.38] = \text{rmax}\{\widetilde{\partial}_{A\cap B}(3), \widetilde{\partial}_{A\cap B}(2)\}.\end{aligned}$$

TABLE 9. Tabular representation of the union $\mathcal{A} \Cup \mathcal{B}$

X	$\varphi_{A\vee B}(x)$	$\widetilde{\varphi}_{A\cup B}(x)$	$\partial_{A\wedge B}(x)$	$\widetilde{\partial}_{A\cap B}(x)$
0	0.8	$[0.6, 0.8]$	0.2	$[0.14, 0.25]$
1	0.5	$[0.5, 0.7]$	0.5	$[0.35, 0.45]$
2	0.6	$[0.4, 0.6]$	0.4	$[0.24, 0.38]$
3	0.8	$[0.5, 0.6]$	0.2	$[0.20, 0.25]$
4	0.3	$[0.2, 0.3]$	0.7	$[0.53, 0.65]$

Theorem 5.12. *A $T\&F$-structure $(X, \mathcal{A})$ with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is a $T\&F$-subalgebra of X if and only if the sets $U(\varphi_A; \alpha)$, $U(\widetilde{\varphi}_A; \tilde{t})$, $L(\partial_A; \beta)$ and $L(\widetilde{\partial}_A; \tilde{s})$ are subalgebras of X for all $\alpha, \beta \in [0,1]$ and $\tilde{t} = [t^-, t^+]$, $\tilde{s} = [s^-, s^+] \in int([0,1])$.*

Proof. Assume that $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X. Let $x_1, y_1 \in U(\varphi_A; \alpha)$, $x_2, y_2 \in U(\widetilde{\varphi}_A; \alpha)$, $x_3, y_3 \in L(\partial_A; \beta)$ and $x_4, y_4 \in L(\widetilde{\partial}_A; \tilde{s})$ for all $\alpha, \beta \in [0,1]$ and $\tilde{t} = [t^-, t^+], \tilde{s} = [s^-, s^+] \in int([0,1])$. Then $\varphi_A(x_1) \geq \alpha$, $\varphi_A(y_1) \geq \alpha$, $\widetilde{\varphi}_A(x_2) \succcurlyeq \tilde{t}$, $\widetilde{\varphi}_A(y_2) \succcurlyeq \tilde{t}$, $\partial_A(x_3) \leq \beta$, $\partial_A(y_3) \leq \beta$, $\widetilde{\partial}_A(x_4) \preccurlyeq \tilde{s}$ and $\widetilde{\partial}_A(y_4) \preccurlyeq \tilde{s}$.

It follows from (5.4) that

$$\begin{aligned}
&\varphi_A(x_1 * y_1) \geq \min\{\varphi_A(x_1), \varphi_A(y_1)\} \geq \alpha,\\
&\widetilde{\varphi}_A(x_2 * y_2) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(x_2), \widetilde{\varphi}_A(y_2)\} \succcurlyeq \tilde{t},\\
&\partial_A(x_3 * y_3) \leq \max\{\partial_A(x_3), \partial_A(y_3)\} \leq \beta,\\
&\widetilde{\partial}_A(x_4 * y_4) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x_4), \widetilde{\partial}_A(y_4)\} \preccurlyeq \tilde{s}.
\end{aligned}$$

Hence $x_1 * y_1 \in U(\varphi_A;\alpha)$, $x_2 * y_2 \in U(\widetilde{\varphi}_A;\alpha)$, $x_3 * y_3 \in L(\partial_A;\beta)$ and $x_4 * y_4 \in L(\widetilde{\partial}_A;\tilde{s})$. Therefore $U(\varphi_A;\alpha)$, $U(\widetilde{\varphi}_A;\tilde{t})$, $L(\partial_A;\beta)$ and $L(\widetilde{\partial}_A;\tilde{s})$ are subalgebras of X.

Conversely, let $(X,\mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ such that $U(\varphi_A;\alpha)$, $U(\widetilde{\varphi}_A;\tilde{t})$, $L(\partial_A;\beta)$ and $L(\widetilde{\partial}_A;\tilde{s})$ are subalgebras of X for all $\alpha,\beta \in [0,1]$ and $\tilde{t} = [t^-,t^+], \tilde{s} = [s^-,s^+] \in int([0,1])$. Let $x,y \in X$ be such that $\varphi_A(x) = \alpha_x$ and $\varphi_A(y) = \alpha_y$. If we take $\alpha := \min\{\alpha_x,\alpha_y\}$, then $x,y \in U(\varphi_A;\alpha)$, and so $x * y \in U(\varphi_A;\alpha)$. Hence

$$\varphi_A(x * y) \geq \alpha := \min\{\alpha_x,\alpha_y\} = \min\{\varphi_A(x), \varphi_A(y)\}.$$

Let $x,y \in X$ be such that $\widetilde{\varphi}_A(x) = \tilde{t}_x$ and $\widetilde{\varphi}_A(y) = \tilde{t}_y$. Taking $\tilde{t} := \mathrm{rmin}\{\tilde{t}_x,\tilde{t}_y\}$ implies that $x,y \in U(\widetilde{\varphi}_A;\tilde{t})$, and so $x * y \in U(\widetilde{\varphi}_A;\tilde{t})$. Thus

$$\widetilde{\varphi}_A(x * y) \succcurlyeq \tilde{t} := \mathrm{rmin}\{\tilde{t}_x,\tilde{t}_y\} = \mathrm{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\}.$$

Similarly, we can verify that $\partial_A(x * y) \leq \max\{\partial_A(x), \partial_A(y)\}$ and $\widetilde{\partial}_A(x * y) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(x), \widetilde{\partial}_A(y)\}$, for all $x,y \in X$. Therefore $(X,\mathcal{A})$ is a $T\&F$-subalgebra of X. □

Given a $T\&F$-structure $(X,\mathcal{A})$ with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$, let $(X,\mathcal{A}^*)$ be a $T\&F$-structure in which $\mathcal{A}^* := (\varphi^*_A,\ \widetilde{\varphi}^*_A,\ \partial^*_A,\ \widetilde{\partial}^*_A)$ is given as follows:

$$\varphi^*_A : X \to [0,1],\ x \mapsto \begin{cases} \varphi_A(x) & \text{if } x \in U(\varphi_A;\alpha), \\ m & \text{otherwise,} \end{cases}$$

$$\widetilde{\varphi}^*_A : X \to int([0,1]),\ x \mapsto \begin{cases} \widetilde{\varphi}_A(x) & \text{if } x \in U(\widetilde{\varphi}_A;\tilde{t}), \\ \tilde{p} & \text{otherwise,} \end{cases}$$

$$\partial^*_A : X \to [0,1],\ x \mapsto \begin{cases} \partial_A(x) & \text{if } x \in L(\partial_A;\beta), \\ n & \text{otherwise,} \end{cases}$$

$$\widetilde{\partial}^*_A : X \to int([0,1]),\ x \mapsto \begin{cases} \widetilde{\partial}_A(x) & \text{if } x \in L(\widetilde{\partial}_A;\tilde{s}), \\ \tilde{q} & \text{otherwise} \end{cases}$$

where $\tilde{t} = [t^-,t^+], \tilde{s} = [s^-,s^+], \tilde{p} = [p^-,p^+], \tilde{q} = [q^-,q^+] \in int([0,1])$ and $\alpha,\beta,m,n \in [0,1]$ with $\varphi_A(x) > m$, $\widetilde{\varphi}_A(x) \succ \tilde{p}$, $\partial_A(x) < n$ and $\widetilde{\partial}_A(x) \prec \tilde{q}$.

Theorem 5.13. *If $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X with*

$$\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A),$$

then so is $(X, \mathcal{A}^*)$ with $\mathcal{A}^* := (\varphi_A^*,\, \widetilde{\varphi}_A^*,\, \partial_A^*,\, \widetilde{\partial}_A^*)$.

Proof. Assume that $(X, \mathcal{A})$ is a $T\&F$-subalgebra of X with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$. Then the sets $U(\varphi_A; \alpha)$, $U(\widetilde{\varphi}_A; \tilde{t})$, $L(\partial_A; \beta)$ and $L(\widetilde{\partial}_A; \tilde{s})$ are subalgebras of X for all $\alpha, \beta \in [0,1]$ and $\tilde{t} = [t^-, t^+]$, $\tilde{s} = [s^-, s^+] \in int([0,1])$ by Theorem 5.12. Let $x, y \in X$. If $x, y \in U(\varphi_A; \alpha)$, then $x * y \in U(\varphi_A; \alpha)$, and so

$$\varphi_A^*(x * y) = \varphi_A(x * y) \geq \min\{\varphi_A(x), \varphi_A(y)\} = \min\{\varphi_A^*(x), \varphi_A^*(y)\}.$$

If $x \notin U(\varphi_A; \alpha)$ or $y \notin U(\varphi_A; \alpha)$, then $\varphi_A(x) = m$ or $\varphi_A(y) = m$. Hence

$$\varphi_A^*(x * y) \geq m = \min\{\varphi_A^*(x), \varphi_A^*(y)\}.$$

If $x, y \in U(\widetilde{\varphi}_A; \tilde{t})$, then $x * y \in U(\widetilde{\varphi}_A; \tilde{t})$, and thus

$$\widetilde{\varphi}_A^*(x * y) = \widetilde{\varphi}_A(x * y) \succcurlyeq \text{rmin}\{\widetilde{\varphi}_A(x), \widetilde{\varphi}_A(y)\} = \text{rmin}\{\widetilde{\varphi}_A^*(x), \widetilde{\varphi}_A^*(y)\}.$$

Assume that $x \notin U(\widetilde{\varphi}_A; \tilde{t})$ or $y \notin U(\widetilde{\varphi}_A; \tilde{t})$, then $\widetilde{\varphi}_A(x) = \tilde{p}$ or $\widetilde{\varphi}_A(y) = \tilde{p}$. Hence $\widetilde{\varphi}_A^*(x * y) \succcurlyeq \tilde{p} = \text{rmin}\{\widetilde{\varphi}_A^*(x), \widetilde{\varphi}_A^*(y)\}$. Similarly, we can verify that $\varphi_A^*(x * y) \leq \max\{\partial_A^*(x), \partial_A^*(y)\}$ and $\widetilde{\varphi}_A^*(x * y) \preccurlyeq \text{rmax}\{\widetilde{\partial}_A^*(x), \widetilde{\partial}_A^*(y)\}$, for all $x, y \in X$. Therefore $(X, \mathcal{A}^*)$ is a $T\&F$-subalgebra of X. □

The following example shows that the converse of Theorem 5.13 is not true in general.

Example 5.14. Let $\mathbb{Z}_6 := \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ in which a binary operation $*$ is defined by

$$(\forall \bar{m}, \bar{n} \in \mathbb{Z}_6)(\bar{m} * \bar{n} = \overline{6 + m - n}). \tag{5.7}$$

Then $(\mathbb{Z}_6, *, \bar{0})$ is a BCI-algebra. Let $(\mathbb{Z}_6, \mathcal{A})$ be a $T\&F$-structure with $\mathcal{A} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ given as follows:

$$\varphi_A : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = \bar{0}, \\ 0.7 & \text{if } x \in \{\bar{2}, \bar{4}\} \\ 0.6 & \text{if } x \in \{\bar{1}, \bar{3}\} \\ 0.3 & \text{if } x = \bar{5}. \end{cases}$$

$$\widetilde{\varphi}_A : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} [0.4, 0.9] & \text{if } x = \bar{0}, \\ [0.1, 0.4] & \text{if } x = \bar{1}, \\ [0.3, 0.7] & \text{if } x = \bar{2}, \\ [0.1, 0.4] & \text{if } x = \bar{3}, \\ [0.2, 0.5] & \text{if } x = \bar{4}, \\ [0.1, 0.4] & \text{if } x = \bar{5}, \end{cases}$$

$$\partial_A : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} 0.2 & \text{if } x = \bar{0}, \\ 0.4 & \text{if } x \in \{\bar{2}, \bar{4}\} \\ 0.6 & \text{if } x \in \{\bar{1}, \bar{5}\} \\ 0.8 & \text{if } x = \bar{3}. \end{cases}$$

$$\widetilde{\partial}_A : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} [0.25, 0.47] & \text{if } x = \bar{0}, \\ [0.33, 0.53] & \text{if } x \in \{\bar{1}, \bar{2}\} \\ [0.27, 0.53] & \text{if } x = \bar{3} \\ [0.35, 0.56] & \text{if } x = \bar{4} \\ [0.38, 0.62] & \text{if } x = \bar{5}. \end{cases}$$

Then $U(\varphi_A; \alpha) = \{\bar{0}, \bar{2}, \bar{4}\}$ for $\alpha \in (0.6, 0.7]$, $U(\widetilde{\varphi}_A; [0.2, 0.5]) = \{\bar{0}, \bar{2}, \bar{4}\}$, $L(\partial_A; \beta) = \{\bar{0}, \bar{2}, \bar{4}\}$ for $\beta \in [0.4, 0.6)$ and $L(\widetilde{\varphi}_B; [0.27, 0.53]) = \{\bar{0}, \bar{3}\}$. Let $(\mathbb{Z}_6, \mathcal{A}^*)$ be a $T\&F$-structure with $\mathcal{A}^* := (\varphi_A^*, \widetilde{\varphi}_A^*, \partial_A^*, \widetilde{\partial}_A^*)$ given by

$$\varphi_A^* : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} \varphi_A(x) & \text{if } x \in U(\varphi_A; \alpha), \\ 0 & \text{otherwise}, \end{cases}$$

$$\widetilde{\varphi}_A^* : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} \widetilde{\varphi}_A(x) & \text{if } x \in U(\widetilde{\varphi}_A; [0.2, 0.5]), \\ [0,0] & \text{otherwise}, \end{cases}$$

$$\partial_A^* : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} \partial_A(x) & \text{if } x \in L(\partial_A; \beta), \\ 1 & \text{otherwise}, \end{cases}$$

$$\widetilde{\partial}_A^* : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} \widetilde{\partial}_A(x) & \text{if } x \in L(\widetilde{\varphi}_B; [0.27, 0.53]), \\ [1,1] & \text{otherwise}, \end{cases}$$

that is,

$$\varphi_A : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = \bar{0}, \\ 0.7 & \text{if } x \in \{\bar{2}, \bar{4}\} \\ 0 & \text{otherwise}, \end{cases}$$

$$\widetilde{\varphi}_A : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} [0.4, 0.9] & \text{if } x = \bar{0}, \\ [0.3, 0.7] & \text{if } x = \bar{2}, \\ [0.2, 0.5] & \text{if } x = \bar{4}, \\ [0, 0] & \text{otherwise,} \end{cases}$$

$$\partial_A : \mathbb{Z}_6 \to [0,1],\ x \mapsto \begin{cases} 0.2 & \text{if } x = \bar{0}, \\ 0.4 & \text{if } x \in \{\bar{2}, \bar{4}\} \\ 1 & \text{otherwise,} \end{cases}$$

$$\widetilde{\partial}_A : \mathbb{Z}_6 \to int([0,1]),\ x \mapsto \begin{cases} [0.25, 0.47] & \text{if } x = \bar{0}, \\ [0.27, 0.53] & \text{if } x = \bar{3} \\ [1, 1] & \text{otherwise.} \end{cases}$$

Then $(\mathbb{Z}_6, \mathcal{A}^*)$ is a $T\&F$-subalgebra of $\mathbb{Z}_6$. But $(\mathbb{Z}_6, \mathcal{A})$ is not a $T\&F$-subalgebra of $\mathbb{Z}_6$ since

$$\begin{aligned} &\varphi_A(\bar{3} * \bar{4}) = \varphi_A(\bar{5}) = 0.3 < 0.6 = \min\{\varphi_A(\bar{3}), \varphi_A(\bar{4})\}, \\ &\widetilde{\varphi}_A(\bar{0} * \bar{2}) = \widetilde{\varphi}_A(\bar{4}) = [0.2, 0.5] \prec [0.3, 0.7] = \text{rmin}\{\widetilde{\varphi}_A(\bar{0}), \widetilde{\varphi}_A(\bar{2})\}, \\ &\partial_A(\bar{1} * \bar{4}) = \partial_A(\bar{3}) = 0.8 > 0.6 = \max\{\partial_A(\bar{1}), \partial_A(\bar{4})\}, \end{aligned}$$

and/or $\widetilde{\partial}_A(\bar{2} * \bar{3}) = \widetilde{\partial}_A(\bar{5}) = [0.38, 0.62] \succ [0.33, 0.53] = \text{rmax}\{\widetilde{\partial}_A(\bar{2}), \widetilde{\partial}_A(\bar{3})\}$.

Theorem 5.15. *Given a $T\&F$-subalgebra $(X, \mathcal{A})$ of a BCI-algebra X with $\mathcal{A} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$, let $(X, \mathcal{A}^*)$ be a $T\&F$-structure with $\mathcal{A}^* := (\varphi_A^*,\ \widetilde{\varphi}_A^*,\ \partial_A^*,\ \widetilde{\partial}_A^*)$ defined by $\varphi_A^*(x) = \varphi_A(0 * x)$, $\widetilde{\varphi}_A^*(x) = \widetilde{\varphi}_A(0 * x)$, $\partial_A^*(x) = \partial_A(0 * x)$ and $\widetilde{\partial}_A^*(x) = \widetilde{\partial}_A(0 * x)$ for all $x \in X$. Then $(X, \mathcal{A}^*)$ is a $T\&F$-subalgebra of X.*

Proof. Note that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. We have

$$\begin{aligned} \varphi_A^*(x * y) &= \varphi_A(0 * (x * y)) = \varphi_A((0 * x) * (0 * y)) \\ &\geq \min\{\varphi_A(0 * x), \varphi_A(0 * y)\} \\ &= \min\{\varphi_A^*(x), \varphi_A^*(y)\}, \end{aligned}$$

$$\begin{aligned} \widetilde{\varphi}_A^*(x * y) &= \widetilde{\varphi}_A(0 * (x * y)) = \widetilde{\varphi}_A((0 * x) * (0 * y)) \\ &\succeq \text{rmin}\{\widetilde{\varphi}_A(0 * x), \widetilde{\varphi}_A(0 * y)\} \\ &= \text{rmin}\{\widetilde{\varphi}_A^*(x), \widetilde{\varphi}_A^*(y)\} \end{aligned}$$

$$\begin{aligned}\partial_A^*(x*y) &= \partial_A(0*(x*y)) = \partial_A((0*x)*(0*y)) \\ &\leq \max\{\partial_A(0*x), \partial_A(0*y)\} \\ &= \max\{\partial_A^*(x), \partial_A^*(y)\}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\partial}_A^*(x*y) &= \widetilde{\partial}_A(0*(x*y)) = \widetilde{\partial}_A((0*x)*(0*y)) \\ &\preceq \mathrm{rmax}\{\widetilde{\partial}_A(0*x), \widetilde{\partial}_A(0*y)\} \\ &= \mathrm{rmax}\{\widetilde{\partial}_A^*(x), \widetilde{\partial}_A^*(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $(X, \mathcal{A}^*)$ is a $T\&F$-subalgebra of X. □

Theorem 5.16. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $(Y, \mathcal{B})$ is a $T\&F$-subalgebra of Y with $\mathcal{B} := (\varphi_B,\ \widetilde{\varphi}_B,\ \partial_B,\ \widetilde{\partial}_B)$, then $(X, f^{-1}(\mathcal{B}))$ is a $T\&F$-subalgebra of X with $f^{-1}(\mathcal{B}) := (f^{-1}(\varphi_B),\ f^{-1}(\widetilde{\varphi}_B),\ f^{-1}(\partial_B),\ f^{-1}(\widetilde{\partial}_B))$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}f^{-1}(\varphi_B)(x*y) &= \varphi_B(f(x*y)) = \varphi_B(f(x)*f(y)) \\ &\geq \min\{\varphi_B(f(x)), \varphi_B(f(y))\} \\ &= \min\{f^{-1}(\varphi_B)(x), f^{-1}(\varphi_B)(y)\},\end{aligned}$$

$$\begin{aligned}f^{-1}(\widetilde{\varphi}_B)(x*y) &= \widetilde{\varphi}_B(f(x*y)) = \widetilde{\varphi}_B(f(x)*f(y)) \\ &\succeq \mathrm{rmin}\{\widetilde{\varphi}_B(f(x)), \widetilde{\varphi}_B(f(y))\} \\ &= \mathrm{rmin}\{f^{-1}(\widetilde{\varphi}_B)(x), f^{-1}(\widetilde{\varphi}_B)(y)\},\end{aligned}$$

$$\begin{aligned}f^{-1}(\partial_B)(x*y) &= \partial_B(f(x*y)) = \partial_B(f(x)*f(y)) \\ &\leq \max\{\partial_B(f(x)), \partial_B(f(y))\} \\ &= \max\{f^{-1}(\partial_B)(x), f^{-1}(\partial_B)(y)\}.\end{aligned}$$

and

$$\begin{aligned}f^{-1}(\widetilde{\partial}_B)(x*y) &= \widetilde{\partial}_B(f(x*y)) = \widetilde{\partial}_B(f(x)*f(y)) \\ &\preceq \mathrm{rmax}\{\widetilde{\partial}_B(f(x)), \widetilde{\partial}_B(f(y))\} \\ &= \mathrm{rmax}\{f^{-1}(\widetilde{\partial}_B)(x), f^{-1}(\widetilde{\partial}_B)(y)\},\end{aligned}$$

Hence $(X, f^{-1}(\mathcal{B}))$ is a $T\&F$-subalgebra of X. □

6. Conclusion

This paper proposes the introduction of True-False structures, a new type of structure constructed using (interval-valued) fuzzy sets. The application of these structures to groups and BCK/BCI-algebras is explored. The notions of (limited) $T\&F$-groups and (limited) $T\&F$-subalgebras are introduced, which are groups and subalgebras equipped with True-False structures. Various properties and characterizations of these structures are investigated. One of the main results is proof that the intersection of two $T\&F$-groups is also a $T\&F$-group. Similarly, the intersection of two $T\&F$-subalgebras is also a $T\&F$-subalgebra, demonstrating the closure properties of these structures under intersection. However, examples are provided to illustrate that the union of two $T\&F$-groups (respectively, $T\&F$-subalgebras) may not necessarily be a $T\&F$-group (respectively, $T\&F$-subalgebra), highlighting the non-closure properties of these structures under union. The study of True-False structures in the context of groups and BCK/BCI-algebras aims to enhance the understanding and application of fuzzy set theory in these areas. The findings contribute to the development of new mathematical structures and provide insights into the behavior of fuzzy sets in group theory and algebra.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions.

This work is supported by Foreign Export Program of China (Grant No. DL 20230410021).

References

1. M. Aaly Kologani, X. L. Xin, Y. B. Jun and M. Mohseni Takallo, Positive implicative equality algebras and equality algebras with some types, *Journal of Algebraic Hyperstructures and Logical Algebras*, **3**(2) (2022), 69–86.
2. K. T. Atanassov, More on intuitionistic fuzzy sets, *Fuzzy Sets and systems*, **33**(1) (1989), 37–45.
3. R. Biswas, Rosenfeld's fuzzy subgroups with intervalvalued membership functions, *Fuzzy Sets and Systems*, **87** (1994), 87–90.
4. R. A. Borzooei, M. Mohseni Takallo, Y. B. Jun and M. Aaly Kologani, Positive implicative True-False ideals in BCK-algebras, *Journal of Mahani Mathematical Research Center*, **11**(3) (2022), 69–85.
5. P. Hemavathi, P. Muralikrishna and K. Palanivel, On interval valued intuitionistic fuzzy β-subalgebras, *Afr. Mat.*, **29**(1-2) (2018), 249–262.
6. S. M. Hong, Y. B. Jun, S. J. Kim and G. I. Kim, Fuzzy BCI-subalgebras with interval-valued membership functions, *Int. J. Math. Math. Sci.*, **25**(2) (2001), 135–143.

7. Y. Huang, *BCI-algebra,* Science Press: Beijing, China, 2006.
8. K. Iséki, On BCI-algebras, *Math. Seminar Notes*, **8** (1980), 125–130.
9. K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japon.*, **23** (1978), 1–26.
10. Y. B. Jun, Interval-valued fuzzy subalgebras/ideals in BCK-algebras, *Sci. Math.*, **3**(3) (2000), 435–444.
11. Y. B. Jun and S. M. Hong, Fuzzy subalgebras of BCK/BCI-algebras redefined, *Math. Japon. Online*, **4** (2001), 769–775.
12. Y. B. Jun and K. H. Kim, Interval-valued fuzzy subalgebras of BCK-algebras, *J. Fuzzy Math.*, **14**(4) (2006), 999–1003.
13. Y. B. Jun and K. H. Kim, Intuitionistic fuzzy ideals of BCK/BCI-algebras, *Internat. J. Math. Math. Sci.*, **24**(12) (2000), 839–849.
14. Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, *Ann. Fuzzy Math. Inform.*, **4**(1) (2012), 83–98.
15. J. Meng, On ideals in BCK-algebras, *Math. Japon.*, **40** (1994), 143–154.
16. J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoon Sa Co.: Seoul, Korea, 1994.
17. M. Mohseni Takallo and M. Aaly Kologani, MBJ-neutrosophic filters of equality algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **1**(2) (2020), 57–75.
18. M. Mohseni Takallo, R. A. Borzooei, M. Aaly Kologani and Y. B. Jun, Commutative True-False ideals in BCI/BCK-algebras, *Algebraic structures and their applications*, **10**(2) (2023), 155–172.
19. M. Mohseni Takallo, R. A. Borzooei, G. R. Rezaei and Y. B. Jun, True-False ideals of BCK/BCI-algebras, *J. Discrete Math. Sci. Cryptogr.*, to appear.
20. M. Mohseni Takallo and Y. B. Jun, Commutative neutrosophic quadruple ideals of neutrosophic quadruple BCK-algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **1**(1) (2020), 95–105.
21. A. Rosenfeld, Fuzzy groups, *J. Math. Anal. App.*, **35** (1971), 512–517.
22. O. Xi, Fuzzy BCK-algebra, *Math. Japon.*, **36**(5) (1991), 935–942.
23. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.
24. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, *Inform. Sci.*, **8** (1975), 199–249.

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Journal of Algebraic Systems

TRUE-FALSE STRUCTURES AND ITS APPLICATION IN GROUPS AND BCK/BCI-ALGEBRAS

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ساختارهای درست-نادرست و کاربرد آن در گروه ها و BCK/BCI-جبرها

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با استفاده از مفهوم مجموعه‌های فازی با ارزش بازه‌ای، ساختار جدیدی به نام ساختارهای درست-نادرست معرفی می‌شود. کاربرد آن در گروه‌ها وBCK/BCI-جبرها مورد بحث قرار گرفته است. معرفی TF-زیرگروه‌های (محدود) و TF-زیرجبرها (محدود) همراه با بررسی خواص مختلف مرتبط انجام شده است. خصوصیات TF-زیرگروه‌ها (محدود) و TF-زیرجبرها (محدود) ارائه شده است، و نشان داده شده است که اشتراک دو TF-زیرگروه (TF-زیرجبر)، یک TF-زیر گروه (TF-زیرجبر) را تشکیل می‌دهد. علاوه بر این، اجتماع دو TF-زیرگروه (TF-زیرجبر) بررسی می‌شوند.

کلمات کلیدی: ساختار درست-نادرست، TF-زیرگروه‌های (محدود)، TF-زیرجبرهای (محدود).