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TOTAL NEAR-RING GRAPH

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Abstract. Let N be a right near-ring and $Z(N)$ be the set of right zero-divisors of N. We define total near-ring graph of N as a graph whose vertex set is the set of all elements of the near-ring N and any two distinct vertices $n_1, n_2 \in N$ are adjacent if and only if $n_1 + n_2$ or $n_2 + n_1 \in Z(N)$. We denote total near-ring graph of N by T_N. In this paper we try to give an overview of the structure of T_N depending upon whether the set of all right zero-divisors $Z(N)$ is an ideal of N or not. We also find the girth and diameter of T_N and its two subgraphs $T_{Z(N)}$ and $T_{Reg(N)}$ for the case when $Z(N)$ is an ideal and not an ideal of the near-ring N.

1. Introduction

Near-rings are one of the generalised structures of rings. It is actually used to describe the formation of the endomorphisms of various mathematical structures adequately. Let $M(G)$ be the set of all maps of an additive (not necessarily abelian) group G into itself. The concept of near-ring arises when we define addition and multiplication on the set $M(G)$ as $(f+g)(a) = f(a)+g(a)$ and $(fg)(a) = f(g(a))$, for all $a \in G$ and $f, g \in M(G)$. An algebraic system $(N, +, .)$ is called a near-ring if $(N, +)$ is a group (not necessarily abelian), $(N, .)$ is a semigroup and $(a + b).c = a.c + b.c$ for all $a, b, c \in N$. This near-ring is termed as right near-ring. A normal subgroup I of $(N, +)$ is called a right ideal of N if $IN \subseteq I$. The set of right zero-divisors Z_r of a near-ring N is defined as

$$Z(N) = \{x \in N : nx = 0 \text{ for some } 0 \neq n \in N\}.$$

Throughout this paper by a near-ring we will mean a right near-ring unless otherwise stated.

The idea of relating a graph with ring was first introduced by Istvan Beck in 1988 [3]. He introduced the zero-divisor graph of a commutative ring as a simple graph whose vertex set consists of all the elements of a commutative ring and two distinct vertices are adjacent if and only if product of them is zero. He also gave the idea of colouring of a commutative ring with the

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help of this graph. In 1999, Anderson and Livingston [2] again defined zero-divisor graph by slightly modifying Beck's definition. They defined zero-divisor graph $\Gamma(R)$ of a ring R as the graph with $Z(R)^*$, the set of nonzero zero-divisors of R as vertices and any two distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$.

The concept of total graph of a commutative ring R, denoted by $T(\Gamma(R))$ was first introduced by D. F. Anderson and A. Badawi [1]. The total graph was defined as the graph with all elements of the ring R as vertices where two distinct $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$ where $Z(R)$ is the set of all zero-divisors of R.

In this paper, we introduce the Total near-ring graph. We define Total near-ring graph denoted by T_N as the graph with all elements of the near-ring N as vertices and any two vertices n_1, n_2 are adjacent if and only if n_1+n_2 or $n_2 + n_1 \in Z(N)$. Let $T_{Z(N)}$ and $T_{Reg(N)}$ be the induced subgraphs of T_N whose vertex sets are $Z(N)$ and $Reg(N)$ be the set of all non zero-divisors of N.

Let G be a graph. The graph G is said to be connected if there is a path between any two distinct vertices of G. On the other side, the graph G is called totally disconnected if no two vertices of G are adjacent. For vertices x and y of G, the distance between x and y denoted by $d(x; y)$ is defined as the length of the shortest path from x to y; $d(x; y) = \infty$, if there is no such path. The diameter of G is

$$diam(G) = sup\{d(x; y) \colon\ x, y \text{ are vertices of } G\}.$$

The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G; $gr(G) = \infty$ if G contains no cycle. A dominating set for a graph G is a subset D of the vertex set such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

For usual graph-theoretic terms and definitions, one can look at [6]. General references for the algebraic part of this paper are [4, 5, 7, 8] and [9].

2. When $Z(N)$ is an ideal of N

In this section, we consider the case when $Z(N)$ is an ideal of N. We wish to find the diameter and girth of the two subgraphs $T_{Z(N)}$ and $T_{Reg(N)}$. The first result in this section is independent of whether $Z(N)$ is an ideal of N or not. Throughout this paper we use the notations $\mid Z(N) \mid= \alpha$ and $\mid N/Z(N) \mid= \beta$. We allow α, β to be infinite cardinals.

Theorem 2.1. *Let N be a near-ring. The graph $T_{Z(N)}$ is always connected and $diam(T_{Z(N)}) = \{\infty, 1, 2\}$.*

Proof. If $Z(N) = \{0\}$, then clearly $diam(T_{Z(N)}) = \infty$. Let n_1, $n_2 \in Z(N)$ be any two distinct vertices of $T_{Z(N)}$. If $n_1 + n_2$ or $n_2 + n_1 \in Z(N)$, then we are done i.e., $diam(T_{Z(N)}) = 1$. Let n_1, n_2 be not adjacent. Then there exists a path of length 2 as $n_1 - 0 - n_2$ in the graph $T_{Z(N)}$. Thus $diam(T_{Z(N)}) = 2$ and hence $T_{Z(N)}$ is always connected and $diam(T_{Z(N)}) = \{\infty, 1, 2\}$. □

Remark 2.2. If N is commutative, then $Z(\Gamma(N))$ defined in [1] is isomorphic to $T_{Z(N)}$.

The following theorem is an analog of Theorem 2.1[1].

Theorem 2.3. *Let N be a near-ring and $Z(N)$ be an ideal of N. Then $T_{Z(N)}$ is a complete subgraph of T_N and $T_{Z(N)}$ is always disjoint from $T_{Reg(N)}$.*

Proof. Let $0 \in Z(N)$ and for any $0 \neq n \in Z(N)$, we have

$$0 + n = n + 0 = n \in Z(N).$$

Thus 0 is adjacent to every vertex of $T_{Z(N)}$. Next let n_1, n_2 be any two non-zero distinct vertices of $T_{Z(N)}$.Then clearly $n_1 + n_2$ or $n_2 + n_1$ is a zero-divisor of N. Hence all the vertices of $T_{Z(N)}$ are adjacent to each other and thus form a complete subgraph $K_{|Z(N)|}$. The next part of the theorem is clear from the definitions of $T_{Z(N)}$ and $T_{Reg(N)}$. □

Theorem 2.4. *Let N be a near-ring and $(N, +)$ be an abelian group. If $Z(N)$ is an ideal of N, then the graph $T_{Reg(N)}$ is disjoint union of complete graphs and complete bipartite graphs.*

Proof. First assume that $n \in Reg(N)$ such that n be the additive inverse of its own. Then $n + n = 0$. Therefore, each coset $n + Z(N)$ forms a complete subgraph K_α of $T_{Reg(N)}$. Again, let us suppose that $n_1 \in Reg(N)$ such that n_1 has an additive inverse other than itself. Then there exists an $n_2 \in Reg(N), n_1 \neq n_2$ such that $n_1 + n_2 = n_2 + n_1 = 0$. Then no elements of the coset $n_1 + Z(N)$ is adjacent to each other. Since

$$(n_1 + z_1) + (n_1 + z_2) = (n_1 + n_1) + (z_1 + z_2) \in Z(N)$$

if and only if $n_1 + n_1 = 0$, which is not possible. Again the cosets $n_1 + Z(N)$ and $n_2 + Z(N)$ are disjoint and each element of $n_1 + Z(N)$ is adjacent to each element of $n_2 + Z(N)$ and thus form a complete bipartite graph $K_{\alpha,\alpha}$.Thus the graph $T_{Reg(N)}$ is the disjoint union of complete graphs and complete bipartite graphs.

Let there be l elements of N which are additive inverse of its own. Thus we can write $T_{Reg(N)} = lK_\alpha \cup \frac{|N| - \alpha - l}{2} K_{\alpha,\alpha}$. □

Remark 2.5. Theorem 2.4 is an analog of Theorem 2.2[1]. But we can observe the difference between the results. In case of Total near-ring graph $T_{Reg}(N)$ can be a disjoint union of K_α's and $K_{\alpha,\alpha}$'s whereas $Reg(\Gamma(R))$ is disjoint union of either complete graphs K_α's or $K_{\alpha,\alpha}$'s.

Example 2.6. Consider the set $\mathbb{Z}_{12} = \{0, 1, 2, ..., 11\}$. Let $(\mathbb{Z}_{12}, +)$ be a group under $'+'$ modulo 12. Define $'.'$ on $\mathbb{Z}_{12}$ by $a.b = a$ for all $a \in \mathbb{Z}_{12}$. Clearly $(\mathbb{Z}_{12}, +, .)$ is a near-ring. Here $Z(N) = \{0\}$. The subgraph $T_{Reg(N)}$ is disjoint union of K_1 and $5K_{1,1}$'s which can be observed from the following figure.

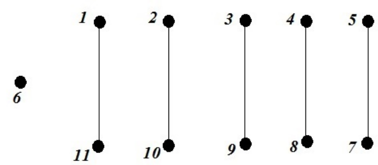

Fig 1: The subgraph $T_{Reg(\mathbb{Z}_{12})}$

The following result is a direct consequence of Theorem 2.3 and Theorem 2.4.

Corollary 2.7. *Let N be a near ring and $Z(N)$ be an ideal of N. Then the following results hold:*

(i) *The graph T_N is planar only if $|Z(N)| \leqslant 2$.*
(ii) *The subgraphs $T_{Z(N)}$ and $T_{Reg(N)}$ are both planar for $|Z(N)| \leqslant 4$ and $|Reg(N)| \leqslant 2$ respectively.*

Theorem 2.8. *If N is a near-ring and $(N, +)$ is an abelian group. If $Z(N)$ is an ideal of N such that $|Z(N)| = \alpha$, then $\gamma(T_N) = 1 + |N| - \alpha$.*

Proof. Suppose that l elements of N are additive self-inverse. Then by Theorem 2.4 we have, $T_{Reg(N)} = lK_\alpha \cup \frac{|N| - \alpha - l}{2} K_{\alpha,\alpha}$ which gives

$$\begin{aligned}
\gamma(T_{Reg(N)}) &= l\gamma(K_\alpha) + \frac{|N| - \alpha - l}{2}\gamma(K_{\alpha,\alpha}) \\
&= l \times 1 + (\frac{|N| - \alpha - l}{2}) \times 2 \\
&= l + |N| - \alpha - l \\
&= |N| - \alpha.
\end{aligned}$$

Also by Theorem 2.3, $T_{Z(N)}$ is complete and is always disjoint from $T_{Reg(N)}$ which yields $\gamma(T_{Z(N)}) = 1$ and consequently

$$\gamma(T_N) = \gamma(T_{Z(N)} \cup T_{Reg(N)}) = \gamma(T_{Z(N)}) + \gamma(T_{Reg(N)}) = 1 + \mid N \mid - \alpha.$$

□

However, it is observed that if the near-ring N is not abelian under $'+'$ operation, then the above theorem is not applicable always. Let us consider an example to verify this.

Example 2.9. Consider the near-ring $N = \{0, a, b, c, x, y\}$ under the addition and multiplication defined by the following tables:

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	0	b	c	c	b
c	0	0	c	b	b	c
x	0	0	x	y	y	x
y	0	0	y	x	x	y

Here $Z(N) = \{0, a\}$.
The total near-ring graph T_N has been shown in the following figure.

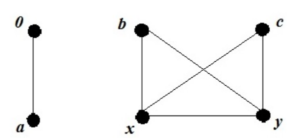

Fig 2: The total near-ring graph T_N

Remark 2.10. Let $Z(N)$ be an ideal of the near-ring N. Then from the above two examples and from the Theorem 2.2[1] it is clear that the graph $T_{Reg(N)}$ may not be isomorphic to the graph $Reg(\Gamma(N))$ [1].

Theorem 2.11. *Let N be an abelian near-ring such that $Z(N)$ be an ideal of N. Then,*

(i) $T_{Reg(N)}$ *is connected if and only if either* $N/Z(N) \cong \mathbb{Z}_2$ *or* $N/Z(N) \cong \mathbb{Z}_3$.

(ii) $T_{Reg(N)}$ *(and hence* $T_{Z(N)}$ *and* T_N*) is totally disconnected if and only if* $Z(N) = 0$ *and for any* $n \in N$, $n + n = 0$.

Proof. (i) It is clear from the Theorem 2.4 that $T_{Reg(N)}$ is connected if and only if the graph is either a single complete graph K_α or a complete bipartite graph $K_{\alpha,\alpha}$. Now in the first case to get a single K_α the set $Reg(N)$ must contains only one element n_1(say)which is self additive inverse. Therefore, $n_1+n_1 = 0$. This implies $N/Z(N) \cong \mathbb{Z}_2$. Similarly, for the second case a single $K_{\alpha,\alpha}$ implies that there are only two distinct elements say, $n_1, n_2 \in Reg(N)$ which are additive inverse of each other. Thus, $n_1 + n_2 = 0$. This gives $\beta = |N/Z(N)| = 3$ which implies $N \cong N/Z(N) \cong \mathbb{Z}_3$.

(ii) The first part is clear. Let us prove the converse part. If $Z(N) = 0$, then clearly $T_{Z(N)}$ is a singletone graph. Again since $n + n = 0$ for every $n(\neq 0) \in N$, so the subgraph $T_{Reg(N)}$ is totally disconnected graph or disjoint union of $\mid N \mid -1$ K_1's. This means that the graph T_N is also disconnected. □

Theorem 2.11 can be compared to Theorem 2.4[1].

Theorem 2.12. *Let* N *be an abelian near-ring and* $Z(N)$ *be an ideal of* N*. For* $\mid N \mid \geq 2$*, we have the following*

$$diam(T_{Reg(N)}) = \begin{cases} 0 & \text{if } N \cong \mathbb{Z}_2, \\ 1, & if \mid Z(N) \mid \geqslant 2 \text{ and } N/Z(N) \cong \mathbb{Z}_2 \text{ or } N \cong \mathbb{Z}_3, \\ 2, & \text{if } N/Z(N) \cong \mathbb{Z}_3 \text{ and } \mid Z(N) \mid \geqslant 2. \end{cases}$$

Proof. The proofs are clear from Theorem 2.11. □

Next, we will find the girth of the two subgraphs $T_{Reg(N)}$ and $T_{Z(N)}$.

Theorem 2.13. *Let* N *be a near-ring and* $Z(N)$ *be an ideal of* N*. Then we have the following:*

(i) $gr(T_{Reg(N)}) = 3, 4$ *or* ∞.
(ii) $gr(T_{Z(N)}) = 3$ *or* ∞.

Proof. (i) From Theorem 2.4, we have seen that the subgraph $T_{Reg(N)}$ is disjoint union of complete graphs K_α's and complete bipartite graphs $K_{\alpha,\alpha}$'s, so the result is clear.

(ii) If $Z(N)$ is an ideal of the near-ring N, then the vertices of the subgraph $T_{Z(N)}$ constitute a complete graph $K_{Z(N)}$ as for any two distinct vertices $n_1, n_2 \in Z(N), n_1 + n_2 \in Z(N)$. Thus the girth of the subgraph $T_{Z(N)}$ is 3. If $\mid Z(N) \mid \leq 2$, then clearly $gr(T_{Z(N)}) = \infty$. □

3. When $Z(N)$ is not an ideal of N

In this section, we consider the case when $Z(N)$ is not an ideal of N. Since the set of zero-divisors is always closed under multiplication, so there must exist distinct $n_1, n_2 \in Z(N)$ such that $n_1 + n_2$ or $n_2 + n_1 \notin Z(N)$. The following lemma will be used in some results of this section.

Lemma 3.1. *If $Z(N)$ is not an ideal of N, then $n + n = 0$, $\forall n \in N$ if and only if $2Z(N) = \{0\}$.*

Proof. If $n + n = 0, \forall n \in N$, then clearly $m + m = 0$, $\forall m \in Z(N)$. Next let us assume that $m + m = 0$, $\forall m \in Z(N)$. Since $Z(N)$ is not an ideal of N, so there exist distinct $n_1, n_2 \in Z(N)$ such that $n_1 + n_2 = n \in Reg(N)$. Now $2n = 2n_1 + 2n_2 = 0$ implies $2n = 0$ and thus $2 = 0$ as $n \in Reg(N)$. Hence $n + n = 0, \forall n \in N$. □

Theorem 3.2. *If $Z(N)$ is not an ideal of N, then we have the following:*

(i) *In the graph T_N, the two subgraphs $T_{Z(N)}$ and $T_{Reg(N)}$ are not disjoint.*
(ii) *If $T_{Reg(N)}$ is connected, then T_N is also connected.*

Proof. (i) Since $Z(N)$ is not an ideal of N, so there exist distinct $n_1, n_2 \in Z(N)$ such that $n_1 + n_2 \in Reg(N)$. Let $n_1 \in N$. Since $(N, +)$ is a group, then there exists $-n_1 \in N$ such that $-n_1$ and $n_1 + n_2$ are adjacent in T_N as $-n_1 + (n_1 + n_2) = n_2 \in Z(N)$. Thus the two subgraphs $T_{Z(N)}$ and $T_{Reg(N)}$ are not disjoint in T_N.

(ii) From Theorem 2.1, it is clear that the subgraph $T_{Z(N)}$ is always connected. Let $T_{Reg(N)}$ be connected. Then from the above result (a), we have seen that there exists a path between $T_{Z(N)}$ and $T_{Reg(N)}$. This proves the result. □

Theorem 3.3. *If $Z(N)$ is not an ideal of N, then the following results hold:*

(i) $gr(T_{Z(N)}) = 3$ *or* ∞.
(ii) $gr(T_N) = 3$ *if and only if* $gr(T_{Z(N)}) = 3$.

Proof. (i) $0 \in Z(N)$ and 0 is adjacent to every element of $Z(N)$. Since $Z(N)$ is not an ideal N, so if for every distinct pair of elements $n_1, n_2 \in Z(N)$, we have $n_1 + n_2 \in Reg(N)$, then the subgraph $T_{Z(N)}$ is a star graph with centre 0 containing no cycle. Again if for some distinct pair of elements $n_1, n_2 \in Z(N)$, we have $n_1 + n_2 \in Z(N)$, then we can construct a 3-cycle $0 - n_1 - n_2 - 0$ in the graph $T_{Z(N)}$. Thus $gr(T_{Z(N)}) = 3$ or ∞.

(ii) It is clear that if $gr(T_{Z(N)}) = 3$, then $gr(T_N) = 3$. We need only to show that $gr(T_{Z(N)}) = 3$ when $gr(T_N) = 3$. If $n + n \neq 0$ for some $n \in Z(N)$

then, we can construct a 3-cycle $0-n-(-n)-0$ in $T_{Z(N)}$. Next, let $n+n=0$ $\forall n \in Z(N)$. Then from Lemma 3.1, we get $n+n=0$ $\forall n \in N$. Let there be a 3-cycle in T_N as $n_1-n_2-n_3-n_1$. Let $l=n_1+n_2$, $m=n_2+n_3$, $p=n_3+n_1$. Now, $l+m=n_1+(n_2+n_2)+n_3=n_1+n_3$ (or n_3+n_1) $\in Z(N)$. Thus we get a 3-cycle in $T_{Z(N)}$ as $0-l-m-0$. Thus the result follows. □

Example 3.4. Consider the near-ring $N=\{0,a,b,c\}$ under the addition and multiplication defined by the following tables [3]:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Here $Z(N)=\{0,a,b\}$, which is not an ideal of N. From the figure of the subgraph $T_{Z(N)}$ shown below it is clear that $gr(T_{Z(N)})=\infty$.

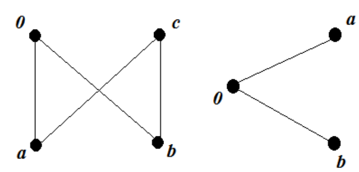

Fig 3: The graph T_N and its subgraph $T_{Z(N)}$

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TOTAL NEAR–RING GRAPH

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گراف شبه–حلقه کامل

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فرض کنید N یک شبه–حلقه راست و $Z(N)$ مجموعه‌ی تمام مقسوم علیه‌های صفر راست آن باشد. گراف شبه–حلقه کامل را به صورت گرافی تعریف می‌کنیم که مجموعه‌ی رئوس آن برابر با N می‌باشد و دو رأس متمایز $n_1, n_2 \in N$ با هم مجاورند اگر و تنها اگر $n_1 + n_2 \in Z(N)$ یا $n_2 + n_1 \in Z(N)$. به علاوه، گراف شبه–حلقه کامل N را با نماد T_N نمایش می‌دهیم. در این مقاله، نمایی کلی از ساختار T_N ارائه می‌دهیم که وابسته به این است که مجموعه‌ی همه‌ی مقسوم علیه‌های صفر راست $Z(N)$ ایده‌آلی از N تشکیل می‌دهد یا نه. همچنین، قطر و کمر گراف T_N و دو زیرگراف $T_{Z(N)}$ و $T_{Reg(N)}$ برای حالت‌هایی که $Z(N)$ ایده‌آلی از N است یا ایده‌آلی از آن نمی‌باشد، را بدست می‌آوریم.

کلمات کلیدی: شبه–حلقه، مقسوم علیه‌های صفر راست، ایده‌آل راست، گراف همبند، قطر، کمر.