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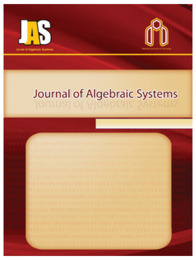

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GENUS OF COMMUTING GRAPHS OF CERTAIN FINITE GROUPS

P. BHOWAL AND R. K. NATH*

Abstract. The commuting graph of a finite group G is a graph whose vertex set is the set of non-central elements of G and two distinct vertices are adjacent if they commute. In this article, we compute genus of commuting graphs of certain classes of finite non-abelian groups and characterize those groups such that their commuting graphs have genus $4, 5$ and 6.

1. Introduction

Let G be any finite non-abelian group with center $Z(G)$. The commuting graph of G, denoted by $\mathcal{C}(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices g and h are adjacent if $gh = hg$. The origin of this graph lies in a work of Brauer and Fowler [6]. A lot of work has been done on commuting graphs of finite groups over the years.

In 2015, Afkhami, Farrokhi and Khashyarmanesh [1] and in 2016, Das and Nongsiang [7] have characterized finite non-abelian groups such that their commuting graphs are planar or toroidal. Recently, Nongsiang [19] has characterized groups G such that $\mathcal{C}(G)$ is double-toroidal or triple-toroidal. It is worth recalling that "the genus of a graph is the smallest non-negative integer k such that the graph can be embedded on the surface obtained by attaching k handles to a sphere". If $\gamma(\Gamma)$ denotes the genus of a graph Γ, having a subgraph Γ_0, then it can be easily visualized that

$$\gamma(\Gamma) \geq \gamma(\Gamma_0). \tag{1.1}$$

Also, [22, Theorem 6-38] yields

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{1.2}$$

"A graph is called planar, toroidal, double-toroidal and triple-toroidal if its genus is $0, 1, 2$ and 3 respectively".

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In this article, we compute $\gamma(\mathcal{C}(G))$ for the classes of finite groups such that their central quotient, $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$ (where q is a prime throughout this article), $D_{2m} = \langle f, g : f^m = g^2 = 1, gfg^{-1} = f^{-1}\rangle$ (where $m \geq 2$) or $Sz(2) = \langle f, g : f^5 = g^4 = 1, g^{-1}fg = f^2\rangle$. Also, we find conditions such that $\gamma(\mathcal{C}(G)) = 4, 5$ or 6 for the above mentioned groups. Consequently, we characterize groups of order q^3, the meta-abelian groups

$$M_{2mn} = \langle f, g : f^m = g^{2n} = 1, gfg^{-1} = f^{-1}\rangle,$$

D_{2m}, $Q_{4m} = \langle f, g : f^{2m} = 1, g^2 = f^m, gfg^{-1} = f^{-1}\rangle$ and

$$U_{6n} = \langle f, g : f^{2n} = g^3 = 1, f^{-1}gf = g^{-1}\rangle$$

such that their commuting graphs have genus $4, 5$ or 6. Spectral aspects of $\mathcal{C}(G)$ for these classes of groups have been described in [8, 9, 11, 10, 12, 14, 18, 21].

2. Main results

To prove the subsequent theorems, the following lemma is useful.

Lemma 2.1. [3, Corollary 2] *If Γ is the disjoint union of K_m and K_n, then $\gamma(\Gamma) = \gamma(K_m) + \gamma(K_n)$.*

Theorem 2.2. *If G $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$, then $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = (q+1)\left\lceil \tfrac{1}{12}((q-1)n-3)((q-1)n-4)\right\rceil$$

according as $(q-1)n \leq 2$ or $(q-1)n \geq 3$, where $n = |Z(G)|$.

Proof. Note that [9, Theorem 2.1] yields $\mathcal{C}(G) = (q+1)K_{(q-1)n}$. Therefore, $\gamma(\mathcal{C}(G)) = 0$ when $n(q-1) \leq 2$. If $(q-1)n \geq 3$, then by (1.2) and Lemma 2.1, we get

$$\gamma(\mathcal{C}(G)) = (q+1)\gamma(K_{n(q-1)}) = (q+1)\left\lceil \tfrac{1}{12}((q-1)n-3)(n(q-1)-4)\right\rceil.$$

□

Corollary 2.3. *If $|G| = q^3$, then $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = (q+1)\left\lceil \tfrac{1}{12}((q-1)q-3)((q-1)q-4)\right\rceil$$

according as $q = 2$ or $q \geq 3$.

Proof. Evidently $|Z(G)| = q$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Therefore, $q(q-1) = 2$ or $q(q-1) \geq 6$ according as $q = 2$ or $q \geq 3$. Hence, Theorem 2.2 leads to the conclusion. □

Corollary 2.4. *For any 4-centralizer finite group G, $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = 3\left\lceil \tfrac{1}{12}(n-3)(n-4)\right\rceil$$

according as $n \leq 2$ or $n \geq 3$, where $n = |Z(G)|$.

Proof. Theorem 2 [4] yields $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, Theorem 2.2 leads to the conclusion. □

Corollary 2.5. *If G is any $(q+2)$-centralizer and $|G| = q^m$ (where $m \in \mathbb{N}$), then $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = (q+1)\left\lceil \tfrac{1}{12}((q-1)n-3)((q-1)n-4)\right\rceil$$

according as $(q-1)n \leq 2$ or $(q-1)n \geq 3$, where $n = |Z(G)|$.

Proof. Note that [2, Lemma 2.7] yields $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Hence, Theorem 2.2 leads to the conclusion. □

Corollary 2.6. *For any finite 5-centralizer group G, $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = 4\left\lceil \tfrac{1}{12}(2n-4)(2n-3)\right\rceil$$

according as $n = 1$ or $n \geq 2$, where $n = |Z(G)|$.

Proof. Note that [4, Theorem 4] yields $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, Theorem 2.2 leads to the conclusion. □

Corollary 2.7. *If q is the smallest prime divisor of the order of G and $\Pr(G) = \frac{q^2+q-1}{q^3}$, then $\gamma(\mathcal{C}(G)) = 0$ or*

$$\gamma(\mathcal{C}(G)) = (q+1)\left\lceil \tfrac{1}{12}((q-1)n-3)((q-1)n-4)\right\rceil$$

according as $(q-1)n \leq 2$ or $(q-1)n \geq 3$, where $n = |Z(G)|$.

Proof. Note that [15, Theorem 3] yields $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$. Hence, Theorem 2.2 leads to the conclusion. □

Theorem 2.8. *If $\frac{G}{Z(G)} \cong D_{2m}$ $(m \geq 2)$, then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1, m = 2, 3 \text{ and } m = n = 2 \\ \left\lceil \frac{1}{12}((m-1)n-3)((m-1)n-4)\right\rceil, & \\ & \text{when } n = 1, m \geq 4 \text{ and } n = 2, m \geq 3 \\ \left\lceil \frac{1}{12}((m-1)n-3)((m-1)n-4)\right\rceil + m\left\lceil \frac{1}{12}(n-3)(n-4)\right\rceil, & \\ & \text{when } n \geq 3, m \geq 2, \end{cases}$$

where $n = |Z(G)|$.

Proof. Note that [9, Theorem 2.5] yields $\mathcal{C}(G) = K_{(m-1)n} \sqcup mK_n$. Therefore,

$$\mathcal{C}(G) = \begin{cases} K_1 \sqcup 2K_1, & \text{when } n = 1 \text{ and } m = 2 \\ K_2 \sqcup 3K_1, & \text{when } n = 1 \text{ and } m = 3 \\ K_2 \sqcup 2K_2, & \text{when } n = m = 2 \end{cases}$$

and so $\gamma(\mathcal{C}(G)) = 0$ in these cases. We also have

$$\mathcal{C}(G) = \begin{cases} K_{m-1} \sqcup mK_1, & \text{when } n = 1 \text{ and } m \geq 4 \\ K_{2(m-1)} \sqcup mK_2, & \text{when } n = 2 \text{ and } m \geq 3. \end{cases}$$

In these cases, $(m-1)n \geq 3$ and so (1.2) and Lemma 2.1 yields

$$\gamma(\mathcal{C}(G)) = \lceil \tfrac{1}{12}((m-1)n-3)((m-1)n-4) \rceil.$$

If $n \geq 3$ and $m \geq 2$, then $(m-1)n \geq 3$. Therefore, by (1.2) and Lemma 2.1 we get the required expression for $\gamma(\mathcal{C}(G))$. □

Corollary 2.9. *Let $G = M_{2mn}$, where $m > 2$ and $n \geq 1$. If $2 \mid m$, then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1, m = 3 \\ \left\lceil \frac{1}{12}((m-1)n-3)((m-1)n-4) \right\rceil, & \\ & \text{when } n = 1, m \geq 5 \text{ or } n = 2, m \geq 3 \\ \left\lceil \frac{1}{12}((m-1)n-3)((m-1)n-4) \right\rceil + m \left\lceil \frac{1}{12}(n-4)(n-3) \right\rceil, & \\ & \text{when } n \geq 3, m \geq 3. \end{cases}$$

If $2 \mid m$, then

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1, m = 4 \\ \left\lceil \frac{1}{12}((m-2)n-3)((m-2)n-4) \right\rceil, & \\ & \text{when } n = 1, m \geq 6 \\ \left\lceil \frac{1}{12}((m-2)n-3)((m-2)n-4) \right\rceil + \frac{m}{2} \left\lceil \frac{1}{12}(2n-3)(2n-4) \right\rceil, & \\ & \text{when } n \geq 2, m \geq 4. \end{cases}$$

Proof. Note that [9, Proposition 2.8] yields $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m depending on m odd or even respectively also $|Z(M_{2mn})| = n$ or $2n$ for m odd or even respectively. Hence, Theorem 2.8 leads to the conclusion. □

Corollary 2.10. *If $G = D_{2m}$ $(m \geq 3)$, then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } m = 3, 4 \\ \left\lceil \frac{1}{12}(m-4)(m-5) \right\rceil & \text{when } 2 \nmid m \text{ and } m \geq 5 \\ \left\lceil \frac{1}{12}(m-5)(m-6) \right\rceil & \text{when } 2 \mid m \text{ and } m \geq 6. \end{cases}$$

Proof. Since $M_{2mn} = D_{2m}$ for $n = 1$, Corollary 2.9 leads to the conclusion. $\square$

Corollary 2.11. *Let $G = Q_{4m}$ $(m \geq 2)$. Then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } m = 2 \\ \left\lceil \frac{1}{12}(2m-5)(2m-6) \right\rceil, & \text{when } m \geq 3. \end{cases}$$

Proof. We have $|Z(Q_{4m})| = 2$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. Hence, Theorem 2.8 leads to the conclusion. $\square$

Corollary 2.12. *For $G = U_{6n}$,*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1, 2 \\ 3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil + \left\lceil \frac{1}{12}(2n-3)(2n-4) \right\rceil, & \text{when } n \geq 3. \end{cases}$$

Proof. We have $Z(U_{6n}) = \langle a^2 \rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$. Hence, Theorem 2.8 leads to the conclusion. $\square$

Corollary 2.13. *If $\Pr(G) \in \left\{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}, \frac{7}{16}\right\}$, then*

$$\begin{aligned}\gamma(\mathcal{C}(G)) \in \Big\{0, 1, 2, 6, &\lceil\frac{1}{2}(2n-1)(3n-2)\rceil + 7\lceil\frac{1}{12}(n-3)(n-4)\rceil, \\ &\lceil\frac{1}{3}(n-1)(4n-3)\rceil + 5\lceil\frac{1}{12}(n-3)(n-4)\rceil, \\ &\lceil\frac{1}{4}(n-1)(3n-4)\rceil + 4\lceil\frac{1}{12}(n-3)(n-4)\rceil, \\ &\lceil\frac{1}{6}(n-2)(2n-3)\rceil + 3\lceil\frac{1}{12}(n-3)(n-4)\rceil, \\ &3\lceil\frac{1}{12}(n-3)(n-4)\rceil, 4\lceil\frac{1}{6}(n-2)(2n-3)\rceil\Big\},\end{aligned}$$

where $\Pr(G)$ is the commuting probability of G and $n = |Z(G)| \geq 3$.

Proof. If $\Pr(G) \in \left\{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}, \frac{7}{16}\right\}$, then [20, pp. 246] and [17, pp. 451] yields $\frac{G}{Z(G)} \cong D_6, D_8, D_{10}, D_{14}, \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

If $\frac{G}{Z(G)} \cong D_6$, then considering $m = 3$ in Theorem 2.8, we get $\gamma(\mathcal{C}(G)) = 0$ whenever $n = 1$. For $n = 2$, $\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(n-2)(2n-3) \right\rceil = 0$. For $n \geq 3$ we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(n-2)(2n-3) \right\rceil + 3\left\lceil \frac{1}{12}(n-4)(n-3) \right\rceil.$$

If $\frac{G}{Z(G)} \cong D_8$, then considering $m = 4$ in Theorem 2.8, we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(3n-3)(3n-4) \right\rceil,$$

if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 0$ or 1 according as $n = 1$ or 2. For $n \geq 3$,
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{4}(n-1)(3n-4) \right\rceil + 4\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$
If $\frac{G}{Z(G)} \cong D_{10}$, then considering $m = 5$ in Theorem 2.8, we get
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(4n-3)(4n-4) \right\rceil,$$
if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 0$ or 2 according as $n = 1$ or 2. For $n \geq 3$ we get
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{3}(4n-3)(n-1) \right\rceil + 5\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$
If $\frac{G}{Z(G)} \cong D_{14}$, then considering $m = 7$ in Theorem 2.8, we get
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(6n-3)(6n-4) \right\rceil,$$
if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 1$ or 6 according as $n = 1$ or 2. For $n \geq 3$ we get
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{2}(2n-1)(3n-2) \right\rceil + 7\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$
If $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then considering $q = 2$ in Theorem 2.2 we get $\gamma(\mathcal{C}(G)) = 0$ if $n = 1, 2$. For $n \geq 3$ we get $\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-4)(n-3) \right\rceil$.

If $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then considering $q = 3$ in Theorem 2.2 we get $\gamma(\mathcal{C}(G)) = 0$ if $n = 1$ or 2. If $n \geq 3$, then we get $\gamma(\mathcal{C}(G)) = 4\left\lceil \frac{1}{6}(n-2)(2n-3) \right\rceil$. □

Theorem 2.14. *If $\frac{G}{Z(G)} \cong Sz(2)$, then*
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{3}(n-1)(4n-3) \right\rceil + 5\left\lceil \tfrac{1}{4}(n-1)(3n-4) \right\rceil,$$
where $n = |Z(G)|$.

Proof. Note that [11, Theorem 2.2] yields $\mathcal{C}(G) = K_{4n} \sqcup 5K_{3n}$. Therefore by (1.2) and Lemma 2.1,
$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \gamma(K_{4n}) + 5\gamma(K_{3n}) \\ &= \left\lceil \frac{1}{12}(4n-3)(4n-4) \right\rceil + 5\left\lceil \frac{1}{12}(3n-3)(3n-4) \right\rceil \\ &= \left\lceil \frac{1}{3}(n-1)(4n-3) \right\rceil + 5\left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil.\end{aligned}$$
□

Corollary 2.15. *If $G = Sz(2)$, then $\gamma(\mathcal{C}(G)) = 0$.*

Proof. We have $|Z(Sz(2))| = 1$ and so Theorem 2.14 leads to the conclusion. □

Theorem 2.16. *If $G = V_{8n} = \langle f, g : f^{2n} = g^4 = 1, g^{-1}fg^{-1} = gfg = f^{-1}\rangle$, then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1, 2 \\ \left\lceil \frac{1}{6}(2n-3)(4n-5) \right\rceil, & \text{when } n \geq 3 \text{ and } 2 \nmid n \\ \left\lceil \frac{1}{3}(n-2)(4n-7) \right\rceil, & \text{when } n \geq 4 \text{ and } 2 \mid n. \end{cases}$$

Proof. In case $2 \nmid n$, [16, Example 2.4] yields $\mathcal{C}(G) = K_{2(2n-1)} \sqcup 2nK_2$. For $n = 1$, $2(2n-1) = 2$ and so $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 3$, then (1.2) and Lemma 2.1 yields

$$\gamma(\mathcal{C}(G)) = \gamma(K_{2(2n-1)}) + 2n\gamma(K_2) = \left\lceil \tfrac{1}{6}(2n-3)(4n-5) \right\rceil.$$

If n is even, then [16, Example 2.4] yields $\mathcal{C}(G) = K_{4(n-1)} \sqcup nK_4$. If $n = 2$, then $4(n-1) = 4$ and so $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 4$, then (1.2) and Lemma 2.1 yields $\gamma(\mathcal{C}(G)) = \gamma(K_{4(n-1)}) + n\gamma(K_4) = \left\lceil \frac{1}{3}(n-2)(4n-7) \right\rceil$. □

Theorem 2.17. *If $G = QD_{2^n} = \langle f, g : f^{2^{n-1}} = g^2 = 1, gfg^{-1} = f^{2^{n-2}-1}\rangle$, where $n \geq 4$, then $\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \right\rceil$.*

Proof. Note that [7, Proposition 4.3] yields $\mathcal{C}(G) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$. Therefore by (1.2) and Lemma 2.1,

$$\gamma(\mathcal{C}(G)) = \gamma(K_{2^{n-1}-2}) + 2^{n-2}\gamma(K_2) = \left\lceil \tfrac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \right\rceil.$$

□

Theorem 2.18. *If $G = SD_{8n}$, then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{when } n = 1 \\ \left\lceil \frac{1}{3}(n-2)(4n-7) \right\rceil, & \text{when } n \geq 3 \text{ and } 2 \nmid n \\ \left\lceil \frac{1}{6}(2n-3)(4n-5) \right\rceil, & \text{when } n \geq 2 \text{ and } 2 \mid n. \end{cases}$$

Proof. If n is odd, then [Line 11, Proof of Theorem 4.2(a)] of [13] yields $\mathcal{C}(G) = K_{4(n-1)} \sqcup nK_4$. If $n = 1$, then $4(n-1) = 0$. Therefore $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 3$, then by (1.2) and Lemma 2.1,

$$\gamma(\mathcal{C}(G)) = \gamma(K_{4(n-1)}) + n\gamma(K_4) = \left\lceil \tfrac{1}{3}(n-2)(4n-7) \right\rceil.$$

If n is even, then [Line 11, Proof of Theorem 4.2(b)] of [13] yields $\mathcal{C}(G) = K_{2(2n-1)} \sqcup 2nK_2$. Therefore by (1.2) and Lemma 2.1,

$$\gamma(\mathcal{C}(G)) = \gamma(K_{2(2n-1)}) + 2n\gamma(K_2) = \left\lceil \tfrac{1}{6}(2n-3)(4n-5) \right\rceil.$$

□

3. Some consequences

Nongsiang and Das [7] characterized the groups whose commuting graphs are planar and toroidal. Nongsiang [19] characterized the groups whose commuting graphs are double-toroidal and triple-toroidal. For a given class of groups we also find the necessary and sufficient condition for the genus of the graphs to be $4, 5$ and 6.

Theorem 3.1. *If G is a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$, then*

(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow q = 3, |Z(G)| = 3$.
(b) $\gamma(\mathcal{C}(G)) \neq 5$.
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow q = 2, |Z(G)| = 8$.
(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $q = 2, |Z(G)| \geq 9; q = 3, |Z(G)| \geq 4;$ $q \geq 5, |Z(G)| \geq 1$.

Proof. Theorem 2.2 yields $\gamma(\mathcal{C}(G)) = (q+1)\left\lceil \frac{1}{12}((q-1)n-3)((q-1)n-4) \right\rceil$, where $n = |Z(G)|$. For $q = 2$, $\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$. Therefore, we get $\gamma(\mathcal{C}(G)) \leq 3$ and $\gamma(\mathcal{C}(G)) = 6$ according as $n \leq 7$ and $n = 8$. If $n \geq 9$, then

$$\tfrac{1}{12}(n-3)(n-4) = \tfrac{1}{12}(n(n-9) + 2n + 12) > 2.$$

Hence, $3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil > 6$. For $q = 3$,

$$\gamma(\mathcal{C}(G)) = 4\left\lceil \tfrac{1}{12}(2n-3)(2n-4) \right\rceil = 4\left\lceil \tfrac{1}{6}(n-2)(2n-3) \right\rceil.$$

Evidently, $\gamma(\mathcal{C}(G)) \leq 3$ whenever $n \leq 2$. For $n = 3$, we have $\gamma(\mathcal{C}(G)) = 4$. If $n \geq 4$, then

$$\tfrac{1}{6}(n-2)(2n-3) = \tfrac{1}{6}(2n(n-4) + n + 6) > 1.$$

Hence, $4\left\lceil \frac{1}{6}(2n-3)(n-2) \right\rceil \geq 8$.

If $q = 5$, then $\mathcal{C}(G) = 6K_{4n}$ and so $\gamma(\mathcal{C}(G)) = 0$ when $n = 1$. If $n \geq 2$, then $6K_{4n}$ has a subgraph $6K_8$. Since $\gamma(6K_8) \geq 7$, therefore by (1.1), $\gamma(\mathcal{C}(G)) \geq 7$.

If $q \geq 7$, then $\mathcal{C}(G)) = (p+1)K_{6n}$ which has a subgraph $8K_6$ for $n \geq 1$. Since $\gamma(8K_6) \geq 7$, therefore by (1.1), $\gamma(\mathcal{C}(G)) \geq 7$. $\square$

Corollary 3.2. *If $|G| = q^3$, then*

(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow q = 3$.
(b) $\gamma(\mathcal{C}(G)) \geq 7$ *whenever* $q \geq 5$.

Corollary 3.2 can be proved by taking the fact that "if G is a non-abelian group of order q^3, then $|Z(G)| = q$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$". Hence, Theorem 3.1 leads to the conclusion.

Theorem 3.3. *If $\frac{G}{Z(G)} \cong D_{2m}$, where $m \geq 2$, then*

(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow m = 6, |Z(G)| = 2; m = 11, |Z(G)| = 1.$
(b) $\gamma(\mathcal{C}(G)) \neq 5.$
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow m = 2, |Z(G)| = 8; m = 4, |Z(G)| = 4; m = 5, |Z(G)| = 3; m = 7, |Z(G)| = 2; m = 13, |Z(G)| = 1.$
(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $m = 2, |Z(G)| \geq 9; m = 3, |Z(G)| \geq 5; m = 4, |Z(G)| \geq 5; m = 5, |Z(G)| \geq 4; m = 6, |Z(G)| \geq 3; m = 7, |Z(G)| \geq 3; m = 8, |Z(G)| \geq 2; m = 9, |Z(G)| \geq 2; m = 10, |Z(G)| \geq 2; m = 11, |Z(G)| \geq 2; m = 12, |Z(G)| \geq 2; m = 13, |Z(G)| \geq 2; m \geq 14, |Z(G)| \geq 1.$

Proof. Note that Theorem 2.8 yields

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}((m-1)n-3)((m-1)n-4) \right\rceil + m \left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

We look at the following scenarios.
Case 1. If $m = 2$, then we have

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil + 2 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil \\ &= 3 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.\end{aligned}$$

Clearly, $\gamma(\mathcal{C}(G)) \leq 3$ whenever $n \leq 7$; and $\gamma(\mathcal{C}(G)) = 6$ when $n = 8$. If $n \geq 9$, then

$$\tfrac{1}{12}(n-3)(n-4) = \tfrac{1}{12}(n^2 - 7n + 12) = \tfrac{1}{12}(n(n-9) + 2n + 12) > 2.$$

Hence, $3 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil > 6$.
Case 2. If $m = 3$, then we have

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(2n-4)(2n-3) \right\rceil + 3 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil \\ &= \left\lceil \frac{1}{6}(n-2)(2n-3) \right\rceil + 3 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.\end{aligned}$$

For $n \leq 4$, it is apparent that $\gamma(\mathcal{C}(G)) \leq 3$. For $n \geq 5$,

$$\tfrac{1}{6}(n-2)(2n-3) = \tfrac{1}{6}(2n^2 - 7n + 6) = \tfrac{2n(n-5)}{6} + \tfrac{n+2}{2} > 3,$$

also $n - 3 > 0$ and $n - 4 > 0$, which gives $\frac{1}{12}(n-3)(n-4) > 0$. Therefore $\left\lceil \frac{1}{6}(n-2)(2n-3) \right\rceil + 3 \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil > 7$.
Case 3. If $m = 4$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(3n-3)(3n-4) \right\rceil + 4\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

For $n \leq 3$, it is apparent that $\gamma(\mathcal{C}(G)) \leq 3$. For $n = 4$, we have $\gamma(\mathcal{C}(G)) = 6$. If $n \geq 5$, then

$$\tfrac{1}{12}(3n-3)(3n-4) = \tfrac{1}{4}(3n^2 - 7n + 4) = \tfrac{3n(n-5)}{4} + (2n+1) \geq 11.$$

Case 4. If $m = 5$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(4n-3)(4n-4) \right\rceil + 5\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

For $n \leq 2$, it is apparent that $\gamma(\mathcal{C}(G)) \leq 3$. For $n = 3$, we have $\gamma(\mathcal{C}(G)) = 6$. If $n \geq 4$, then

$$\tfrac{1}{12}(4n-3)(4n-4) = \tfrac{1}{6}(6n^2 - 14n + 6) = \tfrac{6n(n-4)}{6} + \tfrac{10n+6}{6} \geq \tfrac{23}{3}.$$

Case 5. If $m = 6$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(5n-3)(5n-4) \right\rceil + 6\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

It is apparent that $\gamma(\mathcal{C}(G)) = 1$ or 4 according as $n = 1$ or 2. If $n \geq 3$, then

$$\tfrac{1}{12}(5n-3)(5n-4) = \tfrac{1}{12}(25n^2 - 35n + 12) = \tfrac{25n(n-3)}{12} + \tfrac{40n+12}{12} \geq 11.$$

Case 6. If $m = 7$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(6n-3)(6n-4) \right\rceil + 7\left\lceil \tfrac{1}{12}(n-4)(n-3) \right\rceil.$$

It is apparent that $\gamma(\mathcal{C}(G)) = 1$ or 6 according as $n = 1$ or 2. If $n \geq 3$, then

$$\tfrac{1}{12}(6n-3)(6n-4) = \tfrac{6n^2-7n+2}{2} = \tfrac{6n(n-3)+11n+2}{2} \geq \tfrac{35}{2}.$$

Case 7. If $m = 8$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(7n-3)(7n-4) \right\rceil + 8\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

Evidently, $\gamma(\mathcal{C}(G)) = 1$ for $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(7n-3)(7n-4) = \tfrac{1}{12}(49n^2 - 49n + 12) = \tfrac{49n(n-2)}{12} + \tfrac{49n+12}{12} \geq \tfrac{55}{6}.$$

Case 8. If $m = 9$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(8n-3)(8n-4) \right\rceil + 9\left\lceil \tfrac{1}{12}(n-4)(n-3) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 2$ when $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(8n-3)(8n-4) = \tfrac{1}{12}(64n^2 - 56n + 12) = \tfrac{64n(n-2)}{12} + \tfrac{72n+12}{12} \geq 13.$$

Case 9. If $m = 10$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(9n-3)(9n-4) \right\rceil + 10\left\lceil \tfrac{1}{12}(n-3)(n-4) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 3$ when $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(9n-3)(9n-4) = \tfrac{1}{12}(81n^2-63n+12) = \tfrac{81n(n-2)}{12} + \tfrac{99n+12}{12} \geq \tfrac{35}{2}.$$

Case 10. If $m = 11$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(10n-3)(10n-4) \right\rceil + 11 \left\lceil \tfrac{1}{12}(n-4)(n-3) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 4$ for $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(10n-3)(10n-4) = \tfrac{1}{12}(100n^2-70n+12) = \tfrac{100n(n-2)}{12} + \tfrac{130n+12}{12} \geq \tfrac{68}{3}.$$

Case 11. If $m = 12$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(11n-3)(11n-4) \right\rceil + 12 \left\lceil \tfrac{1}{12}(n-4)(n-3) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 5$ for $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(11n-3)(11n-4) = \tfrac{1}{12}(121n^2-77n+12) = \tfrac{121n(n-2)}{12} + \tfrac{165n+12}{12} \geq \tfrac{55}{2}.$$

Case 12. If $m = 13$, then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(12n-3)(12n-4) \right\rceil + 13 \left\lceil \tfrac{1}{12}(n-4)(n-3) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 6$ for $n = 1$. If $n \geq 2$, then

$$\tfrac{1}{12}(12n-3)(12n-4) = 12n^2-7n+1 = 12n(n-2)+17n+1 \geq 35.$$

Case 13. If $m \geq 14$, then we get $\mathcal{C}(G)) = K_{(m-1)n} \sqcup mK_n$. Therefore $K_{13} \sqcup 14K_1$ is a subgraph of $K_{(m-1)n} \sqcup mK_n$ for every $n \geq 1$. We know that the genus of $K_{13} \sqcup 14K_1$ is equal to 8. Hence by (1.1), $\gamma(\mathcal{C}(G)) \geq 8$. □

Corollary 3.4. *If $G = M_{2mn}$, where $m > 2$, then*

(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow m = 11, n = 1; m = 12, n = 1.$
(b) $\gamma(\mathcal{C}(G)) \neq 5.$
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow m = 4, n = 4; m = 5, n = 3; m = 7, n = 2; m = 8, n = 2; m = 13, n = 1; m = 14, n = 1.$
(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $m = 3, n \geq 5; m = 4, n \geq 5; m = 5, n \geq 4; m = 6, n \geq 3; m = 7, n \geq 3; m = 8, n \geq 3; m = 9, n \geq 2; m = 10, n \geq 2; m = 11, n \geq 2; m = 12, n \geq 2; m = 13, n \geq 2; m = 14, n \geq 2; m \geq 15, n \geq 1.$

Corollary 3.4 can be proved by noting the fact that if $G = M_{2mn}$, then $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m depending on m odd or even respectively also $|Z(M_{2mn})| = n$ or $2n$ for m odd or even respectively. Hence, Theorem 3.3 leads to the conclusion.

Corollary 3.5. *If $G = D_{2m}$, then*

(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow m = 11, 12.$

(b) $\gamma(\mathcal{C}(G)) \neq 5$.
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow m = 13, 14$.
(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $m \geq 15$.

Corollary 3.5 can be proved by taking the fact that $M_{2mn} = D_{2m}$ if $n = 1$. Hence, Corollary 3.4 leads to the conclusion.

Corollary 3.6. *If* $G = Q_{4m}$, *then*
(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow m = 6$.
(b) $\gamma(\mathcal{C}(G)) \neq 5$.
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow m = 7$.
(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $m \geq 8$.

Corollary 3.6 can be proved by noting the fact that if $G = Q_{4m}$, then $|Z(Q_{4m})| = 2$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. Hence, Theorem 3.3 leads to the conclusion.

Corollary 3.7. *If* $G = U_{6n}$, *then* $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ *also* $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n \geq 5$.

Corollary 3.7 can be proved by noting the fact that if $G = U_{6n}$, then $|Z(U_{6n})| = n$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$. Hence, Theorem 3.3 leads to the conclusion.

Theorem 3.8. *If* $\frac{G}{Z(G)} \cong Sz(2)$, *then* $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ *also* $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n \geq 2$.

Proof. Theorem 2.14 yields
$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{3}(4n-3)(n-1) \right\rceil + 5 \left\lceil \tfrac{1}{4}(n-1)(3n-4) \right\rceil,$$
where $n = |Z(G)|$. Clearly, $\gamma(\mathcal{C}(G)) = 0$ when $n = 1$. If $n \geq 2$, then
$$\tfrac{1}{3}(4n-3)(n-1) = \tfrac{4n(n-2)}{3} + \tfrac{n+3}{3} > 1,$$
also $n - 1 > 0$ and $3n - 4 > 0$, so $\frac{1}{2}(n-1)(3n-4) > 0$. Therefore
$$\left\lceil \tfrac{1}{3}(4n-3)(n-1) \right\rceil + 5 \left\lceil \tfrac{1}{4}(3n-4)(n-1) \right\rceil > 7.$$
$\square$

Theorem 3.9. *If* $G = V_{8n}$, *then*
(a) $\gamma(\mathcal{C}(G)) = 4 \Leftrightarrow n = 3$.
(b) $\gamma(\mathcal{C}(G)) \neq 5$.
(c) $\gamma(\mathcal{C}(G)) = 6 \Leftrightarrow n = 4$.
(d) $\gamma(\mathcal{C}(G)) > 18$ *whenever* $n \geq 5$.

Proof. Note that Theorem 2.16 yields $\gamma(\mathcal{C}(G)) = 0$ for $n = 1, 2$.
Case 1. n is odd. In this case, Theorem 2.16 yields

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{6}(4n-5)(2n-3) \right\rceil \text{ whenever } n \geq 3.$$

Obviously, $\gamma(\mathcal{C}(G)) = 4$ when $n = 3$. For $n \geq 5$ then

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{6}(4n-5)(2n-3) \right\rceil \\ &= \left\lceil \frac{1}{3}(8n(n-5) + 18n + 15) \right\rceil \\ &> 18.\end{aligned}$$

Case 2. n is even. In this case, Theorem 2.16 yields

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{3}(4n-7)(n-2) \right\rceil \text{ whenever } n \geq 4.$$

Clearly, $\gamma(\mathcal{C}(G)) = 6$ for $n = 4$. If $n \geq 6$, then

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{3}(4n-7)(n-2) \right\rceil = \left\lceil \tfrac{1}{3}(4n(n-6) + 9n + 14) \right\rceil > 22.$$

□

Theorem 3.10. *If $G = QD_{2^n}$ or SD_{8n}, then $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ also $\gamma(\mathcal{C}(G)) \geq 7$ for $n \geq 5$ or $n \geq 4$ respectively.*

Proof. If $G = QD_{2^n}$, then Theorem 2.17 gives

$$\gamma(\mathcal{C}(G)) = \left\lceil \tfrac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \right\rceil.$$

Obviously, $\gamma(\mathcal{C}(G)) = 1$ when $n = 4$. If $n \geq 5$, then $(2^{n-1}-5) \geq 11$ and $(2^{n-1}-6) \geq 10$. So $\frac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \geq \frac{110}{12}$. Therefore

$$\left\lceil \tfrac{1}{6}(2^{n-1}-5)(2^{n-1}-6) \right\rceil \geq 10$$

and the result follows. If $G = SD_{8n}$, then Theorem 2.18 yields

$$\gamma(\mathcal{C}(G)) = \begin{cases} \left\lceil \frac{1}{12}(4n-7)(4n-8) \right\rceil & \text{if } n \text{ is odd} \\ \left\lceil \frac{1}{12}(4n-5)(4n-6) \right\rceil & \text{if } n \text{ is even.} \end{cases}$$

Obviously, $\gamma(\mathcal{C}(G)) \leq 3$ when $n = 1, 3$. If $n \geq 5$ and n is odd, then

$$\tfrac{1}{12}(4n-7)(4n-8) = \tfrac{1}{12}(16n(n-5) + 20n + 56) \geq 13.$$

Again for $n = 2$ we get $\gamma(\mathcal{C}(G)) = 1$. Finally, if n is even and $n \geq 4$, then

$$\tfrac{1}{12}(4n-5)(4n-6) = \tfrac{1}{12}(16n(n-4) + 20n + 30) \geq \tfrac{55}{6}.$$

Hence the result follows. □

It is observed that $\gamma(\mathcal{C}(G)) \neq 5$ for all the groups considered in our study. It may be interesting to provide examples of groups G such that $\gamma(\mathcal{C}(G)) = 5$. In general we pose the following question:

"Which positive integers can be realized as genus of commuting graphs of some finite non-abelian groups?"

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