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A NEW CLASS OF SMALL SUBMODULES

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Abstract. Let R be a commutative ring with identity $1 \neq 0$ and M a nonzero unital R-module. In this paper, we introduce a new notion of submodules in M, namely T-semi-annihilator small submodules of M with respect to an arbitrary submodule T of M. A submodule N of M is T-semi-annihilator small in M provide that for each submodule X of M with $T \subseteq N + X$ implies that $\mathrm{Ann}(X) \ll (T : M)$. In addition, we investigate some results concerning to this new class of submodules. Among various results, we prove that for a faithful finitely generated multiplication module M, the submodule N of M is a T-semi-annihilator small submodule of M if and only if $(N : M)$ is a $(T : M)$-semi-annihilator small ideal of R. Finally, we explore the properties and the behaviour of this structure under ring homomorphism, localization, direct sums and tensor product of them with a faithfully flat R-module.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity $1 \neq 0$ and M a nonzero unital R-module. We use the notations $\subseteq$ and $\leq$ to denote inclusion and submodule. A nonempty subset S of R is said to be a *multiplicatively closed set* (briefly, m.c.s.) of R if $0 \notin S$, $1 \in S$ and $st \in S$ for each $s, t \in S$. The set of all submodules of M is denoted by $\mathrm{L}(M)$ and also $\mathrm{L}^*(M) = \mathrm{L}(M) \setminus \{0, M\}$ will denote the set of all non-trivial proper submodules of M. Also, for a ring R, the set of all ideals of R is denoted by $\mathbb{I}(R)$ and $\mathbb{I}^*(R) = \mathbb{I}(R) \setminus \{0, R\}$ will denote the set of all non-trivial proper ideals of R. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. Let M be an R-module and $N \leq M$, the *colon ideal* of M into N is defined as $(N :_R M) = \{r \in R : rM \subseteq N\}$. If there is no ambiguity for the ring we will write $(N : M)$. The annihilator of M which is denoted by $\mathrm{Ann}_R(M)$ is $(0 :_R M)$ and so $(N :_R M) = \mathrm{Ann}_R(M/N)$. Also, $\mathrm{Max}(M)$ and $\mathrm{Min}(M)$ will denote the set of all maximal and minimal submodules of M, respectively. Recall that, $(R, \mathfrak{m})$ is a quasi-local ring if $\mathfrak{m}$ is the only maximal ideal of R.

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A proper submodule P of M is a *prime submodule*, if for $r \in R$ and $m \in M$, whenever $rm \in P$ implies that either $r \in (P :_R M)$ or $m \in P$. If P is prime, then the ideal $\mathfrak{p} := (P :_R M)$ is a prime ideal of R. In this case, P is said to be $\mathfrak{p}$-*prime submodule* of M, see [12]. Equivalently, for ideal I of R and $m \in M$ whenever $Im \subseteq P$, then $I \subseteq \mathrm{Ann}_R(M/P)$ or $m \in P$. If Q is a maximal submodule of M, then Q is a prime submodule and $(Q : M) := \mathfrak{m}$ is a maximal ideal of R. In this case, we say Q is an $\mathfrak{m}$-*maximal submodule* of M, see [11, p. 61].

A module M on a ring R (not necessarily commutative) is called *prime* if for every nonzero submodule K of M, $\mathrm{Ann}(K) = \mathrm{Ann}(M)$. An R-module M is called a *multiplication module*, if every submodule N of M has the form $N = IM$ for some ideal I of R, and in this case, $N = (N :_R M)M$, see [5]. An R-module M is an r-multiplication (r-m, for short) module, if for every proper ideal I of R there exists a proper submodule N of M with $N = IM$. Equivalently, for every proper ideal I of R, $IM \neq M$, see [13, Definition 3.1].

Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a direct summand of M, denoted by $N \leq^{\oplus} M$, and we will write $M = N \oplus K$. A submodule N of M is called *small* (*superfluous*), denoted by $N \ll M$, if for every submodule L of M, $N + L = M$, implies that $L = M$. Clearly, the zero submodule of every nonzero module is superfluous. Dually, a submodule N of M is called *essential*, denoted by $N \leq^{e} M$, if for every submodule X of M, $N \cap X = 0$, provide that $X = 0$. Note that $0 \leq^{e} M$ if and only if $M = 0$. Also, note that N always has at least one essential extension, since $N \leq^{e} N$. In particular, an ideal I of R is small (resp., essential) if I is a small (resp., essential) submodule of R as an R-module. We denote by $I \subseteq^{e} R$ (resp., $I \subseteq^{\oplus} R$) an essential ideal (resp., a direct summand) of R. Gilmer [9, p.60] defined the concept of *cancellation ideal* to be an ideal I of R which satisfies the following: whenever $AI = BI$ with A and B are ideals of R implies $A = B$. An R-module M is called a *cancellation module* whenever $IM = JM$ with I and J are ideals of R implies that $I = J$.

By [3, Proposition 9.13], the Jacobson radical of M, denoted by $\mathrm{J}(M)$ is the intersection of all maximal submodules of M and also it is the sum of all small submodules of M, i.e., $\mathrm{J}(M) = \bigcap_{\mathfrak{m} \in \mathrm{Max}(M)} \mathfrak{m} = \sum_{N \ll M} N$. If M does not have maximal submodules, we put $\mathrm{J}(M) = M$. Consequently, if $\mathrm{J}(M)$ is a small submodule of M, then $\mathrm{J}(M)$ is the largest small submodule of M. Moreover, if M is a finitely generated nonzero module, then $M \neq \mathrm{J}(M)$. Dually, for any R-module M, the sum of all minimal submodules of M is called the *socle* of M, denoted $\mathrm{Soc}(M)$. If M has no minimal submodules, we have $\mathrm{Soc}(M) = 0$. The module M is said to be *semisimple* provided $\mathrm{Soc}(M) = M$. For more details we refer the reader to [3, 5, 6, 9, 14, 15]. It is well known that if M is a semisimple module, then the zero submodule is the only small submodule of M and also M is the only essential submodule of M.

The concept of small submodules has been generalized by some researches, for this see [2, 6, 8, 14]. In [2], the authors introduced the concept of *annihilator-small submodules* of any right R-module M. For a unitary right R-module M on an associative ring R with identity, a submodule K of M_R is annihilator-small if $K + T = M$, such that T is a submodule of M_R, implies that $\ell_S(T) = 0$, where $\ell_S(T)$ indicates the left annihilator of T over $S = \mathrm{End}(M_R)$. In [6] the authors introduced the concept of small submodules with respect to an arbitrary submodule. Recall that a submodule N of M is called, T-small in M denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N + X$, implies that $T \subseteq X$. In [10], the author introduced the concept of a semiannihilator small submodule N in M such that N is called *semiannihilator small* (*sa-small* for short), denoted by $N \ll^{sa} M$, if for every submodule L of M with $N + L = M$, implies that $\mathrm{Ann}(L) \ll R$.

In this paper, we introduce a new class of submodules in M, namely T-semi-annihilator small submodules of M with respect to an arbitrary submodule T of M. A submodule N of M is T-semi-annihilator small in M provide that for each submodule X of M with $T \subseteq N + X$, implies that $\mathrm{Ann}(X) \ll (T : M)$. We investigate some results concerning to this class of submodules and we explore the properties and the behaviour of this structure under ring homomorphism, localization, direct sums and tensor product of them with a faithfully flat R-module.

2. Preliminaries

In this section, we will provide the definitions and results which are necessary in the next section. We recall that R is a *von Neumann regular ring* (associative, with 1, not necessarily commutative) if for every element a of R, there is an element $b \in R$ with $a = aba$. These rings are characterized by the fact that every R-module is flat.

Definition 2.1. Let N be a submodule of an R-module M.

(i) N is called an *R-annihilator-small* (briefly, *R-a-small*) *submodule* of M, denoted by $N \ll^a M$, if $N + X = M$ for some submodule X of

M, implies that $\mathrm{Ann}_R(X) = 0$. We denote the set of all R-a-small submodules of M by $\mathrm{S}^a(M)$, see [1, Definition 2.1].

(ii) N is called *semiannihilator small* (*sa-small* for short) submodule, denoted by $N \ll^{sa} M$, if for every submodule X of M with $N + X = M$, implies that $\mathrm{Ann}(X) \ll R$. We denote the set of all sa-small submodules of M by $\mathrm{S}^{sa}(M)$. Particularly, an ideal I of R is a sa-small ideal, if it is a sa-small submodule of R as an R-module, see [10].

(iii) M is called a *semiannihilator hollow* (*sa-hollow* for short) module if every proper submodule of M is sa-small in M, i.e., $\mathrm{L}(M) \setminus \{M\} = \mathrm{S}^{sa}(M)$, see [10, p. 17].

Clearly, every R-a-small submodule is also a sa-small submodule. Obviously, if M is a faithful R-module, then every small submodule is R-a-small submodule in M, and so is sa-small in M. Example 2.2 (*ii*) shows that being faithful to the module, is a necessary condition for this ruling. The converse is not true, see Example 2.2 (*iii*).

Example 2.2. (i) Consider $M = \mathbb{Z}_8$ as a $\mathbb{Z}_8$-module. Then

$$\mathrm{S}^a(\mathbb{Z}_8) = \mathrm{S}^{sa}(\mathbb{Z}_8) = \{\langle\bar{0}\rangle, \langle\bar{2}\rangle, \langle\bar{4}\rangle\}.$$

(ii) Consider the uniserial module $M = \mathbb{Z}_{2^n}$ as a $\mathbb{Z}$-module. Then every nonzero proper submodule of M is of the form $N = \langle\overline{2^k}\rangle$ such that $1 \le k \le n-1$ and also $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{2^n}) = 2^n\mathbb{Z} \not\ll \mathbb{Z}$. Obviously, 0 is not sa-small in M. Therefore $\mathrm{S}^a(M) = \mathrm{S}^{sa}(M) = \emptyset$, while $\mathrm{S}(M) = \mathrm{L}(M) \setminus \{M\}$ where $\mathrm{S}(M)$ is the set of all small submodules of M.

(iii) Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathrm{S}(\mathbb{Z}) = \{0\}$. Suppose that $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$. Thus $(n, m) = 1$, if $n \neq 1$, then $\mathrm{Ann}_{\mathbb{Z}}(m\mathbb{Z}) = 0$ and so $\mathrm{S}^a(\mathbb{Z}) = \mathrm{S}^{sa}(\mathbb{Z}) = \mathrm{L}(\mathbb{Z}) \setminus \{\mathbb{Z}\}$. In fact, $\mathbb{Z}$ is sa-hollow as a $\mathbb{Z}$-module. This implies that a sa-small submodule need not be small.

Note 2.3. We know that an ideal I of R is small in R if and only if $I \subseteq \mathrm{J}(R)$. Therefore a submodule N of a module M is sa-small in M if $N + X = M$ for a submodule X of M, implies that $\mathrm{Ann}(X) \subseteq \mathrm{J}(R)$.

Definition 2.4. An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = \mathrm{Ann}_M(I)$. An R-module M satisfies the *double annihilator condition* (DAC for short), if for each ideal I of R, $I = \mathrm{Ann}_R(\mathrm{Ann}_M(I))$. Also, M is said to be a *strong comultiplication module*, if M is a comultiplication module which satisfies DAC, see [4, Definition 2.1].

For example, the $\mathbb{Z}$ -module $\mathbb{Z}_{2^\infty}$ is a comultiplication module since all of its proper submodules are of the form $(0 :_M 2^k\mathbb{Z})$ for $k = 0, 1, \ldots$. It is clear that M is comultiplication if and only if for every submodule N of M, $\mathrm{Ann}_M(\mathrm{Ann}_R(N)) = N$. Note that, if M is a strong comultiplication R-module, then there exists exactly one ideal I of R with $N = (0 :_M I)$. In the following, we recall some of basic important properties of small submodules which are needed in the rest of the article.

Theorem 2.5. *Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then,*

(i) *$N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$, see* [3, Proposition 5.17].
(ii) *$H + K \ll M$ if and only if $H \ll M$ and $K \ll M$, see* [3, Proposition 5.17].
(iii) *If $K \ll M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll f(M)$ and $f(K) \ll N$. In particular, if $K \ll M \leq N$, then $K \ll N$, see* [3, Lemma 5.18] *and* [15, Remark 2.8, (4)].
(iv) *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then*
  (a) *$K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$, see* [3, Proposition 5.20].
  (b) *$K_1 \oplus K_2 \leq^e M_1 \oplus M_2$ if and only if $K_1 \leq^e M_1$ and $K_2 \leq^e M_2$, see* [3, Proposition 5.20].
(v) *Each finite sum of small submodules of M is a small submodule in M, see* [15, Remark 2.8].
(vi) *If $N \leq X \leq^\oplus M$, then $N \ll X$ if and only if $N \ll M$, see* [15, Remark 2.8].

3. Sa-small submodules w.r.t. an arbitrary submodule

In this section, we introduce the concept of T-sa-small submodule in M with respect to an arbitrary submodule T of M as a special case of sa-small submodule and discuss some of its basic properties. Moreover, we investigate some new other properties of the sa-small submodules and we will generalize these properties to T-sa-small submodules and we supply some examples.

We begin this section with the following definition.

Definition 3.1. Let M be an R-module and $T, N \leq M$ with $T \nsubseteq N$.

(i) N is a *T-semi-annihilator small* (briefly, T-sa-small) submodule of M, denoted by $N \ll^{sa}_T M$, provided for every submodule $X \leq M$ with

$T \subseteq N + X$ implies that $\mathrm{Ann}(X) \ll (T :_R M) = \mathrm{Ann}_R(M/T)$. Equivalently, if for a submodule X of M, $\mathrm{Ann}(X)$ is not small in $(T :_R M)$, then $T \nsubseteq N+X$. The set of all T-sa-small submodules of M is denoted by $\mathrm{S}^{sa}_T(M)$. In particular, for an arbitrary ideal A of R, we say that an ideal I of R is an *A-sa-small ideal* of R if I is an A-sa-small submodule of R as an R-module.

(ii) We say that M is a *T-sa-hollow module* if every submodule N of M is T-sa-small. The sum of all T-sa-small submodules of M is denoted by $\mathrm{J}^{sa}_T(M)$.

(iii) Let $f : M \to N$ be an R-epimorphism and $T \leq M$, we say that f is a *T-sa-small epimorphism*, in case $\mathrm{Ker}(f) \ll^{sa}_T M$.

Note 3.2. Let M be an R-module and $T \leq M$.

(i) Assume that $N \in \mathrm{S}^{sa}_T(M)$. If $T \subseteq N + X$ for some submodule X of M, then clearly $\mathrm{Ann}(M) \ll (T : M)$ because $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(X)$. In fact, if $N \ll^{sa}_T M$ and $\mathrm{Ann}(M) \not\ll (T : M)$, then for every proper submodule X of M, $T \nsubseteq N + X$.

(ii) For every submodule K of M, $K \ll^{sa}_T M$ if and only if R-epimorphism $p_K : M \to M/K$ is a T-sa-small epimorphism.

The following example shows that the concepts of small and T-sa-small submodules are different from each other in general.

Example 3.3. (i) Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathbb{Z}$ is a T-sa-hollow $\mathbb{Z}$-module for every submodule $T = k\mathbb{Z}$ $(k \geq 0)$ of $\mathbb{Z}$, because if $k\mathbb{Z} \subseteq m\mathbb{Z}+n\mathbb{Z} = (m, n)\mathbb{Z}$, then $0 = \mathrm{Ann}_{\mathbb{Z}}(n\mathbb{Z}) \ll (k\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = k\mathbb{Z}$. Therefore $\mathrm{S}^{sa}_T(\mathbb{Z}) = \mathrm{L}(\mathbb{Z})$. Note that $\mathbb{Z}$ is a faithful prime $\mathbb{Z}$-module and 0 is the only small submodule of $\mathbb{Z}$.

(ii) Let p be an arbitrary prime number and n be any positive integer. Then $M = \mathbb{Z}_{p^n}$ is a uniserial $\mathbb{Z}$-module of length n. Clearly, all proper submodules of M are small in M, but $\mathrm{S}^{sa}_T(M) = \emptyset$ for every submodule T of M, because let $T = \langle\overline{p^k}\rangle$ for some $1 \leq k < n$, then $T \subseteq \langle\overline{p^{n-1}}\rangle + \langle\overline{p^{k-1}}\rangle$, but $\mathrm{Ann}_{\mathbb{Z}}(\langle\overline{p^{k-1}}\rangle) = p^{n-k+1}\mathbb{Z} \not\ll (\langle\overline{p^k}\rangle :_{\mathbb{Z}} \mathbb{Z}_{p^n}) = p^{n-k}\mathbb{Z}$.

Note that if $T \in \mathrm{L}^*(M)$ and $T \leq N \leq M$, then $N \notin \mathrm{S}^{sa}_T(M)$, otherwise, since $T \subseteq N + 0$, by assumption, $R = \mathrm{Ann}(0) \ll (T :_R M)$ implies that $(T :_R M) = R = 0$ which is not possible.

Proposition 3.4. *Let M be a nonzero R-module and $T, N \leq M$ with $T \nsubseteq N$. If there exists a submodule $T' \subseteq T$ with $T' \nsubseteq N$ such that $N \ll^{sa}_{T'} M$, then*

$N \ll_T^{sa} M$. In particular, if for some $m \in T \setminus N$ we have $N \ll_{Rm}^{sa} M$, then $N \ll_T^{sa} M$.

Proof. Assume that $T \subseteq N + X$ for some submodule X of M. Then $T' \subseteq N + X$ and by assumption, $\mathrm{Ann}(X) \ll (T' : M)$. By Theorem 2.5 (iii), $\mathrm{Ann}(X) \ll (T : M)$ since $(T' : M) \subseteq (T : M)$. By taking $T = Rm$, the second part will be obtained. □

The following theorem states the conditions (i) and (ii) under which a module M is T-sa-hollow for a submodule T of M.

Theorem 3.5. *Let M be a nonzero R-module and $T \leq M$. Then the following assertions hold.*

(i) *If M is a faithful prime R-module, then M is T-sa-hollow.*
(ii) *If $\mathrm{S}_0^{sa}(M) \neq \emptyset$, then M is a faithful prime module. Furthermore, M is T-sa-hollow.*

Proof. (i) Suppose that N is an arbitrary submodule of M and $T \subseteq N + X$ for some submodule $X \leq M$. Then by assumption, $\mathrm{Ann}(X) = \mathrm{Ann}(M) = 0$ which is small in $(T : M)$, so $N \leq_T^{sa} M$, as needed.

(ii) Let N be a 0-sa-small submodule of M. Then for all nonzero submodules X of M, since $0 \subseteq N + X$, by assumption, $\mathrm{Ann}(X) \ll \mathrm{Ann}(M)$. Now, since $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(X)$, $\mathrm{Ann}(X) = \mathrm{Ann}(M)$. This is impossible unless $\mathrm{Ann}(X) = \mathrm{Ann}(M) = 0$ for every nonzero submodule X of M, because the nonzero ideal $\mathrm{Ann}(X)$ is never small in itself. This implies that M is a faithful prime module. The second part is a result from (i). □

Note 3.6. Let M be an R-module and take $T = M$ Then the notions of T-sa-small submodules and sa-small submodules are equal.

In the following theorem, we prove that in a local ring the concepts of small ideals, a-small ideals and sa-small ideals are identical.

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a local ring. Then*
$$\mathrm{S}(R) = \mathrm{S}^a(R) = \mathrm{S}^{sa}(R) = \mathrm{L}(R) \setminus \{R\}.$$

Proof. We prove that every proper ideal of R is small in R. Assume that I is a proper ideal of R with $I + X = R$ for some ideal X of R. Then $\mathfrak{m} + X = R$ and this implies that $X = R$, because otherwise $X \subseteq \mathfrak{m}$ and so $\mathfrak{m} + X = \mathfrak{m} \subsetneq R$ which is a contradiction. Hence $I \in \mathrm{S}(R)$. Similarly, we conclude that $I + X = R$ implies that $\mathrm{Ann}(X) = 0$ and so $\mathrm{Ann}(X) \ll R$, as needed. □

Proposition 3.8. *Let K and N be submodules of an R-module M such that $K + N = M$. Then the following assertions hold.*

(i) *If* $\mathrm{Ann}(K) \neq 0$, *then* $N \not\ll^{a} M$.
(ii) *If* $\mathrm{Ann}(K) \not\ll R$, *then* $N \not\ll^{sa} M$.

Proof. The proofs are straightforward. □

Proposition 3.9. *Let M be a faithful R-module and $N \in \mathrm{L}(M)$. If* $\mathrm{Ann}(N) \subseteq^{e} R$, *then* $N \ll^{a} M$.

Proof. Suppose that $N + K = M$ for some submodule K of M. Since $\mathrm{Ann}(N) \cap \mathrm{Ann}(K) = \mathrm{Ann}(N + K) = \mathrm{Ann}(M) = 0$, hence $\mathrm{Ann}(K) = 0$, as needed. □

Theorem 3.10. *Let M be an R-module and $T \leq M$. Then the following assertions hold.*

(i) *Every T-sa-small submodule of M is a sa-small submodule of M.*
(ii) *Assume that M is a prime cancellation multiplication module and $N \leq T \leq^{\oplus} M$. If $N \ll^{sa} M$, then $N \ll^{sa}_{T} M$.*

Proof. (i) Assume that $N \ll^{sa}_{T} M$ and $N + K = M$ for some submodule K of M. Since $T \subseteq N + K$ and $N \ll^{sa}_{T} M$, $\mathrm{Ann}(K) \ll (T : M) \leq R$. By Theorem 2.5 (iii), $\mathrm{Ann}(K) \ll R$ and so $N \ll^{sa} M$.

(ii) Suppose that $M = T + T'$ for some submodule T' of M and $T \subseteq N + K$ for some submodule K of M. This implies that $M = T + T' \subseteq N + K + T'$ and so $N + K + T' = M$. By assumption, $\mathrm{Ann}(K + T') \ll R$. Since M is prime,

$$\mathrm{Ann}(K + T') = \mathrm{Ann}(K) \subseteq (T : M) \subseteq R.$$

Since M is a cancellation multiplication module, $(T : M) \subseteq^{\oplus} R$ and by Theorem 2.5 (vi), $\mathrm{Ann}(K) \ll (T : M)$, as needed. □

The following example shows that in general the concepts of small submodules and sa-small submodules are independent from each other. In (iii), we show that the converse of Theorem 3.10 (i), is not true in general. Also, (iv) shows that for R-modules M, M', if $f : M \to M'$ is an R-epimorphism, then the image of a T-sa-small submodule of M need not be an $f(T)$-sa-small submodule in M'.

Example 3.11. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ be the rings of integers and integers modulo n, respectively.

(i) Consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}$-module. Since $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_6) = 6\mathbb{Z}$ is not a small ideal of $\mathbb{Z}$, hence $\langle\bar{0}\rangle \notin \mathrm{S}^{sa}(\mathbb{Z}_6)$ whereas $\langle\bar{0}\rangle \in \mathrm{S}(\mathbb{Z}_6)$. We note that the only nonzero proper submodules of $\mathbb{Z}_6$ are $N = \langle\bar{2}\rangle$ and $L = \langle\bar{3}\rangle$ where $N + L = \mathbb{Z}_6$ and both of $\mathrm{Ann}_{\mathbb{Z}}(N) = 3\mathbb{Z}$ and $\mathrm{Ann}_{\mathbb{Z}}(L) = 2\mathbb{Z}$ are not small ideals of $\mathbb{Z}$. It concludes that $\mathrm{S}^{sa}(\mathbb{Z}_6) = \emptyset$ whereas $\mathrm{S}(\mathbb{Z}_6) = \{\langle\bar{0}\rangle\}$.

(ii) We take the $\mathbb{Z}$-module $M = 2\mathbb{Z} \times \mathbb{Z}_8$. Then $N = \langle(0, \bar{0})\rangle$ is a sa-small submodule of M but N is not a T-sa-small submodule of M for submodule $T = \langle(0, \bar{4})\rangle$ of M, since $T \subseteq N + \langle(0, \bar{2})\rangle$ whereas $\mathrm{Ann}_{\mathbb{Z}}(\langle(0, \bar{2})\rangle) = 4\mathbb{Z}$ is not small in $(T :_{\mathbb{Z}} M) = 0$.

(iii) Consider the natural $\mathbb{Z}$-epimorphism $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}_8$ where $\pi(n) = \overline{n}$. Take $N = 0$ and $T = 2\mathbb{Z}$. Clearly, $0 \ll^{sa}_{2\mathbb{Z}} \mathbb{Z}$ because we have $2\mathbb{Z} \subseteq 0+2\mathbb{Z}$ and also $2\mathbb{Z} \subseteq 0 + \mathbb{Z}$, then

$$0 = \mathrm{Ann}_{\mathbb{Z}}(2\mathbb{Z}) \ll (2\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 2\mathbb{Z},$$

$$0 = \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}) \ll (2\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 2\mathbb{Z}.$$

But $\pi(N) = \pi(0) = \langle\bar{0}\rangle$ is not $\pi(T)$-sa-small submodule of $\mathbb{Z}_8$ since $\pi(T) = \langle\bar{2}\rangle$ and $\langle\bar{2}\rangle \subseteq \langle\bar{0}\rangle + \mathbb{Z}_8$ whereas $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_8) = 8\mathbb{Z}$ is not small in $(\langle\bar{2}\rangle :_{\mathbb{Z}} \mathbb{Z}_8) = 2\mathbb{Z}$.

Theorem 3.12. *Let M be an R-module and $N \leq K \leq M$. Then we have:*

(i) *Let $T \leq T' \lneq M$. If $N \ll^{sa}_{T} M$, then $N \ll^{sa}_{T'} M$.*

(ii) *If $K \ll^{sa}_{T} M$, then $N \ll^{sa}_{T} M$. If M is a faithful prime R-module, the converse is also true.*

(iii) *Let $T \subseteq K$. Then $N \ll^{sa}_{T} M$ if and only if $N \ll^{sa}_{T} K$.*

Proof. (i) Assume that $T' \subseteq N + X$ for some submodule X of M. Then $T \subseteq N + X$ and so $\mathrm{Ann}(X) \ll (T : M) \leq (T' : M)$. By Theorem 2.5 (iii), $\mathrm{Ann}(X) \ll (T' : M)$ as we needed.

(ii) Assume that $T \subseteq N+X$ for some submodule X of M. Then $T \subseteq K+X$. By assumption, $\mathrm{Ann}(X) \ll (T : M)$ and so $N \ll^{sa}_{T} M$. Conversely, let $N \ll^{sa}_{T} M$ and $T \subseteq K + X$ for some submodule X of M. By assumption, $\mathrm{Ann}(X) = \mathrm{Ann}(M) = 0$ is small in $(T : M)$, as needed.

(iii) ($\Rightarrow$) It is clear.

($\Leftarrow$) Let $T \subseteq N + L$ for some submodule L of M. Then by the modular law $T \subseteq (N + L) \cap K = N + (L \cap K)$. Since $N \ll^{sa}_{T} K$,

$$\mathrm{Ann}(L) \subseteq \mathrm{Ann}(L \cap K) \ll (T :_R M)$$

and this implies that $\mathrm{Ann}(L) \ll (T :_R M)$ and so the proof is complete. □

Proposition 3.13. *Let $M = N \oplus K$ be a multiplication R-module such that N, K are finitely generated nonzero submodules of M. Then N and K are not sa-small in M.*

Proof. By [7, Corollary 2.3], $\mathrm{Ann}(N)+\mathrm{Ann}(K) = R$. If $N \ll^{sa} M$, then since $M = N + K$, $\mathrm{Ann}(K) \ll R$ and so $\mathrm{Ann}(N) = R$. It concludes that $N = 0$ which is impossible. □

Proposition 3.14. *Let M be an R-module. Then the following assertions hold.*

(i) $0 \in \mathrm{S}^{sa}(M)$ *if and only if* $\mathrm{Ann}(M) \subseteq \mathrm{J}(R)$.
(ii) *If R is a field, then every proper submodule of M is sa-small.*
(iii) *If M is a prime module and* $\mathrm{Ann}(M) \subseteq \mathrm{J}(R)$, *then every proper submodule of M is sa-small. Furthermore,* $\mathrm{S}(M) \subseteq \mathrm{S}^{sa}(M)$.

Proof. (i) The proof is straightforward.

(ii) Suppose that $N \lneq M$ such that $N + X = M$ for some submodule X of M. If $\mathrm{Ann}(X) = 0$, then $\mathrm{Ann}(X) \ll R$ and the proof is complete. If $\mathrm{Ann}(X) = R$, then $X = 0$ and hence $N = M$, which is a contradiction.

(iii) The proof is clear. □

Theorem 3.15. *Let M be an R-module and N a proper submodule of M which is not sa-small in M. Then R is not a field.*

Proof. Since $N \not\ll^{sa} M$, hence there exists a submodule X of M such that $N + X = M$ and $\mathrm{Ann}(X) \not\ll R$. Then clearly, $\mathrm{Ann}(X)$ is a nontrivial ideal of R and the proof is complete. □

Recall that a ring is perfect if and only if it satisfies DCC on its principal ideals.

Proposition 3.16. *Let M be a nonzero finitely generated R-module. Then the following statements are equivalent:*

(i) *The zero submodule is a sa-small submodule of M;*
(ii) $\mathrm{Ann}(M) \subseteq \mathrm{J}(R)$*;*
(iii) *M is r-m;*
(iv) *$M_{\mathfrak{m}} \neq 0$ for every $\mathfrak{m} \in \mathrm{Max}(R)$;*
(v) *R is a perfect quasi-local ring.*

Proof. (i) $\Leftrightarrow$ (ii): By Proposition 3.14 (i).

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): By [13, Proposition 3.6].

(iii) $\Leftrightarrow$ (v): By [13, Corollary 3.4]. □

The following corollary follows by [13, Proposition 3.3].

Corollary 3.17. *Let M be a nonzero finitely generated module on a quasi-local ring $(R, \mathfrak{m})$. Then the following assertions are equivalent to above equivalences:*

(i) $\mathrm{Max}(M) \neq \emptyset$;

(ii) *There is an $m \in M$ such that $Rm \notin \mathrm{S}(M)$.*

In the following theorem (i), we present a condition that the ring R not to be semisimple. Also, in (ii), we show that in a sa-hollow ring R, the only nonzero idempotent element is the identity element 1_R.

Recall that an element $x \in R$ is a zero-divisor, if there exists a nonzero $y \in R$ such that $xy = 0$. We denote the set of zero-divisors from a ring R by $\mathrm{Z}(R)$.

Theorem 3.18. *Let R be a ring. Then the following statements are true:*

(i) *Let R be a sa-hollow ring. If $x \in \mathrm{Z}(R)$, then $R \neq Rx + Ry$ for some $y \in R$. Moreover, $e = 1$ is the only nonzero idempotent element of R.*

(ii) *If R is a von Neumann regular ring, then none of the nonzero finitely generated ideals of R is a sa-small ideal of R.*

(iii) *Suppose that M is a faithful multiplication module on an integral domain R. Then every proper submodule of M is sa-small in M.*

Proof. (i) Assume that $x \in \mathrm{Z}(R)$. Then there exists an element $0 \neq y \in R$ such that $xy = 0$. Let $R = Rx + Ry$.Then $Ry \subseteq \mathrm{Ann}(Rx) \ll R$. This implies that $Ry \ll R$, hence $Rx = R$ and so $x \in \mathrm{U}(R)$ which is a contradiction. Furthermore, assume that e is an idempotent element of R. Since $R(1-e) + Re = R$, by assumption, $R(1-e) \subseteq \mathrm{Ann}(Re) \ll R$. Thus, $R(1-e) \ll R$ and so $Re = R$. It concludes that $e = 1$.

(ii) The proof follows from the fact that since R is a von Neumann regular ring, hence every finitely generated ideal I of R is a direct summand of R such that $I = Re$ with $e^2 = e$. Thus $R = Re \oplus R(1-e)$. By proof of (i), I can not be a sa-small ideal of R.

(iii) Clearly, the zero submodule is sa-small in M, because $0 + M = M$ and we have $\mathrm{Ann}(M) = 0$ is small in R. Now, assume that N is a nonzero submodule of M and $N + K = M$ for some submodule K of M. Then $K = IM$ for some nonzero ideal I of R. Let $r \in \mathrm{Ann}(K)$. Then $r(IM) = 0$, so $rI = 0$. It concludes that $r = 0$ since R is an integral domain. Hence $\mathrm{Ann}(K) = 0$ which is a small ideal of R, as needed. □

Proposition 3.19. *Let M be a strong comultiplication module and $N + L = M$ for submodules N, K of M. Then $N \ll^{sa} M$ if and only if $L \leq^e M$.*

Proof. Suppose that $N + L = M$. Then by assumption, $\mathrm{Ann}(L) \ll R$. By [16, Theorem 2.5], $L = (0 :_M \mathrm{Ann}_R(L))$ is an essential submodule of M if and only if $L \leq^e M$. □

The following proposition describes the relation between T-sa-small submodules of an R-module M and $(T : M)$-sa-small ideals of R for an arbitrary submodule T of M.

Proposition 3.20. *Let M be a multiplication R-module. If $N \ll^{sa} M$ (resp., $N \ll^{sa}_T M$), then $(N : M) \ll^{sa} R$ (resp., $(N : M) \ll^{sa}_{(T:M)} R$). The converse is true if M is also a finitely generated faithful module. Furthermore, in this case, $\mathrm{J}^{sa}_T(M) = \mathrm{J}^{sa}_{(T:M)}(R)M$.*

Proof. ($\Rightarrow$) Let $(N : M) + J = R$ for some ideal J of R. Thus $(N : M)M + JM = M$. Since M is multiplication, $N + JM = M$ and so $\mathrm{Ann}(JM) \ll R$. From $\mathrm{Ann}(J) \subseteq \mathrm{Ann}(JM)$, we infer that $\mathrm{Ann}(J) \ll R$ and so $(N : M) \ll^{sa} R$, as we needed.

($\Leftarrow$) Suppose that $(N : M) \ll^{sa} R$ and $N + K = M$ for some submodule K of M. By [7, p. 756], $(N : M)M + (K : M)M = RM$. Since M is a finitely generated faithful multiplication module, M is a cancellation module and so $(N : M) + (K : M) = R$. By assumption, $\mathrm{Ann}(K : M) \ll R$. Since M is a faithful module, $\mathrm{Ann}(K : M) = \mathrm{Ann}(K) \ll R$.

Similarly, assume that $N \ll^{sa}_T M$ and $(T : M) \subseteq (N : M) + J$ for some ideal J of R. Since M is a multiplication module,

$$T = (T : M)M \subseteq (N : M)M + JM = N + JM.$$

By assumption, $\mathrm{Ann}(J) \subseteq \mathrm{Ann}(JM) \ll (T :_R M)$ and so

$$\mathrm{Ann}(J) \ll (T :_R M)$$

as we needed. Conversely, let $(N : M) \ll^{sa}_{(T:M)} R$ and $T \subseteq N + K$ for some submodule K of M. Therefore

$$T = (T : M)M \subseteq (N : M)M + (K : M)M = ((N : M) + (K : M))M$$

and so $(T : M) \subseteq (N : M) + (K : M)$ because M is a cancellation module. Since $(N : M) \ll^{sa}_{(T:M)} R$, we infer that $\mathrm{Ann}(K : M) \ll (T : M)$. Since M is faithful, $\mathrm{Ann}(K:M)=\mathrm{Ann}(K)\ll(T:M)$ and the proof is complete. For the second part we note that

$$\mathrm{J}_T^{sa}(M):=\sum_{N\ll_T^{sa}M}N=\sum_{N\ll_T^{sa}M}(N:M)M$$
$$:=\left(\sum_{(N:M)\ll_{(T:M)}^{sa}R}(N:M)\right)M=\mathrm{J}_{(T:M)}^{sa}(R)M.$$

□

Corollary 3.21. *Let M be a faithful finitely generated multiplication R-module and $T\le M$. Then M is a T-sa-hollow module if and only if R is a $(T:M)$-sa-hollow ring.*

Proposition 3.22. *Let M be an R-module with submodules $N\le K$ and $T\le M$. Then the following statements are true:*

- (i) *Assume that M and M' are R-modules and $f:M\to M'$ is an R-epimorphism. If $N'\ll_{T'}^{sa}M'$ for some submodule T' of M', then $f^{-1}(N')\ll_{f^{-1}(T')}^{sa}M$.*
- (ii) *Let $N\lneq T\le M$ be submodules of M. If $K/N\ll_{T/N}^{sa}M/N$, then $K\ll_T^{sa}M$ and $N\ll_T^{sa}M$.*
- (iii) *Let M be a Noetherian R-module and S be a m.c.s. of R. Then $S^{-1}N$ is an $S^{-1}T$-sa-small submodule of $S^{-1}R$-module $S^{-1}M$ if and only if N is a T-sa-small submodule of M.*

Proof. (i) Suppose that $f^{-1}(T')\subseteq f^{-1}(N')+L$ for some submodule L of M. Since f is an R-epimorphism, hence

$$T'=f(f^{-1}(T'))\subseteq f(f^{-1}(N')+L)\subseteq N'+f(L).$$

Since $N'\ll_{T'}^{sa}M'$, hence $\mathrm{Ann}(f(L))\ll(T':M')$ and so

$$\mathrm{Ann}(L)\subseteq\mathrm{Ann}(f(L))\ll(T':_R M')=(f^{-1}(T'):_R M).$$

It implies that $\mathrm{Ann}(L)\ll(f^{-1}(T'):_R M)$ and the proof is complete.

(ii) Assume that $K/N\ll_{T/N}^{sa}M/N$ and also, $T\subseteq K+L$ for some submodule L of M. Then $T/N\subseteq(K+L)/N=K/N+L/N$ implies that

$$\mathrm{Ann}(L/N)\ll(T/N:M/N)=(T:M).$$

Since $\mathrm{Ann}(L)\subseteq\mathrm{Ann}(L/N)\ll(T:M)$, hence $\mathrm{Ann}(L)\ll(T:M)$. It concludes that $K\ll_T^{sa}M$ and by (ii), $N\ll_T^{sa}M$.

(iii) Since X is a finitely generated submodule of M, hence we have $S^{-1}(0:_R X)=(S^{-1}0:_{S^{-1}R} S^{-1}X)$. Suppose that $T\subseteq N+X$, we must show that $\mathrm{Ann}_R(X)\ll(T:_R M)$. Note that

$$S^{-1}T\subseteq S^{-1}(N+X)=S^{-1}N+S^{-1}X.$$

By assumption, $\mathrm{Ann}_{S^{-1}R}(S^{-1}X)\ll(S^{-1}T:_{S^{-1}R} S^{-1}M)$. Therefore

$$\begin{aligned}\mathrm{Ann}_{S^{-1}R}(S^{-1}X):&=(S^{-1}0:_{S^{-1}R} S^{-1}X)=S^{-1}(0:_R X)\\&\ll S^{-1}(T:_R M)=(S^{-1}T:_{S^{-1}R} S^{-1}M).\end{aligned}$$

This implies that $\mathrm{Ann}_R(X)\ll(T:_R M)$. Similarly, one can check that the converse is also true. □

Corollary 3.23. *Let $f: M\to M'$ be an R-epimorphism and $T'\leq M'$. If M' is a T'-sa-hollow module, then M is an $f^{-1}(T')$-sa-hollow module.*

Proof. Assume that K is a submodule of M. Then $f(K)\ll^{sa}_{T'} M'$ since M' is a T'-sa-hollow module. By Proposition 3.22 (i), $f^{-1}(f(K))$ is an $f^{-1}(T')$-sa-small submodule of M. Since $K\subseteq f^{-1}(f(K))$, by Proposition 3.22 (i), K is also an $f^{-1}(T')$-sa-small submodule of M. □

The following example shows that the converse of Proposition 3.22 (ii) is not true.

Example 3.24. Consider the $\mathbb{Z}$-module $M=\mathbb{Z}$. Take $T=2\mathbb{Z}$, $K=4\mathbb{Z}$ and $N=8\mathbb{Z}$. Then $8\mathbb{Z}$ is a $2\mathbb{Z}$-sa-small submodule of $\mathbb{Z}$, because if $2\mathbb{Z}\subseteq 8\mathbb{Z}+k\mathbb{Z}$, for some submodule $k\mathbb{Z}$ of $\mathbb{Z}$, then either $(k,8)=2$ or $(k,8)=1$. In any case, $0=\mathrm{Ann}_{\mathbb{Z}}(k\mathbb{Z})\ll(2\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z})=2\mathbb{Z}$, but $K/N=4\mathbb{Z}/8\mathbb{Z}$ is not a $2\mathbb{Z}/8\mathbb{Z}$-sa-small submodule of $\mathbb{Z}/8\mathbb{Z}$, because if

$$2\mathbb{Z}/8\mathbb{Z}\subseteq 4\mathbb{Z}/8\mathbb{Z}+k\mathbb{Z}/8\mathbb{Z}$$

for some submodule $k\mathbb{Z}/8\mathbb{Z}$ of $\mathbb{Z}/8\mathbb{Z}$, then $2\mathbb{Z}\subseteq 4\mathbb{Z}+k\mathbb{Z}=(4,k)\mathbb{Z}$. If $k=2$, then $\mathrm{Ann}_{\mathbb{Z}}(2\mathbb{Z}/8\mathbb{Z})=4\mathbb{Z}$ is not small in $(2\mathbb{Z}/8\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}/8\mathbb{Z})=2\mathbb{Z}$. If $k=1$, then $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z})=8\mathbb{Z}$ is not small in $(2\mathbb{Z}/8\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}/8\mathbb{Z})=2\mathbb{Z}$.

We recall that for an ideal I of a ring R the *radical* of I is defined by $rad(I)=\{x\in R\mid x^n\in I \text{ for some } n\in N\}$. Let N be a proper submodule of M. Then the *prime radical* of N, denoted by $rad(N)$, is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, then $rad(N)$ is defined to be M.

Lemma 3.25. *If $I\ll^{sa} R$, then $rad(I)\ll^{sa} R$.*

Proof. Let $rad(I)+J=R$ for some ideal J of R. Since

$$rad(I) + J \subseteq rad(I) + rad(J),\ rad(I) + rad(J) = R.$$

This implies that $rad(I + J) = R$ and so $I + J = R$. Hence $\mathrm{Ann}(J) \ll R$, since $I \ll^{sa} R$ and so $rad(I) \ll^{sa} R$. □

Theorem 3.26. *Let M be a finitely generated faithful multiplication R-module. If $N \ll^{sa} M$, then $rad(N) \ll^{sa} M$.*

Proof. By [12, Theorem 4], $(rad(N) : M) = rad(N : M)$. Since $N \ll^{sa} M$, by Proposition 3.20, $(N : M) \ll^{sa} R$ and by Lemma 3.25, $rad(N : M) \ll^{sa} R$. It concludes that $(rad(N) : M) \ll^{sa} R$. Again using Proposition 3.20, we find that $rad(N) \ll^{sa} M$. □

In the following theorem, we will present a condition that a sa-small submodule of M is not a direct summand of M.

Proposition 3.27. *Let N be a nonzero sa-small submodule of M such that $N + K = M$ for some submodule K of M. If $(N : K) + (K : N) = R$, then the sum $N + K$ is not direct.*

Proof. Contrary, assume that $N \cap K = 0$. Then clearly $(N : K) = \mathrm{Ann}(K)$ and $(K : N) = \mathrm{Ann}(N)$. Thus $(N : K)+(K : N) = \mathrm{Ann}(K)+\mathrm{Ann}(N) = R$. By assumption, $N \in \mathrm{S}^{sa}(M)$ and $N + K = M$, hence $\mathrm{Ann}(K) \ll R$ and so $\mathrm{Ann}(N) = R$. This implies that $N = 0$ which is a contradiction. □

We recall that two ideals I and J of R are comaximal ideals of R, whenever $I + J = R$.

Corollary 3.28. *Let I and J be nonzero comaximal ideals of R. If either I or J is sa-small ideal in R, then R is not semisimple.*

Corollary 3.29. *Every sa-hollow semisimple ring is a field.*

Proof. Assume that I is a nonzero ideal of R. By assumption, $I \subseteq^{\oplus} R$. Then by the proof of Proposition 3.27, I is not a sa-small ideal of R which is a contradiction. □

Theorem 3.30. *Let $f : M \to N$ be an R-monomorphism and $T \leq M$. If $K \ll^{sa}_{T} M$, then $f(K) \ll^{sa}_{f(T)} f(M)$.*

Proof. Assume that $f(T) \subseteq f(K) + L$ for some submodule L of $f(M)$. We must show that $\mathrm{Ann}(L) \ll (f(T) :_R f(M))$. Take $t \in T$, then $f(t) = f(k)+s$ for some $k \in K$ and $s \in L$. Hence $f(t - k) \in L$ and so $t - k \in f^{-1}(L)$. This implies that $t \in K + f^{-1}(L)$. Therefore $T \subseteq K + f^{-1}(L)$ and by assumption, $\mathrm{Ann}(f^{-1}(L)) \ll (T :_R M)$. Now, we claim that $(T :_R M) = (f(T) :_R f(M))$.

Clearly, if $r \in (T :_R M)$, then $rM \subseteq T$ and so $f(rM) = rf(M) \subseteq f(T)$ which implies that $r \in (f(T) :_R f(M))$. Conversely, let $r \in (f(T) :_R f(M))$, then $rf(M) = f(rM) \subseteq f(T)$. Since f is a monomorphism, hence

$$rM \subseteq f^{-1}(f(rM)) \subseteq f^{-1}(f(T)) = T.$$

□

Recall that an R-epimorphism $f : M \to N$ is a *T-sa-small epimorphism* for some submodule $T \leq M$, whenever $\mathrm{Ker}(f) \ll_T^{sa} M$.

Theorem 3.31. *Let $f : N \to K$ be an R-monomorphism and $T \leq N$. If $g : K \to M$ is an $f(T)$-sa-small epimorphism, then $g \circ f : N \to M$ is also a T-sa-small epimorphism.*

Proof. Assume that $T \subseteq \mathrm{Ker}(g \circ f) + X$ for some submodule X of N. We show that $\mathrm{Ann}(X) \ll (T : N)$. Since $\mathrm{Ker}(g \circ f) = f^{-1}(\mathrm{Ker}(g))$, we have

$$f(T) \subseteq f(\mathrm{Ker}(g \circ f)) + f(X) \subseteq \mathrm{Ker}(g) + f(X).$$

Since $\mathrm{Ker}(g) \ll_{f(T)}^{sa} K$, by assumption, $\mathrm{Ann}(f(X)) \ll (f(T) :_R K)$.

We have $(f(T) :_R K) \subseteq (T :_R N)$, because if $r \in (f(T) :_R K)$ and $x \in N$, then $rf(x) = f(rx) \in f(T)$. Since f is monomorphism, hence $rx \in f^{-1}(f(T)) = T$. It concludes that $r \in (T :_R N)$. Thus

$$\mathrm{Ann}(X) \ll (T :_R N),$$

since $\mathrm{Ann}(X) \subseteq \mathrm{Ann}(f(X))$, and the proof is complete. □

We recall that an R-module F is called *flat* if whenever $N \to K \to L$ is an exact sequence of R-modules, then $F \otimes N \to F \otimes K \to F \otimes L$ is an exact sequence as well. An R-module F is called *faithfully flat*, whenever $N \to K \to L$ is an exact sequence of R-modules if and only if

$$F \otimes N \to F \otimes K \to F \otimes L$$

is an exact sequence.

Theorem 3.32. *Let M be an R-module and F be a faithfully flat R-module and $N, T \leq M$. Then the following statements are true:*

(i) *If $F \otimes N$ is a sa-small submodule of $F \otimes M$, then N is a sa-small submodule of M.*
(ii) *If $F \otimes N$ is a $F \otimes T$-sa-small submodule of $F \otimes M$, then N is a T-sa-small submodule of M.*

Proof. (i) Let $N + K = M$ for some submodule K of M. Then

$$F \otimes (N + K) = F \otimes N + F \otimes K = F \otimes M.$$

By assumption, $\operatorname{Ann}(K) = \operatorname{Ann}(F \otimes K) \ll R$ and so $N \ll^{sa} M$, as we needed.

(ii) Let $T \subseteq N + K$. Then $F \otimes T \subseteq F \otimes (N + K) = F \otimes N + F \otimes K$. Now, since $F \otimes N \ll^{sa}_{F \otimes T} F \otimes M$, hence $\operatorname{Ann}(F \otimes K) \ll (F \otimes T :_R F \otimes M)$. It concludes that $\operatorname{Ann}(K) \ll (T :_R M)$ and so $N \ll^{sa}_T M$. □

Conclusions

Investigating the properties of small submodules and new generalizations of them has been of interest for many years, see [2, 3, 6]. This motivates the research in the direction of finding a new concept namely the semiannihilator small submodules of an R-module M with respect to an arbitrary submodule T of M which is related to small ideals of R. We call them T-sa-small submodules of M such that this class of submodules of M is in general different from the classical concept of small submodules and also class of sa-small submodules, but in certain conditions they are equivalent. For example, if we take $T = M$, then the notions of T-sa-small submodules and sa-small submodules are the same. In Theorem 3.7, we proved that in a local ring R the concepts of small, R-a-small and sa-small ideals are the same. In Theorem 3.9, we proved that in a faithful R-module M all submodules N of M such that $\operatorname{Ann}_R(N) \subseteq^e R$ are R-a-small submodules of M.

We showed that every T-sa-small submodule of M is a sa-small submodule of M, see Theorem 3.10 (i). Among various results, in Theorem 3.18 (ii), we proved that, if R is a von Neumann regular ring, then none of the finitely generated ideals of R is a sa-small ideal of R. In addition, we proved that in a faithful finitely generated multiplication R-module M there is a bijection between the class of T-sa-small submodules of M and the class of $(T : M)$-sa-small ideals of R (see Proposition 3.20). In Proposition 3.27, we proved that if an R-module M has a nonzero sa-small submodule such as N, where for some submodule K of M, $N + K = M$ and $(N : K) + (K : N) = R$, then M is not semisimple.

Several properties, examples and characterizations of annihilator small submodules and T-sa-small submodules have been investigated. Moreover, we investigated the properties and the behaviour of this structure under ring homomorphism, localization, direct sums and tensor product of them with a faithfully flat R-module, see Proposition 3.22, Example 3.24 and Theorems 3.26, 3.30, 3.31, 3.32.

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A NEW CLASS OF SMALL SUBMODULES

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یک رده‌بندی جدید از زیرمدول‌های کوچک

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فرض کنید R یک حلقه‌ی جابجایی یکدار با $1 \neq 0$ و M یک R-مدول یکانی غیرصفر باشد. در این مقاله، یک مفهوم جدید از زیرمدول‌های M را معرفی می‌کنیم که زیرمدول‌های T-نیم-پوچساز از M نسبت به یک زیرمدول دلخواه T از M نامیده می‌شوند. یک زیرمدول N از M یک زیرمدول T-نیم-پوچساز کوچک است در صورتی که برای هر زیرمدول X از M با $T \subseteq N + X$ ایجاب کند که $\mathrm{Ann}(X) \ll (T : M)$. علاوه بر این، تعدادی نتایج مرتبط با این رده‌ی جدید از این زیرمدول‌ها را بررسی می‌کنیم.

در میان نتایج مختلف، ما ثابت می‌کنیم که برای یک مدول ضربی متناهی مولد وفادار M، زیرمدول N از M یک زیرمدول T-نیم-پوچساز کوچک M است اگر و تنها اگر $(N : M)$ یک ایده‌آل $(T : M)$-نیم-پوچساز کوچک R باشد. سرانجام، ما خواص و رفتار این ساختار را تحت همریختی حلقه، موضعی سازی، جمع‌های مستقیم و ضرب تانسوری آن‌ها با یک R-مدول تخت وفادار را بررسی می‌کنیم.

کلمات کلیدی: زیرمدول کوچک، زیرمدول نیم-پوچساز کوچک، مدول ضربی.