

## (WEAKLY) $(s, n)$ -CLOSED HYPERIDEALS IN COMMUTATIVE MULTIPLICATIVE HYPERRINGS

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**ABSTRACT.** A multiplicative hyperring is a well-known type of algebraic hyperstructures which extends a ring to a structure in which the addition is an operation but the multiplication is a hyperoperation. Let  $G$  be a commutative multiplicative hyperring and  $s, n \in \mathbb{Z}^+$ . A proper hyperideal  $Q$  of  $G$  is called (weakly)  $(s, n)$ -closed if  $(0 \notin a^s \subseteq Q) \Rightarrow a^n \subseteq Q$  for  $a \in G$  implies  $a^n \subseteq Q$ . In this paper, we aim to investigate (weakly)  $(s, n)$ -closed hyperideals and give some results explaining the structures of these notions.

### 1. INTRODUCTION

The idea of 2-absorbing ideals, as an extension of prime ideals, was introduced by Badawi in [11]. This concept was generalized to a notion called  $n$ -absorbing ideal by Anderson and Badawi in [8]. More generally, for  $s, n \in \mathbb{Z}^+$ , they defined the concept of  $(s, n)$ -closed ideals in [9].

Hyperstructures, which are a generalization of classical algebraic structures, take an important place in both pure and applied mathematics. These structures were first introduced by Marty in 1934 [24]. He published some notes on hypergroups as a generalization of groups. Later on, many authors have worked on this new topic of modern algebra and developed it [13, 14, 12, 16, 15, 25, 27, 33]. Similar to hypergroups, hyperrings are algebraic structures, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. An important type of a hyperring called the multiplicative hyperring was introduced by Rota in 1982 [29]. In this hyperstructure, the multiplication is a hyperoperation, while the addition is an operation. Many illustrations and results of the multiplicative hyperring can be seen in [2, 1, 7, 23, 22, 21, 32]. The notion of primeness of hyperideal in a multiplicative hyperring was conceptualized by Procesi and Rota in [28]. Dasgupta extended the prime and primary hyperideals in multiplicative hyperrings in [18]. A generalization of prime hyperideals called

2-absorbing hyperideals was introduced in [6] and this concept was further generalized to a notion called  $n$ -absorbing hyperideals in [5].

In this paper, we introduce the notions of  $(s, n)$ -closed hyperideal and weakly  $(s, n)$ -closed hyperideal of a commutative multiplicative hyperring. Many specific results are presented to explain the structures of these notions. Additionally, we study the relationship between the new hyperideals and classical hyperideals and explore some ways to connect them.

## 2. SOME BASIC DEFINITIONS

In this section, we give some definitions and results which we need to develop our paper. A hyperoperation “ $\circ$ ” on nonempty set  $A$  is a mapping from  $A \times A$  into  $P^*(A)$  where  $P^*(A)$  is the family of all nonempty subsets of  $A$ . If “ $\circ$ ” is a hyperoperation on  $A$ , then  $(A, \circ)$  is called hypergroupoid. The hyperoperation on  $A$  can be extended to subsets of  $A$ . Let  $A_1, A_2$  be two subsets of  $A$  and  $a \in A$ , then  $A_1 \circ A_2 = \cup_{a_1 \in A_1, a_2 \in A_2} a_1 \circ a_2$ , and  $A_1 \circ a = A_1 \circ \{a\}$ . A hypergroupoid  $(A, \circ)$  is called a semihypergroup if  $\cup_{v \in b \circ c} a \circ v = \cup_{u \in a \circ b} u \circ c$  for all  $a, b, c \in A$  which means  $\circ$  is associative. A semihypergroup is said to be a hypergroup if  $a \circ A = A = A \circ a$  for all  $a \in A$ . A nonempty subset  $B$  of a semihypergroup  $(A, \circ)$  is called a subhypergroup if we have  $a \circ B = B = B \circ a$  for all  $a \in B$  [19].

**Definition 2.1.** [19] An algebraic structure  $(G, +, \circ)$  is said to be commutative multiplicative hyperring if

- (1)  $(G, +)$  is a commutative group;
- (2)  $(G, \circ)$  is a semihypergroup;
- (3) for all  $a, b, c \in G$ , we have  $a \circ (b + c) \subseteq a \circ b + a \circ c$  and  $(b + c) \circ a \subseteq b \circ a + c \circ a$ ;
- (4) for all  $a, b \in G$ , we have  $a \circ (-b) = (-a) \circ b = -(a \circ b)$ ;
- (5) for all  $a, b \in G$ ,  $a \circ b = b \circ a$ .

If in (3) the equality holds then we say that the multiplicative hyperring is strongly distributive. Let  $(\mathbb{Z}, +, \cdot)$  be the ring of integers. Corresponding to every subset  $X \in P^*(\mathbb{Z})$  with  $|X| \geq 2$ , there exists a multiplicative hyperring  $(\mathbb{Z}_X, +, \circ)$  with  $\mathbb{Z}_X = \mathbb{Z}$  and for any  $a, b \in \mathbb{Z}_X$ ,  $a \circ b = \{a.x.b \mid x \in X\}$  [18].

**Definition 2.2.** [3] An element  $e$  in  $G$  is an identity element if  $a \in a \circ e$  for all  $a \in G$ . Moreover, an element  $e$  in  $G$  is a scalar identity element if  $a = a \circ e$  for all  $a \in G$ .

Throughout this paper,  $G$  denotes a commutative multiplicative hyperring with identity 1.

**Definition 2.3.** [19] A non-empty subset  $H$  of  $G$  is a hyperideal if

- (i)  $a - b \in H$  for all  $a, b \in H$ ;
- (ii)  $r \circ a \subseteq H$  for all  $a \in H$  and  $r \in G$ .

**Definition 2.4.** [18] A proper hyperideal  $P$  of  $G$  is called a prime hyperideal if  $a \circ b \subseteq P$  for  $a, b \in G$  implies that  $a \in P$  or  $b \in P$ .

The intersection of all prime hyperideals of  $G$  containing a hyperideal  $I$  is called the prime radical of  $I$ , denoted by  $rad(I)$ . If the multiplicative hyperring  $G$  does not have any prime hyperideal containing  $I$ , we define  $rad(I) = G$ . Let  $\mathbf{C}$  be the class of all finite products of elements of  $G$  that is  $\mathbf{C} = \{r_1 \circ r_2 \circ \dots \circ r_n : r_i \in G, n \in \mathbb{N}\} \subseteq P^*(G)$ . A hyperideal  $I$  of  $G$  is said to be a  $\mathbf{C}$ -hyperideal of  $G$  if, for any  $A \in \mathbf{C}$ ,  $A \cap I \neq \emptyset$  implies  $A \subseteq I$ . Let  $I$  be a hyperideal of  $G$ . Then,  $D \subseteq rad(I)$  where  $D = \{r \in G \mid r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$  [18]. The equality holds when  $I$  is a  $\mathbf{C}$ -hyperideal of  $G$  [17, Proposition 3.2]. Recall that a hyperideal  $I$  of  $G$  is called a strong  $\mathbf{C}$ -hyperideal if for any  $E \in \mathfrak{U}$ ,  $E \cap I \neq \emptyset$ , then  $E \subseteq I$ , where  $\mathfrak{U} = \{\sum_{i=1}^n A_i \mid A_i \in \mathbf{C}, n \in \mathbb{N}\}$  and  $\mathbf{C} = \{r_1 \circ r_2 \circ \dots \circ r_n \mid r_i \in G, n \in \mathbb{N}\}$  (for more details see [17]).

**Definition 2.5.** [3] A proper hyperideal  $I$  of  $G$  is maximal in  $G$  if for any hyperideal  $J$  of  $G$  with  $I \subset J \subseteq G$  then  $J = G$ .

Also, we say that  $G$  is a local multiplicative hyperring, if it has just one maximal hyperideal.

**Definition 2.6.** [3] Let  $P$  and  $Q$  be hyperideals of  $G$ . We said that  $P$  and  $Q$  are coprime if  $P + Q = G$ .

### 3. $(s, n)$ -CLOSED HYPERIDEALS

In this section, we define the concept of  $(s, n)$ -closed hyperideal and present some basic properties of the  $(s, n)$ -closed hyperideals.

**Definition 3.1.** Let  $s, n \in \mathbb{Z}^+$ . A proper hyperideal  $Q$  of a commutative multiplicative hyperring  $G$  is called  $(s, n)$ -closed if  $a^s \subseteq Q$  for  $a \in G$  implies  $a^n \subseteq Q$ .

It is clear that every  $(s, n)$ -closed hyperideal of  $G$  is  $(s', n')$ -closed for all  $s, s', n, n' \in \mathbb{Z}^+$  with  $s' \leq s$  and  $n' \geq n$ .

**Example 3.2.** Consider the set of all integers  $\mathbb{Z}$  with ordinary addition and the hyperoperation  $\circ$  defined as  $\alpha \circ \beta = \{2\alpha\beta, 4\alpha\beta\}$  for all  $\alpha, \beta \in \mathbb{Z}$ .

Then  $Q = \langle 105 \rangle$  is an  $(s, 3)$ -closed hyperideal of the multiplicative hyperring  $(\mathbb{Z}, +, \circ)$  for all  $s \in \mathbb{Z}^+$ .

Clearly, every prime hyperideal of  $G$  is an  $(s, n)$ -closed for all  $s, n \in \mathbb{Z}^+$ . However, the next example shows that the converse is not true, in general.

**Example 3.3.** Consider the multiplicative hyperring  $(\mathbb{Z}, +, \circ)$  where  $+$  is the ordinary addition and the hyperoperation  $\circ$  is defined as  $a \circ b = \{2ab, 3ab\}$  for all  $a, b \in \mathbb{Z}$ . Then  $Q = \langle 6 \rangle$  is an  $(3, 2)$ -closed hyperideal of  $\mathbb{Z}$ , but it is not prime.

A proper hyperideal  $I$  of  $G$  refers to an  $n$ -absorbing hyperideal of  $G$  if  $x_1 \circ \cdots \circ x_{n+1} \subseteq I$  for  $x_1, \dots, x_{n+1} \in G$  implies that there exist  $n$  of the  $x_i$ 's whose product is a subset of  $I$  [5].

**Theorem 3.4.** *Let  $Q$  be an  $n$ -absorbing  $\mathbf{C}$ -hyperideal of  $G$ . Then  $Q$  is an  $(s, n)$ -closed hyperideal for each  $s \in \mathbb{Z}^+$ .*

*Proof.* Assume that  $Q$  is an  $n$ -absorbing hyperideal of  $G$  and  $a^s = a^n \circ a^{s-n} \subseteq Q$  for  $a \in G$  and  $s, n \in \mathbb{Z}^+$  with  $n < s$ . Then we have  $a^n \circ b \subseteq Q$  for every  $b \in a^{s-n}$ . Since  $Q$  is an  $n$ -absorbing hyperideal of  $G$ , we get  $a^n \in Q$  or  $a^{n-1} \circ b \in Q$ . In the first case, we are done. In the second case, we have  $a^{s-1} \subseteq Q$ . Consider  $a^{s-1} = a^n \circ a^{s-n-1} \subseteq Q$  and pick  $c \in a^{s-n-1}$ . By continuing the process mentioned, we obtain  $a^n \subseteq Q$ . Hence  $Q$  is  $(s, n)$ -closed for  $s > n$ . If  $s \leq n$ , then  $Q$  is  $(s, n)$ -closed clearly. Thus  $Q$  is an  $(s, n)$ -closed hyperideal for each  $s \in \mathbb{Z}^+$ .  $\square$

**Theorem 3.5.** *Let  $Q_1, \dots, Q_t$  be prime hyperideals of  $G$ . Then  $Q_1 \circ \cdots \circ Q_t$  is an  $(s, n)$ -closed hyperideal of  $G$  for all  $s, n \in \mathbb{Z}^+$  with  $1 \leq t$  and  $\min\{s, t\} \leq n$ .*

*Proof.* Assume that  $a^s \subseteq Q_1 \circ \cdots \circ Q_t$  for  $a \in G$ . This means  $a^s \subseteq Q_i$  for each  $i \in \{1, \dots, t\}$ . Since  $Q_i$  is a prime hyperideal of  $G$ , we have  $a \in Q_i$ . Hence  $a^t \subseteq Q_1 \circ \cdots \circ Q_t$ . This implies that  $a^n \subseteq Q_1 \circ \cdots \circ Q_t$  for all  $\min\{s, t\} \leq n$ .  $\square$

**Theorem 3.6.** *Let  $Q_1, \dots, Q_t$  be hyperideals of  $G$  such that for each  $i \in \{1, \dots, t\}$ ,  $Q_i$  is  $(s_i, n_i)$ -closed with  $s_i, n_i \in \mathbb{Z}^+$ . Then*

- (i)  $Q_1 \circ \cdots \circ Q_t$  is  $(s, n)$ -closed for  $s, n \in \mathbb{Z}^+$  such that  $s \leq \min\{s_1, \dots, s_t\}$  and  $n \geq \min\{s, n_1 + \cdots + n_t\}$ .
- (ii)  $Q_1 \cap \cdots \cap Q_t$  is  $(s, n)$ -closed for  $s, n \in \mathbb{Z}^+$  such that  $s \leq \min\{s_1, \dots, s_t\}$  and  $n \geq \min\{s, \max\{n_1, \dots, n_t\}\}$ .

*Proof.* (i) Assume that  $a^s \subseteq Q_1 \circ \cdots \circ Q_t$  for  $s \leq \min\{s_1, \dots, s_t\}$  and  $a \in G$ . This means  $a^s \subseteq Q_i$  for all  $i \in \{1, \dots, t\}$  and so  $a^{s_i} \subseteq Q_i$ . Since  $Q_i$  is  $(s_i, n_i)$ -closed, we get  $a^{n_i} \subseteq Q_i$ . Then we conclude that

$$a^{n_1} \circ \cdots \circ a^{n_t} = a^{n_1 + \cdots + n_t} \subseteq Q_1 \circ \cdots \circ Q_t.$$

This implies that  $a^n \subseteq Q_1 \circ \cdots \circ Q_t$  for  $n_1 \cdots + n_t \leq n$ . Consequently, we have  $a^n \subseteq Q_1 \circ \cdots \circ Q_t$  for all  $n \geq \min\{s, n_1 + \cdots + n_t\}$ .

(ii) Suppose that  $a^s \subseteq Q_1 \cap \cdots \cap Q_t$  for  $s \leq \min\{s_1, \dots, s_t\}$  and  $a \in G$ . Hence  $a^s \subseteq Q_i$  for all  $i \in \{1, \dots, t\}$  and then  $a^{s_i} \subseteq Q_i$ . Since  $Q_i$  is  $(s_i, n_i)$ -closed, we obtain  $a^{n_i} \subseteq Q_i$ . Thus we have  $a^n \subseteq Q_i$  for every  $i \in \{1, \dots, t\}$  and  $n \geq \max\{n_1, \dots, n_t\}$ . Hence  $a^n \subseteq Q_1 \cap \cdots \cap Q_t$  for all  $n \geq \min\{s, \max\{n_1, \dots, n_t\}\}$ .  $\square$

**Example 3.7.** Let  $(\mathbb{Z}, +, \cdot)$  be the ring of integers and  $A \in P^*(\mathbb{Z})$  with  $|A| \geq 2$ . Consider the multiplicative hyperring  $(\mathbb{Z}_A, +, \circ)$  where  $+$  is ordinary addition,  $\mathbb{Z}_A = \mathbb{Z}$  and  $a \circ b = \{axb \mid x \in A\}$  for all  $a, b \in \mathbb{Z}_A$ .

(i) Let  $A = \{5, 7\}$ . Put  $Q = \langle 858 \rangle$ ,  $Q_1 = \langle 6 \rangle$ ,  $Q_2 = \langle 11 \rangle$  and  $Q_3 = \langle 13 \rangle$ . Since  $Q_1$ ,  $Q_2$  and  $Q_3$  are  $(s_1, 2)$ -closed,  $(s_2, 1)$ -closed and  $(s_3, 1)$ -closed hyperideals of  $\mathbb{Z}_A$ , respectively, with  $s_1 \geq 3$  and  $s_2, s_3 \geq 2$ , we conclude that  $Q = Q_1 \circ Q_2 \circ Q_3$  is an  $(s, n)$ -closed hyperideal of  $\mathbb{Z}_A$  such that  $s \leq \min\{s_1, s_2, s_3\}$  and  $n \geq \min\{s, 4\}$ .

(ii) Let  $A = \{17, 19\}$ . Put  $Q = \langle 2310 \rangle$ ,  $Q_1 = \langle 6 \rangle$ ,  $Q_2 = \langle 35 \rangle$  and  $Q_3 = \langle 11 \rangle$ . Since  $Q = Q_1 \cap Q_2 \cap Q_3$  and  $Q_1$ ,  $Q_2$  and  $Q_3$  are  $(s_1, 2)$ -closed,  $(s_2, 2)$ -closed and  $(s_3, 1)$ -closed hyperideals of  $\mathbb{Z}_A$ , respectively, with  $s_1, s_2 \geq 3$  and  $s_3 \geq 2$ ,  $Q$  is an  $(s, n)$ -closed hyperideal of  $\mathbb{Z}_A$  such that  $s \leq \min\{s_1, s_2, s_3\}$  and  $n \geq \min\{s, \max\{1, 2\}\}$ .

In view of Theorem 3.6, the following results are obtained.

**Corollary 3.8.** *Let  $Q_1, \dots, Q_t$  be  $(s, n)$ -closed hyperideals of  $G$  for  $s, n \in \mathbb{Z}^+$ . Then  $Q_1 \cap \cdots \cap Q_t$  is an  $(s, n)$ -closed hyperideal of  $G$ .*

Recall from [3] that two hyperideals  $P$  and  $Q$  of  $G$  are coprime if  $P + Q = G$ .

**Corollary 3.9.** *Let  $Q_1, \dots, Q_t$  be  $(s, n)$ -closed hyperideals of  $G$  for  $s, n \in \mathbb{Z}^+$  such that  $Q_1, \dots, Q_t$  are pairwise coprime. Then  $Q_1 \circ \cdots \circ Q_t$  is an  $(s, n)$ -closed hyperideal of  $G$ .*

**Theorem 3.10.** *Let  $P$  and  $Q$  be two hyperideals of  $G$  such that  $Q$  is an  $(s, 2)$ -closed strong **C**-hyperideal for  $s \in \mathbb{Z}^+$ . If  $P^s \subseteq Q$ , then  $P^2 + P^2 \subseteq Q$ .*

*Proof.* Assume that  $a, b \in P$ . Since  $P^s \subseteq Q$ , we have  $a^s, b^s, (a + b)^s \subseteq Q$ . Since  $Q$  is an  $(s, 2)$ -closed hyperideal of  $G$ , we get  $a^2, b^2, (a + b)^2 \subseteq Q$ . Pick  $c \in (a + b)^2$ . Since  $(a + b)^2 \subseteq a^2 + a \circ b + a \circ b + b^2$ , we have  $c \in a^2 + d + b^2$  for some  $d \in a \circ b + a \circ b$  and so  $c - d \in a^2 + b^2 \subseteq Q$ . Then we get  $d \in Q$  as

$c \in (a + b)^2 \subseteq Q$ . Since  $d \in a \circ b + a \circ b$  and  $Q$  is a strong **C**-hyperideal of  $G$ , we obtain  $a \circ b + a \circ b \subseteq Q$ . Hence  $P^2 + P^2 \subseteq Q$ .  $\square$

We define the relation  $\gamma$  on a multiplicative hyperring  $(G, +, \circ)$  as follows:  $x\gamma y$  if and only if  $\{x, y\} \subseteq A$  where  $A$  is a finite sum of finite products of elements of  $G$ .

This means  $x\gamma y$  if and only if there exist  $z_1, \dots, z_n \in G$  that  $\{x, y\} \subseteq \sum_{j \in J} \prod_{i \in I_j} z_i$  and  $I_j, J \subseteq \{1, \dots, n\}$ . The transitive closure of  $\gamma$  is denoted by  $\gamma^*$ . The relation  $\gamma^*$  is the smallest equivalence relation on a multiplicative hyperring  $G$  such that the quotient  $G/\gamma^*$ , the set of all equivalence classes, is a fundamental ring. Let  $\mathbb{A}$  be the set of all finite sums of products of elements of  $G$ . The definition of  $\gamma^*$  can be rewritten on  $G$ , i.e.,  $x\gamma^*y$  if and only if there exist  $z_1, \dots, z_n \in G$  with  $z_1 = x, z_{n+1} = y$  and  $u_1, \dots, u_n \in \mathbb{A}$  such that  $\{z_i, z_{i+1}\} \subseteq u_i$  for  $i \in \{1, \dots, n\}$ . Suppose that  $\gamma^*(x)$  is the equivalence class containing  $x \in G$ . Define  $\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(z)$  for all  $z \in \gamma^*(x) + \gamma^*(y)$  and  $\gamma^*(x) \odot \gamma^*(y) = \gamma^*(w)$  for all  $w \in \gamma^*(x) \circ \gamma^*(y)$ . Then  $(G/\gamma^*, +, \odot)$  is a ring, which is called a fundamental ring of  $G$  [33].

**Theorem 3.11.** *Let  $Q$  be a hyperideal of  $G$ . Then  $Q$  is an  $(s, n)$ -closed hyperideal of  $(G, +, \circ)$  if and only if  $Q/\gamma^*$  is an  $(s, n)$ -closed ideal of  $(G/\gamma^*, \oplus, \odot)$ .*

*Proof.* ( $\implies$ ) Let  $\underbrace{a \odot \dots \odot a}_s \in Q/\gamma^*$  for some  $a \in G/\gamma^*$ . Then there exists  $x \in G$  with  $a = \gamma^*(x)$  and  $\underbrace{a \odot \dots \odot a}_s = \underbrace{\gamma^*(x) \odot \dots \odot \gamma^*(x)}_s = \gamma^*(x^s)$ . Since  $\gamma^*(x^s) \in Q/\gamma^*$ , we obtain  $x^s \subseteq Q$ . Since  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ , we conclude that  $x^n \subseteq Q$ . Thus

$$\underbrace{a \odot \dots \odot a}_n = \underbrace{\gamma^*(x) \odot \dots \odot \gamma^*(x)}_n = \gamma^*(x^n) \in Q/\gamma^*$$

which implies  $Q/\gamma^*$  is an  $(s, n)$ -closed ideal of  $G/\gamma^*$ .

( $\impliedby$ ) Assume that  $x^s \subseteq Q$  for some  $x \in G$ . Therefore  $\gamma^*(x) \in G/\gamma^*$  and  $\underbrace{\gamma^*(x) \odot \dots \odot \gamma^*(x)}_s = \gamma^*(x^s) \in Q/\gamma^*$ . Since  $Q/\gamma^*$  is an  $(s, n)$ -closed ideal of  $G/\gamma^*$ , we get  $\underbrace{\gamma^*(x) \odot \dots \odot \gamma^*(x)}_n = \gamma^*(x^n) \in Q/\gamma^*$  which means  $x^n \subseteq Q$ .

Consequently  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ .  $\square$

Let  $G$  be a multiplicative hyperring. Then we call  $M_m(G)$  as the set of all hypermatrices of  $G$ . Also, for all

$$A = (A_{ij})_{m \times m}, B = (B_{ij})_{m \times m} \in P^*(M_m(G)), A \subseteq B$$

if and only if  $A_{ij} \subseteq B_{ij}$  [3].

**Theorem 3.12.** *Let  $Q$  be a hyperideal of  $G$ . If  $M_m(Q)$  is an  $(s, n)$ -closed hyperideal of  $M_m(G)$ , then  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ .*

*Proof.* Assume that  $a^s \subseteq Q$  for some  $a \in G$ . Then

$$\begin{pmatrix} a^s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(Q).$$

Since

$$\begin{pmatrix} a^s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}}_s$$

and  $M_m(Q)$  is an  $(s, n)$ -closed hyperideal of  $M_m(G)$ , we get

$$\underbrace{\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \cdots \circ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}}_n \subseteq M_m(Q).$$

Therefore

$$\begin{pmatrix} a^n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_m(Q)$$

which implies  $a^n \subseteq Q$ . Thus  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ .  $\square$

Recall from [3] that a non-empty subset  $S$  of  $G$  with scalar identity 1 is said to be a multiplicative closed subset (MCS), if  $S$  contains 1 and  $S$  is closed under the hypermultiplication. Assume that  $S^{-1}G$  is the set  $(G \times S / \sim)$  of equivalence classes where

$$(a_1, t_1) \sim (a_2, t_2) \iff \exists t \in S \text{ such that } t \circ t_1 \circ a_2 = t \circ t_2 \circ a_1.$$

The equivalence class of  $(a, t) \in G \times S$  is denoted by  $\frac{a}{t}$ . The set  $S^{-1}G$  is a hyperring with  $\oplus$  and  $\odot$  defined by



$$\begin{aligned} \frac{a_1}{t_1} \oplus \frac{a_2}{t_2} &= \frac{t_1 \circ a_2 + t_2 \circ a_1}{t_1 \circ t_2} = \left\{ \frac{x+y}{z} \mid x \in t_1 \circ a_2, y \in t_2 \circ a_1, z \in t_1 \circ t_2 \right\} \\ \frac{a_1}{t_1} \odot \frac{a_2}{t_2} &= \frac{a_1 \circ a_2}{t_1 \circ t_2} = \left\{ \frac{x}{y} \mid x \in a_1 \circ a_2, y \in t_1 \circ t_2 \right\} \end{aligned}$$

The localization map  $\phi : G \longrightarrow S^{-1}G$ , defined by  $a \mapsto \frac{a}{1}$ , is a homomorphism of hyperrings. Moreover the localization of a hyperideal is a hyperideal [26].

**Theorem 3.13.** *Let  $Q$  be a  $\mathbf{C}$ -hyperideal of  $G$  and  $S$  a MCS such that  $Q \cap S = \emptyset$ . If  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ , then  $S^{-1}Q$  is an  $(s, n)$ -closed hyperideal of  $S^{-1}G$ .*

*Proof.* Let  $\underbrace{\frac{a}{t} \odot \cdots \odot \frac{a}{t}}_s = \frac{a^s}{t^s} \subseteq S^{-1}Q$  for some  $\frac{a}{t} \in S^{-1}G$ . Then we have

$\frac{x}{y} \in \frac{a^s}{t^s}$  for any  $x \in a^s = \underbrace{a \circ \cdots \circ a}_s$  and  $y \in t^s = \underbrace{t \circ \cdots \circ t}_s$ . Therefore

$\frac{x}{y} = \frac{x'}{y'}$  for some  $x' \in Q$  and  $y' \in S$ . Then there exists  $u \in S$  such that  $u \circ x \circ y' = u \circ x' \circ y$ . This means  $u \circ x \circ y' \subseteq Q$ . Since  $x \in a^s$ , we obtain  $u \circ x \circ y' \subseteq u \circ a^s \circ y'$ . Since  $Q$  is a  $\mathbf{C}$ -hyperideal of  $G$ , we get  $u \circ a^s \circ y' \subseteq Q$  and so  $u^s \circ a^s \circ y'^s = (u \circ a \circ y')^s \subseteq Q$ . Take any  $\alpha \in u \circ a \circ y'$ . Since  $\alpha^s \subseteq Q$  and  $Q$  is an  $(s, n)$ -closed hyperideal of  $G$ , we get  $\alpha^n \subseteq Q$ . Since  $Q$  is a  $\mathbf{C}$ -hyperideal of  $G$  and  $\alpha^n \subseteq (u \circ a \circ y')^n$ , we have  $(u \circ a \circ y')^n \subseteq Q$ . Hence  $\frac{a^n}{t^n} = \frac{u^n \circ a^n \circ y'^n}{u^n \circ t^n \circ y'^n} \subseteq S^{-1}Q$  which implies  $\underbrace{\frac{a}{t} \odot \cdots \odot \frac{a}{t}}_n \subseteq S^{-1}Q$ . Consequently,

$S^{-1}Q$  is an  $(s, n)$ -closed hyperideal of  $S^{-1}G$ .  $\square$

**Example 3.14.** Consider the hyperring of integers  $(\mathbb{Z}, +, \circ)$  such that  $x \circ y = \{xy, 5xy\}$  for any  $x, y \in \mathbb{Z}$ . Let  $S = \{5^n \mid n \geq 0\}$ . Then  $S$  is a multiplicative closed subset of  $\mathbb{Z}$ . Since  $Q = \langle 6 \rangle$  is a  $(3, 2)$ -closed  $\mathbf{C}$ -hyperideal of  $\mathbb{Z}$ ,  $S^{-1}Q$  is a  $(3, 2)$ -closed hyperideal of  $S^{-1}\mathbb{Z}$ .

Let us define  $\mathfrak{C}(Q) = \{(s, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid Q \text{ is } (s, n)\text{-closed}\}$  for some proper hyperideal  $Q$  of  $G$ . Then we have

$$\{(s, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 1 \leq s \leq n\} \subseteq \mathfrak{C}(Q) \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Moreover,  $rad(Q) = Q$  if and only if  $\mathfrak{C}(Q) = \mathbb{Z}^+ \times \mathbb{Z}^+$ .

**Theorem 3.15.** *Let  $Q$  be a proper hyperideal of  $G$  and  $s, n \in \mathbb{Z}^+$ . If  $(s, n), (s+1, n+1) \in \mathfrak{C}(Q)$  with  $s \neq n$ , then  $(s+1, n) \in \mathfrak{C}(Q)$ .*

*Proof.* Let  $s < n$ . Clearly, we have  $(s+1, n) \in \mathfrak{C}(Q)$ . Now, let us consider  $s > n$ . Assume that  $a^{s+1} \subseteq Q$  for some  $a \in G$ . Since  $(s+1, n+1) \in \mathfrak{C}(Q)$ ,



we get  $a^{n+1} \subseteq Q$ . Since  $n+1 \leq s$ , we obtain  $a^s \subseteq Q$  which implies  $a^n \subseteq Q$  as  $(s, n) \in \mathfrak{C}(Q)$ . Hence  $(s+1, n) \in \mathfrak{C}(Q)$ .  $\square$

*Remark 3.16.* Let  $Q$  be a proper hyperideal of  $G$  and  $s, n \in \mathbb{Z}^+$ . If  $(s, n) \in \mathfrak{C}(Q)$ , then  $(s', n') \in \mathfrak{C}(Q)$  for  $s', n' \in \mathbb{Z}^+$  such that  $1 \leq s' \leq s$  and  $n \leq n'$  by being  $Q$  a hyperideal.

**Theorem 3.17.** *Let  $Q$  be a proper  $\mathbf{C}$ -hyperideal of  $G$ . If  $(s, n) \in \mathfrak{C}(Q)$  for  $s, n \in \mathbb{Z}^+$  such that  $n \leq \frac{s}{2}$ , then  $(s+1, n) \in \mathfrak{C}(Q)$  and then  $(t, n) \in \mathfrak{C}(Q)$  for all  $t \in \mathbb{Z}^+$ .*

*Proof.* Assume that  $a^{s+1} \subseteq Q$  for some  $a \in G$ . Therefore we have  $(a^2)^s = a^{2s} \subseteq Q$ . Choose  $x \in a^2$ . So  $x^s \subseteq Q$ . Since  $(s, n) \in \mathfrak{C}(Q)$ , we get  $x^n \subseteq Q$ . From  $x \in a^2$  it follows that  $x^n \subseteq a^{2n}$ . Then we have  $a^{2n} \subseteq Q$  as  $Q$  is a  $\mathbf{C}$ -hyperideal of  $G$ . By the hypothesis, we have  $a^s \subseteq Q$ . Since  $(s, n) \in \mathfrak{C}(Q)$ , we get  $a^n \subseteq Q$  which implies  $(s+1, n) \in \mathfrak{C}(Q)$ . By a similar argument, we have  $(t, n) \in \mathfrak{C}(Q)$  for all  $t \in \mathbb{Z}^+$  such that  $t \geq n$ . Now, by Remark 3.16, we conclude that  $(t, n) \in \mathfrak{C}(Q)$  for all  $t \in \mathbb{Z}^+$ .  $\square$

Assume that  $Q$  is a proper hyperideal of  $G$ . Consider the mappings  $\omega_Q : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ , defined by  $\omega_Q(s) = \min\{n \mid Q \text{ is } (s, n)\text{-closed}\}$  and  $\Omega_Q : \mathbb{Z}^+ \mapsto \mathbb{Z}^+ \cup \{\infty\}$ , defined by  $\Omega_Q(n) = \sup\{s \mid Q \text{ is } (s, n)\text{-closed}\}$ . The rows of  $\mathfrak{C}(Q)$  determine  $\omega_Q$  and the columns of  $\mathfrak{C}(Q)$  determine  $\Omega_Q$ . Since  $(n, n) \in \mathfrak{C}(Q)$  for all  $n \in \mathbb{Z}^+$ , we have  $1 \leq \omega_Q(s) \leq s$ . Moreover, by Remark 3.16, we conclude that  $\omega_Q(s) \leq \omega_Q(s+1)$  and  $\Omega_Q(n) \leq \Omega_Q(n+1)$ .

*Remark 3.18.* Let  $P$  and  $Q$  be two proper hyperideals of  $G$ . It can be easily seen that  $\mathfrak{C}(P) \subseteq \mathfrak{C}(Q)$  for if and only if  $\omega_P(s) \leq \omega_Q(s)$  for all  $s \in \mathbb{Z}^+$  if and only if  $\Omega_P(n) \leq \Omega_Q(n)$  for all  $n \in \mathbb{Z}^+$ .

**Proposition 3.19.** *Let  $Q$  be a proper hyperideal of  $G$  and  $s, n \in \mathbb{Z}^+$ . If  $\omega_Q(s) < s$ , then either  $\omega_Q(s+1) = \omega_Q(s)$  or  $\omega_Q(s+1) \geq \omega_Q(s) + 2$ .*

*Proof.* Let  $\omega_Q(s+1) = \omega_Q(s) + 1$  and  $\omega_Q(s) = n$ . Then  $n < s$  and  $\omega_Q(s+1) = n+1$ . Hence we have  $(s+1, n+1), (s, n) \in \mathfrak{C}(Q)$ . By Theorem 3.15, we conclude that  $(s+1, n) \in \mathfrak{C}(Q)$  which implies  $\omega_Q(s+1) \leq n$  which is a contradiction.  $\square$

**Proposition 3.20.** *Let  $Q$  be a proper hyperideal of  $G$  and  $s, n \in \mathbb{Z}^+$ . If  $\Omega_Q(n) > n$ , then either  $\Omega_Q(n+1) = \Omega_Q(n)$  or  $\Omega_Q(n+1) \geq \Omega_Q(n) + 2$ .*

*Proof.* By using an argument similar to that in the proof of Proposition 3.19, one can easily complete the proof.  $\square$

**Theorem 3.21.** *Let  $P$  and  $Q$  be proper hyperideals of  $G$ . Then*

- (i)  $\omega_{P \cap Q} \leq \omega_P \vee \omega_Q$ .
- (ii)  $\Omega_P \wedge \Omega_Q \leq \Omega_{P \cap Q}$ .

*Proof.* Put  $\omega_P(s) = n'$  and  $\omega_Q(s) = n''$  for  $s \in \mathbb{Z}^+$  such that  $n = \max\{n', n''\}$ . By Remark 3.16 and Corollary 3.8, we have  $(s, n) \in \mathfrak{C}(P) \cap \mathfrak{C}(Q) \subseteq \mathfrak{C}(P \cap Q)$ . Therefore we conclude that  $\omega_{P \cap Q}(s) \leq n$ . Since  $n = (\omega_P \vee \omega_Q)(s)$ , we have  $\omega_{P \cap Q}(s) \leq (\omega_P \vee \omega_Q)(s)$ . (ii) This can be proved in a very similar manner to the way in which (i) was proved.  $\square$

**Proposition 3.22.** *Let  $P$  and  $Q$  be proper hyperideals of  $G$ . Then  $\omega_{P \cap Q} = \omega_P \vee \omega_Q$  if and only if  $\mathfrak{C}(P) \cap \mathfrak{C}(Q) = \mathfrak{C}(P \cap Q)$ .*

*Proof.* ( $\implies$ ) Let  $\omega_{P \cap Q} = \omega_P \vee \omega_Q$ . This means that  $\omega_P \leq \omega_{P \cap Q}$  and  $\omega_Q \leq \omega_{P \cap Q}$ . By Remark 3.18, we get  $\mathfrak{C}(P \cap Q) \subseteq \mathfrak{C}(P) \cap \mathfrak{C}(Q)$ . On the other hand, we have  $\mathfrak{C}(P) \cap \mathfrak{C}(Q) \subseteq \mathfrak{C}(P \cap Q)$  by Corollary 3.8. Consequently,  $\mathfrak{C}(P) \cap \mathfrak{C}(Q) = \mathfrak{C}(P \cap Q)$ .

( $\impliedby$ ) Let  $\mathfrak{C}(P) \cap \mathfrak{C}(Q) = \mathfrak{C}(P \cap Q)$ . This implies that  $\omega_P \leq \omega_{P \cap Q}$  and  $\omega_Q \leq \omega_{P \cap Q}$  by Remark 3.18. Therefore we obtain  $\omega_P \vee \omega_Q \leq \omega_{P \cap Q}$ . On the other hand, by Theorem 3.21, we get  $\omega_{P \cap Q} \leq \omega_P \vee \omega_Q$ . Consequently,  $\omega_{P \cap Q} = \omega_P \vee \omega_Q$ .  $\square$

**Proposition 3.23.** *Let  $P$  and  $Q$  be proper hyperideals of  $G$ . Then  $\Omega_{P \cap Q} = \Omega_P \wedge \Omega_Q$  if and only if  $\mathfrak{C}(P) \cap \mathfrak{C}(Q) = \mathfrak{C}(P \cap Q)$ .*

*Proof.* By an argument similar to that one given in Proposition 3.22, we are done.  $\square$

In view of Proposition 3.22 and Proposition 3.23, the following result is obtained.

**Corollary 3.24.** *Let  $P$  and  $Q$  be proper hyperideals of  $G$ . Then  $\omega_{P \cap Q} = \omega_P \vee \omega_Q$  in and only if  $\Omega_{P \cap Q} = \Omega_P \wedge \Omega_Q$*

#### 4. WEAKLY $(s, n)$ -CLOSED HYPERIDEALS

**Definition 4.1.** Let  $Q$  be a proper hyperideal of  $G$  and  $s, n \in \mathbb{Z}^+$ .  $Q$  refers to a weakly  $(s, n)$ -closed hyperideal if  $0 \notin x^s \subseteq Q$  for  $x \in G$  implies that  $x^n \subseteq Q$ .

**Example 4.2.** Consider the multiplicative hyperring  $(\mathbb{Z}_A, +, \circ)$  where  $\mathbb{Z}_A = \mathbb{Z}$ ,  $A = \{7, 11\}$ ,  $a \circ b = \{axb \mid x \in A\}$  for all  $a, b \in \mathbb{Z}_A$  and  $+$  is ordinary addition. Then  $\langle 390 \rangle$  is a weakly  $(s, 4)$ -closed hyperideal of  $\mathbb{Z}$  for all  $s \in \mathbb{Z}^+$ .

**Theorem 4.3.** *Let  $Q$  be an  $(s, n)$ -closed hyperideal of  $G$ . Then  $Q$  is a weakly  $(s, n)$ -closed of  $G$ .*

*Proof.* It is obvious.  $\square$

The following example shows that the converse of Theorem 4.3 is not always true.

**Example 4.4.** Consider the ring  $(\{\bar{0}, \bar{1}, \bar{2}, \dots, \bar{7}\} = \mathbb{Z}_8, \oplus, \odot)$  such that  $\bar{x} \oplus \bar{y}$  and  $\bar{x} \odot \bar{y}$  are remainder of  $\frac{x+y}{8}$  and  $\frac{x \cdot y}{8}$  where  $+$  and  $\cdot$  are ordinary addition and multiplication for all  $\bar{x}, \bar{y} \in \mathbb{Z}_8$ . Now, we define the hyperoperation  $\bar{x} \circ \bar{y} = \{\overline{xy}, \overline{2xy}, \overline{3xy}, \overline{4xy}, \overline{5xy}, \overline{6xy}, \overline{7xy}\}$ . Then the hyperideal  $Q = \{\bar{0}, \bar{4}\}$  of the multiplicative hyperring  $(\mathbb{Z}_8, \oplus, \circ)$  is weakly  $(3, 1)$ -closed but it is not  $(3, 1)$ -closed.

It can be easily verified that an intersection of weakly  $(s, n)$ -closed hyperideals of  $G$  is weakly  $(s, n)$ -closed. Furthermore, every weakly  $(s, n)$ -closed hyperideal of  $G$  is weakly  $(s, n')$ -closed for all  $n' \in \mathbb{Z}^+$  such that  $n' \geq n$ . Let  $Q$  be a weakly  $(s, n)$ -closed  $\mathbf{C}$ -hyperideal of  $G$ . We define  $x \in G$  to be an  $(s, n)$ -unbreakable-zero element of  $Q$  if  $0 \in x^s$  and  $x^n \notin Q$ . Indeed, it means that  $Q$  is not  $(s, n)$ -closed if and only if  $Q$  has an  $(s, n)$ -unbreakable-zero element.

**Theorem 4.5.** *Let  $Q$  be a weakly  $(s, n)$ -closed strong  $\mathbf{C}$ -hyperideal of  $G$ . If  $x$  is an  $(s, n)$ -unbreakable-zero element of  $Q$ , then  $0 \in (x + a)^s$  for all  $a \in Q$ .*

*Proof.* Assume that  $a \in Q$ . So  $\sum_{i=1}^s \binom{s}{i} x^{s-i} a^i \subseteq Q$ . On the other hand,  $0 + \sum_{i=1}^s \binom{s}{i} x^{s-i} a^i \subseteq x^s + \sum_{i=1}^s \binom{s}{i} x^{s-i} a^i$ . Since  $Q$  is a strong  $\mathbf{C}$ -hyperideal of  $G$ , we have  $x^s + \sum_{i=1}^s \binom{s}{i} x^{s-i} a^i \subseteq Q$  which implies  $(x + a)^s \subseteq Q$  as  $(x + a)^s \subseteq x^s + \sum_{i=1}^s \binom{s}{i} x^{s-i} a^i$ . If  $(x + a)^n \subseteq Q$ , then  $x^n + \sum_{i=1}^n \binom{n}{i} x^{n-i} a^i \subseteq Q$  because  $Q$  is a strong  $\mathbf{C}$ -hyperideal of  $G$ . Since  $\sum_{i=1}^n \binom{n}{i} x^{n-i} a^i \subseteq Q$ , we get  $x^n \subseteq Q$ , a contradiction. Then  $(x + a)^n \not\subseteq Q$ . Since  $Q$  is a weakly  $(s, n)$ -closed of  $G$ , we obtain  $0 \in (x + a)^s$ .  $\square$

Recall from [3] that an element  $x \in G$  is nilpotent if there exists an integer  $t$  such that  $0 \in x^t$ . The set of all nilpotent elements of  $G$  is denoted by  $\Upsilon$ .

**Theorem 4.6.** *If  $Q$  is a weakly  $(s, n)$ -closed strong  $\mathbf{C}$ -hyperideal of  $G$  such that is not  $(s, n)$ -closed, then  $Q \subseteq \Upsilon$ .*

*Proof.* Suppose that  $Q$  is a weakly  $(s, n)$ -closed strong  $\mathbf{C}$ -hyperideal of  $G$ . If  $Q$  is not  $(s, t)$ -closed, then there exists  $x \in G$  such that  $x$  is an  $(s, n)$ -unbreakable-zero element of  $Q$ . Assume that  $a \in Q$ . We have  $0 \in x^s$  which

implies  $0 \in (x + a)^s$  by Theorem 4.5. This means that  $x, x + a \in \Upsilon$ . Since by Theorem 3.5 in [3]  $\Upsilon$  is a hyperideal of  $G$ , we conclude that  $a = (x + a) - x \in \Upsilon$ . This means that  $Q \subseteq \Upsilon$ .  $\square$

A non-empty finite subset  $\xi = \{e_1, \dots, e_n\}$  of a multiplicative hyperring  $G$  is said to be  $i$ -set if  $e_i \neq 0$  for at least one  $1 \leq i \leq n$  and for every  $x \in G$ ,  $x \in \sum_{i=1}^n x \circ e_i$  [30].

**Theorem 4.7.** *Let  $G$  be a strongly distributive multiplicative hyperring with scalar identity  $e$  such that has an  $i$ -set and  $s, n \in \mathbb{Z}^+$  with  $s > n$ . Then every hyperideal of  $G$  contained in  $\Upsilon$  is weakly  $(s, n)$ -closed if and only if  $0 \in x^s$  for all  $x \in \Upsilon$ .*

*Proof.* ( $\implies$ ) Let  $0 \notin x^s$  for some  $x \in \Upsilon$ . Put  $Q = \langle x^s \rangle$ . So  $\langle x^s \rangle \subseteq \Upsilon$ . Since every hyperideal of  $G$  contained in  $\Upsilon$  is weakly  $(s, n)$ -closed,  $Q$  is a weakly  $(s, n)$ -closed hyperideal of  $G$ . Therefore  $0 \notin x^s \subseteq Q$  which implies  $x^n \subseteq Q$  and  $0 \notin x^n$ . Then for every  $a \in x^n$  there exist  $a_1, \dots, a_t \in G$  and  $x_1, \dots, x_t \in x^s$  such that  $a \in \sum_{i=1}^t a_i \circ x_i$ . Hence

$$0 \in \sum_{i=1}^t x_i \circ a_i - a \subseteq \sum_{i=1}^t x^s \circ a_i - x^n = x^n \circ (\sum_{i=1}^t x^{s-n} \circ a_i - e).$$

This means that  $0 \in x^n \circ c - e$  for some  $c \in \sum_{i=1}^t x^{s-n} \circ a_i$ . Since  $x \in \Upsilon$ , we have  $\sum_{i=1}^t x^{s-n} \circ a_i \in \Upsilon$ . Hence  $c - e$  is invertible by Theorem 3.20 in [4]. Thus we have  $0 \in x^n$  which is a contradiction.

( $\impliedby$ ) Let  $0 \in x^s$  for all  $x \in \Upsilon$ . Suppose that  $Q$  is a hyperideal of  $G$  such that  $Q \subseteq \Upsilon$  and  $0 \notin a^s \subseteq Q$  for some  $a \in G$  but  $a^n \notin Q$ . Then there exist  $c \in a^n$  such that  $c \notin Q$ . If  $c \in \Upsilon$ , then  $0 \in c^s \subseteq a^{sn}$  and so  $a \in \Upsilon$ . Hence  $0 \in a^s$ , a contradiction. If  $c \notin \Upsilon$ , then  $0 \notin c^s$ . Since  $c^s \subseteq a^{ns} \subseteq Q$ , we have  $c^s \subseteq \Upsilon$ . Let  $d \in c^s$ . Then  $0 \in d^s \subseteq c^{s^2}$  which means  $c \in \Upsilon$ , a contradiction.  $\square$

Let  $s, n \in \mathbb{Z}^+$ . An element  $a \in G$  is  $(s, n)$ -regular if  $a^n \subseteq a^s \circ b$  for some  $b \in G$ . An element  $a \in G$  is  $(s, n)$ -Regular if  $a^n \subseteq a^s \circ B$  for some subset  $B \neq a^{n-s}$  of  $G$ . Recall from [4] that an element  $a \in G$  is called weak zero divisor if for  $0 \neq b \in G$  we have  $0 \in a \circ b$ . Denote the set of weak zero divisor by  $Z_w(G)$ . Also, recall from [3] that an element  $x \in G$  is said to be invertible if there exists  $y \in G$ , such that  $e \in x \circ y$ . The set of all invertible elements in  $G$  is denoted by  $U(G)$ .

**Theorem 4.8.** *Assume that  $G$  is a strongly distributive multiplicative hyperring with scalar identity  $e$ ,  $a \in G \setminus (Z_w(G) \cup U(G))$  and  $s, n \in \mathbb{Z}^+$ . Then  $a$  is  $(s, n)$ -regular if and only if  $s \leq n$ .*

*Proof.* Let  $a$  be  $(s, n)$ -regular and  $s > n$ . From  $a^n \subseteq a^s \circ b$  for some  $b \in G$ , it follows that  $0 \in a^s \circ b - a^n = a^n \circ (a^{s-n} \circ b - e)$ . By the hypothesis, we have  $0 \in a^{s-n} \circ b - e$  which implies  $e \in a^{s-n} \circ b$ . This means  $e \in a \circ d$  for some  $d \in a^{s-n-1} \circ b$ . Then  $a \in U(G)$  which is a contradiction. The converse is clear.  $\square$

**Theorem 4.9.** *Let  $s, n \in \mathbb{Z}^+$  with  $s > n$ . If  $a$  is  $(s, n)$ -regular, then  $a$  is  $(s+1, n)$ -Regular.*

*Proof.* Assume that  $a$  is  $(s, t)$ -regular such that  $s > n$ . From  $a^n \subseteq a^s \circ b$  for some  $b \in G$ , it follows that

$$a^n \subseteq a^s \circ b = a^n \circ (a^{s-n} \circ b) \subseteq (a^s \circ b) \circ (a^{s-n} \circ b) = a^{s+1} \circ (a^{s-n-1} \circ b^2).$$

Put  $B = a^{s-n-1} \circ b^2$ . Thus we have  $a^n \subseteq a^{s+1} \circ B$  which means  $a$  is  $(s+1, n)$ -Regular.  $\square$

**Theorem 4.10.** *If  $a \in U(G)$ , then  $a$  is  $(s, n)$ -Regular for all  $s, n \in \mathbb{Z}^+$ .*

*Proof.* Let  $a \in U(G)$ . Then there exists  $b \in G$  such that  $e \in a \circ b$ . So we have  $e \in a^{s-n} \circ b^{s-n}$ . Therefore  $a^n \circ e \subseteq a^s \circ b^{s-n}$ . If  $x \in a^n$ , then  $x \in x \circ e \subseteq a^n \circ e$  which implies  $a^n \subseteq a^n \circ e$ . Hence we get  $a^n \subseteq a^s \circ b^{s-n}$ . Put  $B = b^{s-n}$ . Then  $a^n \subseteq a^s \circ B$  which means  $a$  is  $(s, n)$ -Regular.  $\square$

**Theorem 4.11.** *Let  $G$  be a strongly distributive multiplicative hyperring that has an  $i$ -set and  $s, n \in \mathbb{Z}^+$  with  $s > n$ . Then every proper hyperideal of  $G$  is weakly  $(s, n)$ -closed if and only if every non-nilpotent element of  $G$  is  $(s, n)$ -Regular and  $0 \in a^s$  for all  $a \in \Upsilon$ .*

*Proof.* ( $\implies$ ) Since every hyperideal of  $G$  contained in  $\Upsilon$  is weakly  $(s, n)$ -closed, we have  $0 \in a^s$  for all  $a \in \Upsilon$  by Theorem 4.7. Let  $a \in G \setminus \Upsilon$ . Assume that  $a \in U(G)$ . Therefore  $a$  is  $(s, n)$ -Regular by Theorem 4.10. Now, assume that  $a \notin U(G)$ . Put  $Q = \langle a^s \rangle$ . By the hypothesis,  $Q$  is a weakly  $(s, n)$ -closed hyperideal of  $G$ . Therefore  $0 \notin a^s \subseteq Q$  which implies  $a^n \subseteq Q$ . Then for every  $x \in a^n$  there exist  $b_{1_x}, \dots, b_{t_x} \in a^s$  and  $c_{1_x}, \dots, c_{t_x} \in G$  such that  $x \in \sum_{i=1}^{t_x} b_{i_x} \circ c_{i_x} \subseteq \sum_{i=1}^{t_x} a^s \circ c_{i_x} = a^s \circ \sum_{i=1}^{t_x} c_{i_x}$ . Put  $c_x = \sum_{i=1}^{t_x} c_{i_x}$ . Then we obtain  $a^n = \bigcup_{x \in a^n} \{x\} \subseteq \bigcup_{x \in a^n} a^s \circ c_x = a^s \circ \bigcup_{x \in a^n} c_x$ . Now, put  $B = \bigcup_{x \in a^n} c_x$ . Thus,  $a^n \subseteq a^s \circ B$  which means  $a$  is  $(s, n)$ -Regular.

( $\impliedby$ ) Assume that  $Q$  is a proper hyperideal of  $G$  and  $0 \notin a^s \subseteq Q$  for some  $a \in G$ . So  $a \notin \Upsilon$ . By the hypothesis,  $a$  is  $(s, n)$ -Regular. This means  $a^n \subseteq a^s \circ B$  for some subset  $B$  of  $G$ . Since  $a^s \circ B \subseteq Q$ , we have  $a^n \subseteq Q$ . Consequently,  $Q$  is weakly  $(s, n)$ -closed.  $\square$

Assume that  $(G_1, +_1, \circ_1)$  and  $(G_2, +_2, \circ_2)$  are two multiplicative hyperrings. Recall from [19] that a mapping  $\psi$  from  $G_1$  into  $G_2$  is said to be a hyperring good homomorphism if for all  $x, y \in G_1$ ,  $\psi(x +_1 y) = \psi(x) +_2 \psi(y)$  and  $\psi(x \circ_1 y) = \psi(x) \circ_2 \psi(y)$ .

**Theorem 4.12.** *Let  $G_1$  and  $G_2$  be two multiplicative hyperrings,  $\psi : G_1 \longrightarrow G_2$  a hyperring good homomorphism and  $s, n \in \mathbb{Z}^+$ .*

- (1) *If  $Q_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_2$  and  $\psi$  is injective, then  $\psi^{-1}(Q_2)$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$ .*
- (2) *If  $\psi$  is surjective and  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$  with  $\text{Ker}(\psi) \subseteq Q_1$ , then  $\psi(Q_1)$  is a weakly  $(s, n)$ -closed hyperideal of  $G_2$ .*

*Proof.* (1) Let  $0 \notin a^s \subseteq \psi^{-1}(Q_2)$  for some  $a \in G_1$ . Then we get  $0 \notin \psi(a^s) = \psi(a)^s \subseteq Q_2$  as  $\psi$  is injective. Since  $Q_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_2$ , we obtain  $(\psi(a))^n \subseteq Q_2$  which implies  $\psi(a^n) \subseteq Q_2$  which means  $a^n \subseteq \psi^{-1}(Q_2)$ . Thus  $\psi^{-1}(Q_2)$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$ .

(2) Let  $0 \notin b^s \subseteq \psi(Q_1)$  for some  $b \in G_2$ . Then  $\psi(a) = b$  for some  $a \in G_1$  as  $\psi$  is surjective. Therefore  $\psi(a^s) = \psi(a)^s \subseteq \psi(Q_1)$ . Now, pick any  $x \in a^s$ . Then  $\psi(x) \in \psi(a^s) \subseteq \psi(Q_1)$  and so there exists  $y \in Q_1$  such that  $\psi(x) = \psi(y)$ . Then we have  $\psi(x - y) = 0$  which means  $x - y \in \text{Ker}(\psi) \subseteq Q_1$  and so  $x \in Q_1$ . So  $a^s \subseteq Q_1$ . Since  $Q_1$  is weakly  $(s, n)$ -closed and  $0 \notin a^s$ , we get  $a^n \subseteq Q_1$ . This means that  $\psi(a^n) = b^n \subseteq \psi(Q_1)$ . Consequently,  $\psi(Q_1)$  is a weakly  $(s, n)$ -closed hyperideal of  $G_2$ .  $\square$

Now, the following result obtained by the previous theorem directly.

**Corollary 4.13.** *Let  $P$  and  $Q$  be two hyperideals of  $G$  with  $P \subseteq Q$  and  $s, n \in \mathbb{Z}^+$ . If  $Q$  is a weakly  $(s, n)$ -closed hyperideal of  $G$ , then  $Q/P$  is a weakly  $(s, n)$ -closed hyperideal of  $G/P$ .*

Let  $(G_1, +_1, \circ_1)$  and  $(G_2, +_2, \circ_2)$  be two multiplicative hyperrings with non zero identity. The set  $G_1 \times G_2$  with the operation  $+$  and the hyperoperation  $\circ$  defined as

$$(x_1, x_2) + (y_1, y_2) = (x_1 +_1 y_1, x_2 +_2 y_2)$$

$$(x_1, x_2) \circ (y_1, y_2) = \{(x, y) \in R_1 \times R_2 \mid x \in x_1 \circ_1 y_1, y \in x_2 \circ_2 y_2\}$$

is a multiplicative hyperring [32]. Now, we give some characterizations of weakly  $(s, n)$ -closed hyperideals on cartesian product of commutative multiplicative hyperring.



**Theorem 4.14.** *Let  $(G_1, +_1, \circ_1)$  and  $(G_2, +_2, \circ_2)$  be two multiplicative hyperringes with scalar identities  $e_1$  and  $e_2$  respectively,  $Q_1$  a  $\mathbf{C}$ -hyperideal of  $G_1$  and  $s, n \in \mathbb{Z}^+$ . Then the followings are equivalent.*

- (i)  $Q_1 \times G_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1 \times G_2$ .
- (ii)  $Q_1$  is an  $(s, n)$ -closed hyperideal of  $G_1$ .
- (iii)  $Q_1 \times G_2$  is an  $(s, n)$ -closed hyperideal of  $G_1 \times G_2$ .

*Proof.* (i)  $\implies$  (ii) Assume that  $Q_1 \times G_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1 \times G_2$ . By Theorem 4.12 (2), we conclude that  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$ . Suppose that  $Q_1$  is not an  $(s, n)$ -closed hyperideal of  $G_1$ . This implies that  $Q_1$  has an  $(s, n)$ -unbreakable-zero element  $x$ . Hence  $0 \in x^s$  and  $x^n \notin Q_1$ . This implies that  $(0, 0) \notin (x, e_2)^s \subseteq Q_1 \times G_2$  and  $(x, e_2)^n \not\subseteq Q_1 \times G_2$  which is a contradiction since  $Q_1 \times G_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1 \times G_2$ . Thus  $Q_1$  is an  $(s, n)$ -closed hyperideal of  $G_1$ .

(ii)  $\implies$  (iii) The claim follows by Theorem 2.12 in [9] and Theorem 3.11.

(iii)  $\implies$  (i) This follows directly from the denitions.  $\square$

**Lemma 4.15.** *Let  $I_1$  and  $I_2$  be hyperideals of the multiplicative hyperringes  $(G_1, +_1, \circ_1)$  and  $(G_2, +_2, \circ_2)$ , respectively. Then  $I_1$  and  $I_2$  are  $\mathbf{C}$ -hyperideals if and only if  $I_1 \times I_2$  is  $\mathbf{C}$ -hyperideal of  $G_1 \times G_2$ .*

*Proof.* ( $\implies$ ) Let  $I_1$  and  $I_2$  be  $\mathbf{C}$ -hyperideals of  $G_1$  and  $G_2$ , respectively and  $(a_1, b_1) \circ \dots \circ (a_n, b_n) \cap I_1 \times I_2 \neq \emptyset$  for some  $a_1, \dots, a_n \in G_1$  and  $b_1, \dots, b_n \in G_2$ . This means  $(a, b) \in (a_1, b_1) \circ \dots \circ (a_n, b_n)$  for some  $(a, b) \in I_1 \times I_2$ . Therefore we have  $a \in a_1 \circ_1 \dots \circ_1 a_n$  and  $b \in b_1 \circ_2 \dots \circ_2 b_n$ . Since  $I_1$  and  $I_2$  are  $\mathbf{C}$ -hyperideals, we get  $a_1 \circ_1 \dots \circ_1 a_n \subseteq I_1$  and  $b_1 \circ_2 \dots \circ_2 b_n \subseteq I_2$ . This implies that  $(a_1, b_1) \circ \dots \circ (a_n, b_n) \subseteq I_1 \times I_2$ , as needed.

( $\impliedby$ ) Let  $a_1 \circ_1 \dots \circ_1 a_n \cap I_1 \neq \emptyset$  for some  $a_1, \dots, a_n \in G_1$  and

$$b_1 \circ_2 \dots \circ_2 b_n \cap I_2 \neq \emptyset$$

for some  $b_1, \dots, b_n \in G_2$ . This means  $(a_1, b_1) \circ \dots \circ (a_n, b_n) \cap I_1 \times I_2 \neq \emptyset$ . Since  $I_1 \times I_2$  is  $\mathbf{C}$ -hyperideal of  $G_1 \times G_2$ , we have  $(a_1, b_1) \circ \dots \circ (a_n, b_n) \subseteq I_1 \times I_2$  which means  $a_1 \circ_1 \dots \circ_1 a_n \subseteq I_1$  and  $b_1 \circ_2 \dots \circ_2 b_n \subseteq I_2$ , as claimed.  $\square$

**Theorem 4.16.** *Let  $(G_1, +_1, \circ_1)$  and  $(G_2, +_2, \circ_2)$  be two multiplicative hyperringes with scalar identities  $e_1$  and  $e_2$ , repectively, and  $s, n \in \mathbb{Z}^+$ . Then the followings are equivalent.*

- (i)  $Q$  is a weakly  $(s, n)$ -closed  $\mathbf{C}$ -hyperideal of  $G_1 \times G_2$  that is not  $(s, n)$ -closed.
- (ii)  $Q = Q_1 \times Q_2$  for some  $\mathbf{C}$ -hyperideals  $Q_1$  and  $Q_2$  of  $G_1$  and  $G_2$ , respectively, such that one of the following is hold.

- (1)  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$  that is not  $(s, n)$ -closed,  $0 \in b^s$  whenever  $b^s \subseteq Q_2$  for  $b \in G_2$ . If  $0 \notin a^s \subseteq Q_1$  for  $a \in G_1$ , then  $Q_2$  is an  $(s, n)$ -closed hyperideal of  $G_2$ ,
- (2)  $Q_2$  is a weakly  $(s, n)$ -closed hyperideal of  $G_2$  that is not  $(s, n)$ -closed,  $0 \in b^s$  whenever  $b^s \subseteq Q_1$  for  $b \in G_1$ . If  $0 \notin a^s \subseteq Q_2$  for  $a \in G_2$ , then  $Q_1$  is an  $(s, n)$ -closed hyperideal of  $G_1$ .

*Proof.* (i)  $\implies$  (ii) Assume that  $Q$  is a weakly  $(s, n)$ -closed **C**-hyperideal of  $G_1 \times G_2$  that is not  $(s, n)$ -closed. Then  $Q = Q_1 \times Q_2$  for some hyperideals  $Q_1$  and  $Q_2$  of  $G_1$  and  $G_2$ , respectively, such that one of them is weakly  $(s, n)$ -closed but is not  $(s, n)$ -closed by Theorem 4.14. We may assume that  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$  but is not  $(s, n)$ -closed. Then we conclude that  $Q_1$  has an  $(s, n)$ -unbreakable-zero element  $a_1$  which means  $0 \in a_1^s$  and  $a_1^n \not\subseteq Q_1$ . Let  $b^s \subseteq Q_2$  for some  $b \in G_2$ . Since  $(a_1, b)^s \subseteq Q$ , we get  $(0, 0) \in (a_1, b)^s$  which means  $0 \in b^s$ . Suppose that  $0 \notin a^s \subseteq Q_1$  for  $a \in G_1$ . Let  $b \in G_2$  with  $b^s \subseteq Q_2$ . Therefore  $(0, 0) \notin (a, b)^s \subseteq Q$ . This means that  $b^n \subseteq Q_2$  which implies  $Q_2$  is an  $(s, n)$ -closed hyperideal of  $G_2$ . Similiar for the other case.

(ii)  $\implies$  (i) Assume that  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$  that is not  $(s, n)$ -closed. Suppose that  $x$  is an  $(s, n)$ -unbreakable-zero element of  $Q_1$ . This means that  $(x, 0)$  is an  $(s, n)$ -unbreakable-zero element of  $Q$ . Hence  $Q$  is not an  $(s, n)$ -closed. Let  $(0, 0) \notin (a, b)^s \subseteq Q$  for  $a \in G_1$  and  $b \in G_2$ . Since  $0 \in b^s$ , we get  $0 \notin a^s \subseteq Q_1$ . Then  $(a, b)^n \subseteq Q$  as  $Q_1$  is a weakly  $(s, n)$ -closed hyperideal of  $G_1$  and  $Q_2$  is an  $(s, n)$ -closed hyperideal of  $G_2$ . Consequently,  $Q$  is a weakly  $(s, n)$ -closed **C**-hyperideal of  $G_1 \times G_2$  that is not  $(s, n)$ -closed.  $\square$

## 5. CONCLUSION

In this paper, we transfered the idea of  $(s, n)$ -closed ideals in multiplicative hyperrings and named it as  $(s, n)$ -closed hyperideals. Some main results were given to explain the structure of this concept. Moreover, we extended this notion to weakly  $(s, n)$ -closed hyperideals. Some properties of this concept were presented. We explored the behaviour of these hyperideals under homomorphism of hyperrings. We also gave some characterizations of weakly  $(s, n)$ -closed hyperideals on cartesian product of commutative multiplicative hyperrings.

## 6. FUTURE WORK

**Definition 6.1.** Let  $\mathcal{HI}(G)$  be the set of hyperideals of  $G$ ,  $\phi : \mathcal{HI}(G) \longrightarrow \mathcal{HI}(G) \cup \{\emptyset\}$

a function and  $s, n \in \mathbb{Z}^+$ . A proper hyperideal  $Q$  of  $G$  is called a  $\phi - (s, n)$ -closed hyperideal if  $a^s \subseteq Q - \phi(Q)$  for  $a \in G$  implies  $a^n \subseteq Q$ .

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(WEAKLY)  $(s, n)$ -CLOSED HYPERIDEALS IN COMMUTATIVE  
MULTIPLICATIVE HYPERRINGS

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ابرایده‌آل‌های  $(s, n)$ -بسته (ضعیف) در ابرحلقه‌های ضربی جابه‌جایی

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یک ابرحلقه ضربی نوع معروفی از ابرساختارهای جبری است که حلقه را به ساختاری تعمیم می‌دهد که در آن جمع یک عمل دوتایی و ضرب یک ابرعمل دوتایی است. فرض کنید  $G$  یک ابرحلقه ضربی جابه‌جایی و  $s, n$  دو عدد صحیح باشند. ابرایده‌آل سره‌ای  $Q$  از  $G$  را  $(s, n)$ -بسته (ضعیف) می‌نامیم هرگاه برای هر  $a \in G$  اگر  $a^s \subseteq Q$  و  $a^n \subseteq Q$ ، آنگاه  $a^n \subseteq Q$ . در این مقاله قصد داریم به بررسی ابرایده‌آل‌های  $(s, n)$ -بسته (ضعیف) پرداخته و بعضی نتایجی که ساختار این مفهوم را توضیح می‌دهند ارائه دهیم.

کلمات کلیدی: ابرایده‌آل  $(s, n)$ -بسته، ابرایده‌آل  $(s, n)$ -بسته ضعیف، عنصر  $(s, n)$ -ناگسستی-صفر.