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ON TYPE KRULL DIMENSION OF MODULES

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Abstract. In this paper, the concept of type Krull dimension of a module is introduced and some related properties are investigated. Using this concept, we extend some of the basic results about modules with Krull dimension. It is shown that every module with homogeneous type Krull dimension has type Krull dimension equal to zero. Also, it is proved that an R-module M has type Krull dimension if and only if it has type Noetherian dimension. We observe that, every module with Krull dimension has type Krull dimension, but its converse is not true in general. Further, we define *t-Artinian* (resp., *t-Noetherian*) modules and it is shown that if M is a *t-Artinian* module with type Krull dimension, then it has Krull dimension and these two dimensions for M coincide. At the end, we define the concept of α-DICCT modules and it is proved that an R-module M is α-DICCT if and only if it has type Krull (resp., Noetherian) dimension.

1. Introduction

In 1967, Rentschler and Gabriel introduced the concept of the Krull dimension of a module M, which measure its deviation from being Artinian for finite ordinals, in [15], and then by Krause, it was extended to modules over non-commutative rings for each ordinal number, in [11]. Dual of this concept, that is, the Noetherian dimension almost simultaneously, it was introduced and investigated by Karamzadeh [8] and Lemonnier [13] and these are the two most important research fields of non-commutative rings. According to the concept of Krull (resp., Noetherian) dimension of a module, various dimensions are defined on certain modules such as small, essential and parallel modules, etc, see [10, 1, 2, 16]. Motivated by these concepts, in this paper, we define the concept of type Krull (resp., Noetherian) dimension of a module which is measure its deviation (codeviation) type submodules from being Artinian (resp., Noetherian) and extend some of the basic results about modules with this dimension. The double infinite chain condition was introduced by Contessa for modules over commutative rings (DICC-modules for short), see [5, 3, 4]. In [14], Osofsky extended the concept of DICC to

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objects in AB5* category. She characterized DICC objects in this category and obtained some non-commutative generalizations. In [9], Karamzadeh et al. generalized this concept to α-DICC modules over non-commutative rings, where α is an ordinal number. They also gived a structure theorem for α-DICC modules and proved that α-DICC modules are just modules with Krull dimension (equivalently, Noetherian dimension). Hence, DICC modules have Krull dimension and most of the results that Contessa proved, are even true for non-commutative rings with Krull dimension. Most recently, Shirali et al. studied modules that satisfy the double infinite chain condition of parallel submodules. In this paper, we studied modules that satisfy the double infinite chain condition of type submodules. Let us give a brief outline of this article. After reviewing some necessary preliminaries, in Section 3, we investigate some basic properties of modules with type Krull dimension. For instance, we show that if for every type submodule A of M, A or $\frac{M}{A}$ have type Krull dimension, then so does M. Also, we show that, if every non-essential submodule of an R-module M is finitely generated, then M has type Krull dimension equal to zero. In Section 4, we define concept of *t-Artinian* (resp., *t-Noetherian*) modules and we show that a module M is *t-Artinian* if and only if for every non-empty set $\mathcal{F}$ of submodules of M, there exists $N \in \mathcal{F}$ such that for every $N' \subseteq N$, if $N' \in \mathcal{F}$, then $N' \subseteq_t M$. Finally, in the last section we study α-DICCT modules. In particular, it is proved that an R-module M is α-DICCT if and only if for every type submodule A of M, $tk\text{-dim}\,(A) \leq \alpha$ or $tn\text{-dim}\,(\frac{M}{A}) \leq \alpha$ and α is the least ordinal with this property.

Throughout this article, rings are associative with identity and all modules are unitary right modules. If M is an R-module, then $k\text{-dim}\,(M)$, $n\text{-dim}\,(M)$, $tk\text{-dim}\,(M)$ and $tn\text{-dim}\,(M)$ are the Krull dimension, Noetherian dimension, type Krull dimension and type Noetherian dimension of the poset of all submodules of M, respectively. It is convenient that, when we are dealing with the latter dimensions, we may begin our list of ordinals with -1. Now, we recall some definitions that we need throughout this article. The Goldie dimension of an R-module M, denoted by $G\text{-dim}\,(M)$, which is the supremum, α say, of all cardinals k such that M contains a direct sum of k non-zero submodules. Two modules A and B are orthogonal (written $A \perp B$) if they have no non-zero isomorphic submodules; whereas A is parallel to B (written $A \parallel B$) if no non-zero submodule of A is orthogonal to B and symmetrically, no non-zero submodule of B is orthogonal to A. A module M is called atomic if all its non-zero submodules are parallel to each other. A is type submodule of M (written $A \subseteq_t M$) if it has not proper parallel extension. So, if A is not type submodule of M (written $A \nsubseteq_t M$), there exists a submodule B of M such that $A \subset B$ and $A \parallel B$. A submodule E of M is said to be essential (written $E \subseteq_e M$) if $E \cap B \neq 0$, for each non-zero submodule B of M, otherwise E is non-essential. Let S be a submodule of a module M. A submodule C of M is said to be a complement to S in M if C is maximal with respect to the property that $C \cap S = 0$. Also, submodule K of M is called closed, if whenever N is a submodule of M such that K is an essential submodule of N, then $N = K$. The reader is referred to [6, 7, 9, 16] for more details and undefined terms and notations.

2. Preliminaries

This section contains some preliminary results that are needed in the sequel, most of which are in [6].
First, we recall the following definitions.

Definition 2.1. *Two modules A and B are* orthogonal, *written as $A \perp B$, if they do not have non-zero isomorphic submodules. Modules A_1 and A_2 are parallel, denoted as $A_1 \parallel A_2$, if there does not exist a $0 \neq C_2 \subseteq A_2$ with $A_1 \perp C_2$, and also there does not exist a $0 \neq C_1 \subseteq A_1$ such that $C_1 \perp A_2$. An equivalent definition of $A_1 \parallel A_2$ is that for any $0 \neq C_1 \subseteq A_1$, there exist $0 \neq aR \subseteq C_1$ and $0 \neq bR \subseteq A_2$ with $aR \cong bR$, and dually for any $0 \neq C_2 \subseteq A_2$, there exist $0 \neq aR \subseteq A_1$, $0 \neq bR \subseteq C_2$ with $aR \cong bR$.*

Definition 2.2. *A submodule A of a module M is called a type submodule, denoted as $A \subseteq_t M$, if the following equivalent conditions hold:*

(1) *If $A \subseteq B \subseteq M$ with $A \parallel B$, then $A = B$.*
(2) *If $A \subset B \subseteq M$, then $A \perp X$ for some $0 \neq X \subseteq B$.*

Lemma 2.3. [6, Lemma 4.1.5] *Let M be a module with submodules A and B.*

(1) *$M \neq 0$ is an atomic module if and only if (0) and M are the only type submodules of M.*
(2) *If $A \subseteq_t M$ and $A \subseteq B$, then $A \subseteq_t B$.*
(3) *If $A \subseteq_t B \subseteq_t M$, then $A \subseteq_t M$.*
(4) *If $A \subseteq_t M$ and $A \subseteq_e B \subseteq M$, then $A = B$.*

Lemma 2.4. [6, Proposition 4.1.6] *Let M be a module and A a submodule of M. Let $A \subseteq_t M$ and $A \subseteq B \subseteq M$. Then $\frac{B}{A} \subseteq_t \frac{M}{A}$ if and only if $B \subseteq_t M$.*

Definition 2.5. *An R-module M has finite type dimension n, denoted by $t.\dim(M) = n$, if M contains an essential direct sum of n pairwise orthogonal* *atomic submodules of M. If no such n exists, we say that the type dimension of M is infinite, and write $t.\dim(M) = \infty$. If $M = 0$, then $t.\dim(M) = 0$.*

Remark 2.6. For a module M, $t.\dim(M) = \infty$ if and only if there exist an infinite number of pairwise orthogonal non-zero submodules of M.

Let us continue with the following well-known and important result, see [6, Proposition 4.1.12(2)].

Proposition 2.7. *The following statements are equivalent for a module M.*

1. $t.\dim(M) < \infty$.
2. *M has the ascending chain condition (briefly, ACC) on type submodules.*
3. *M has the descending chain condition (briefly, DCC) on type submodules.*

Remark 2.8. Let M be an R-module. If $G\text{-}\dim(M) < \infty$, then the complements in M satisfy $DCC(resp., ACC)$, see [12, Proposition (6.30)′]. Also, we know that type submodules are complement submodules, see [6], and by the previous proposition, $t.\dim(M) < \infty$. But the converse of this fact is not true in general. For example, if $M = \oplus_{i>0}\mathbb{Z}_{p^i}$ as a $\mathbb{Z}$-module where p is a prime number, then $t.\dim(M) = 1$, but $G\text{-}\dim(M) = \infty$.

3. Type Krull dimension and its properties

In this section, we introduce and study the concept of type Krull dimension of an R-module M, which is Krull-like dimension extension of the concept of DCC on type submodules. In other words, it is the deviation of the poset of type submodules of M.
Next, we give our definition of type Krull dimension.

Definition 3.1. *Let M be an R-module. The type Krull dimension of M, denoted by $tk\text{-}\dim(M)$ is defined by transfinite recursion as follows: If $M = 0$, $tk\text{-}\dim(M) = -1$. If α is an ordinal number and $tk\text{-}\dim(M) \not< \alpha$, then $tk\text{-}\dim(M) = \alpha$ provided there is no infinite descending chain of type submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ such that for each $i = 1, 2, \cdots$, $tk\text{-}dim(\frac{M_{i-1}}{M_i}) \not< \alpha$. In otherwise, $tk\text{-}\dim(M) = \alpha$, if $tk\text{-}\dim(M) \not< \alpha$ and for each chain of type submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ there exists an integer t, such that for each $i \geq t$, $tk\text{-}\dim(\frac{M_{i-1}}{M_i}) < \alpha$. A ring R has type Krull dimension, if as an R-module has type Krull dimension. It is possible that there is no ordinal α such that $tk\text{-}\dim(M) = \alpha$, in this case we say M has no type Krull dimension.*

If $tk\text{-dim}\,(M) > \alpha$, then there exists an infinite descending chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of type submodules of M such that $tk\text{-dim}\,(\frac{M_i}{M_{i+1}}) \geq \alpha$ for all i.

Clearly, $tk\text{-dim}\,(M) = 0$ if and only if M satisfies DCC on its type submodules. So $tk\text{-dim}\,(M) = 0$ if and only if $t.\text{dim}\,(M) < \infty$, by Proposition 2.7. In this case, M satisfies ACC on its type submodules.

Remark 3.2. Every atomic module M has type Krull dimension, because $t.\text{dim}\,(M) = 1 < \infty$ and so in view by Proposition 2.7, M satisfies DCC on its type submodules. For example, $M = \bigoplus_{i>0} \mathbb{Z}_{p^i}$, where p is a prime number, as a $\mathbb{Z}$-module, and also, a domain R are atomic modules and so, they have type Krull dimension.

Remark 3.3. Every module with Krull dimension has type Krull dimension, because every module with Krull dimension has finite Goldie dimension, see [7, Proposition 1.4]), and so, it has finite type dimension, by Remark 2.8. Therefore, it has type Krull dimension equal to zero. But the converse of this fact does not necessarily hold; even if we add finite Goldie dimension or finite type dimension conditions to it. For example, $\mathbb{Q}$ is uniform as $\mathbb{Z}$-module and so is atomic, hence it has type Krull dimension, by the previous remark, but it does not have Krull dimension.

By applying Lemma 2.3(3), the proof of the following fact is evident.

Lemma 3.4. *Let M be an R-module with type Krull dimension. Then for each type submodule A of M, A has type Krull dimension and $tk\text{-dim}\,(A) \leq tk\text{-dim}\,(M)$.*

Lemma 3.5. *Let M be an R-module with type Krull dimension. Then for each type submodule A of M, $\frac{M}{A}$ has type Krull dimension and $tk\text{-dim}\,(\frac{M}{A}) \leq tk\text{-dim}\,(M)$.*

Proof. Let $tk\text{-dim}\,(M) = \alpha$ and $\frac{M_0}{A} \supseteq \frac{M_1}{A} \supseteq \frac{M_2}{A} \supseteq \cdots$ be a descending chain of type submodules of $\frac{M}{A}$. Then $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a descending chain of type submodules of M, by Lemma 2.4. Hence, there exists an integer t such that $tk\text{-dim}\,(\frac{M_i/A}{M_{i+1}/A}) = tk\text{-dim}\,(\frac{M_i}{M_{i+1}}) < \alpha$ for all $i \geq t$, thus $tk\text{-dim}\,(\frac{M}{A})$ exists and it is less than or equal to α. $\square$

Proposition 3.6. *Let M be an R-module. If every type submodule of M has type Krull dimension, then so does M and*

$$tk\text{-dim}\,(M) = \sup\{tk\text{-dim}\,(A) : A \subseteq_t M\}.$$

Proof. We note that all of type submodules of M is in the form of a set and hence $\sup\{tk\text{-dim}\,(A) : A \subseteq_t M\}$ exists and we call it α. Give any chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of type submodules of M, there exists k such that $tk\text{-dim}\,(\frac{M_i}{M_{i+1}}) < tk\text{-dim}\,(M_k) \leq \alpha$, for every $i \geq k$. Therefore, M has type Krull dimension equal to α. $\square$

Proposition 3.7. *Let M be an R-module. If for each type submodule A of M, $\frac{M}{A}$ has type Krull dimension, then so does M and*

$$tk\text{-dim}\,(M) \leq \sup\{tk\text{-dim}\,(\tfrac{M}{A}) | A \subseteq_t M\} + 1.$$

Proof. Let $\alpha = \sup\{tk\text{-dim}\,(\frac{M}{A}) | A \subseteq_t M\}$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a infinite descending chain of type submodules of M. Since $\frac{M_i}{M_{i+1}} \subseteq \frac{M}{M_{i+1}}$ and $\frac{M}{M_{i+1}}$ has type Krull dimension, thus $tk\text{-dim}\,(\frac{M_i}{Mi+1})$ exists for all i, by Lemmas 2.4 and 3.4, and so $tk\text{-dim}\,(\frac{M_i}{Mi+1}) \leq \alpha$. This shows that M has type Krull dimension and $tk\text{-dim}\,(M) \leq \alpha + 1$. $\square$

Theorem 3.8. *Let M be an R-module. If for each type submodule A of M, A or $\frac{M}{A}$ has type Krull dimension, then so does M.*

Proof. Let $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be an infinite descending chain of type submodules of M. If there exists $k \in \mathbb{N}$ such that M_k has type Krull dimension, then $\frac{M_i}{M_{i+1}}$ has type Krull dimension, for every $i \geq k$, by Lemma 2.3(2), and then M has type Krull dimension. Otherwise $\frac{M}{M_i}$ has type Krull dimension, for all i and in view of the proof of the previous proposition, we infer that $\frac{M_i}{M_{i+1}}$ has type Krull dimension. Thus, M has type Krull dimension. $\square$

Proposition 3.9. *Let M be an R-module. If for every type submodule A of M, A has Krull dimension, then M has type Krull dimension and $tk\text{-dim}\,(M) \leq \sup\{k\text{-dim}\,(A) | A \subseteq_t M\} + 1$.*

Proof. Let $\alpha = \sup\{k\text{-dim}\,(A) | A \subseteq_t M\} + 1$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be an infinite descending chain of type submodules of M. Since M_i has Krull dimension, by Remark 3.3, it has type Krull dimension. So there exists $n \in \mathbb{N}$ such that for every $i \geq n$, $tk\text{-dim}\,(\frac{M_i}{M_{i+1}}) \leq k\text{-dim}\,(\frac{M_i}{M_{i+1}}) \leq k\text{-dim}\,(M_i) < \alpha$. Thus we have $tk\text{-dim}\,(M) \leq \alpha$. $\square$

Corollary 3.10. *Let M be an R-module. If for every type submodule A of M, A has Krull dimension and $k\text{-dim}\,(A) < \alpha$, then $tk\text{-dim}\,(M) \leq \alpha + 1$.*

Proposition 3.11. *Let M be an R-module. If for every type submodule A of M, $\frac{M}{A}$ has Krull dimension, then M has type Krull dimension and $tk\text{-dim}\,(M) \leq \sup\{k\text{-dim}\,(\frac{M}{A}) | A \subseteq_t M\} + 1$.*

Proof. Let $\alpha = \sup\{k\text{-dim}\,(\frac{M}{A}) | A \subseteq_t M\} + 1$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be an infinite chain of type submodules of M. Since $\frac{M_{i-1}}{M_i} \subseteq \frac{M}{M_i}$ and $\frac{M}{M_i}$ has Krull dimension, so $\frac{M_{i-1}}{M_i}$ has Krull dimension and hence, Remark 3.3 implies that it has type Krull dimension. Thus

$$tk\text{-dim}\,(\tfrac{M_{i-1}}{M_i}) \leq k\text{-dim}\,(\tfrac{M_{i-1}}{M_i}) \leq k\text{-dim}\,(\tfrac{M}{M_i}) < \alpha.$$

So, $tk\text{-dim}\,(\frac{M_{i-1}}{M_i}) < \alpha$ and consequently, $tk\text{-dim}\,(M) \leq \alpha$. □

Definition 3.12. *An R-module M has homogeneous type Krull dimension, if every submodule of M has type Krull dimension.*

Lemma 3.13. *Let M be an R-module. If M has homogeneous type Krull dimension, then so does $\frac{M}{A}$, for every type submodule A of M.*

Proof. Let $\frac{B}{A}$ be a submodule of $\frac{M}{A}$ such that $A \subseteq B \subseteq M$. Since A is type submodule of M, $A \subseteq_t B$, by Lemma 2.3(2). By hypothesis and Lemma 3.5, we infer that $\frac{B}{A}$ has type Krull dimension. □

Theorem 3.14. *If M is an R-module with homogeneous type Krull dimension, then $tk\text{-dim}\,(M) = 0$.*

Proof. It suffices to prove the $t.\text{dim}\,(M) < \infty$. The proof is by induction on $tk\text{-dim}\,(M)$. If $tk\text{-dim}\,(M) = -1$, then $M = 0$ and hence $t.\text{dim}\,(M) = 0$. Now, let $\mathcal{A}$ be a set of ordinal numbers α such that there exists a module with type Krull dimension of α which has no finite type dimension. If $\mathcal{A} \neq \emptyset$, then it has a minimum member such as α, according to well-ordering principle. Let M be a module that $tk\text{-dim}\,(M) = \alpha$ and it has not finite type dimension. Then M contains an infinte direct sum of pairwise orthogonal submodules such as $\bigoplus_{i=1}^{\infty} M_i \subseteq M$. Now, we put $N_n = \bigoplus_{i=1}^{\infty} M_{i2^n}$, for every $n \in \mathbb{N}$. Thus, we have $N_1 \supset N_2 \supset \cdots$ is chain of type submodules of N_1. By assumption, N_1 has type Krull dimension and so $tk\text{-dim}\,(N_1) \leq tk\text{-dim}\,(M) = \alpha$. So, there exists an integer k such that for each $i \geq k$, $tk\text{-dim}\,(\frac{N_i}{N_{i+1}}) < tk\text{-dim}\,(N_1) \leq \alpha$. It is easy to see that $\frac{N_i}{N_{i+1}}$ does not have finite type dimension for each i, but it has type Krull dimension less than of α and this is contradiction with minimality of α. □

Corollary 3.15. *Let M be an R-module with $tk\text{-dim}\,(M) \geq 1$. Then M has a submodule which does not have type Krull dimension and so, it is a non-type submodule.*

Using the previous theorem, the proof of the following fact is evident.

Corollary 3.16. *Let M be an R-module with homogeneous type Krull dimension. Then M satisfies DCC(resp., ACC) on its type submodules.*

Theorem 3.17. *Let M be an R-module.*

1. *Every decomposable submodule of M is Artinian.*
2. *Every proper closed submodule of M is Artinian.*
3. *Every type submodule of M is Artinian.*
4. *M satisfies DCC on its type submodules.*

Then we have $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

Proof. $(1) \Leftrightarrow (2)$ See [17, Theorem 1.4].

$(2) \Rightarrow (3)$ Because type submodules of M are closed in M.

$(3) \Rightarrow (4)$ Let

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

be a descending chain of type submodules of M. By (3), M_1 is Artinian and then the chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ is stopped, as we needed. □

Theorem 3.18. *Let M be an R-module.*

1. *Every decomposable submodule K of M has k-$\dim(K) \leq \alpha$.*
2. *Every closed submodule K of M has tk-$\dim(K) \leq \alpha$.*
3. *Every type submodule K of M has tk-$\dim(K) \leq \alpha$.*
4. *M has type Krull dimension.*

Then we have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

Proof. $(1) \Rightarrow (2)$ Let K be a closed submodule of M and $K_1 \supseteq K_2 \supseteq \cdots$ be a descending chain of type submodules of K. In view of [12, Proposition 6.32], there exists a non-zero submodule A of M such that $A \cap K = 0$ and so, $A \cap K_1 = 0$. By (1), the decomposable submodule $A \oplus K_1$ has Krull dimension and k-$\dim(A \oplus K_1) \leq \alpha$. So Remark 3.3 implies that tk-$\dim(K_1) \leq \alpha$. Hence, there exists an integer n such that tk-$\dim(\frac{K_i}{K_{i+1}}) < \alpha$, for each $i \geq n$. So tk-$\dim(K) \leq \alpha$ and we are done.

$(2) \Rightarrow (3)$ Because type submodules of M are closed in M.

$(3) \Rightarrow (4)$ This is Proposition 3.6. □

Theorem 3.19. *Let M be an R-module. If every decomposable submodule of M is finitely generated, then every type submodule of M is finitely generated.*

Proof. Let N be any non-zero type submodule of M. Then N is a complement submodule, by Remark 2.8, and so there exists a non-zero submodule L of M such that $N \cap L = 0$. By hypothesis, the decomposable submodule $N \oplus L$ is finitely generated and hence N is finitely generated. □

Proposition 3.20. *If R-module M satisfies DCC(resp., ACC) on its non-essential submodules, then every proper type submodule of M is Artinian (resp., Noetherian).*

Proof. Let A be a proper type submodule of M and

$$A_1 \supseteq A_2 \supseteq \cdots (A_1 \subseteq A_2 \subseteq \cdots)$$

is a descending (ascending) chain of submodules of A. By Lemma 2.3(4), A is non-essential submodule of M, so A_i is non-essential submodule of M. Therefore, the above chain is stopped. □

Theorem 3.21. *If every non-essential submodule of an R-module M is finitely generated, then $tk\text{-}\dim(M) = 0$.*

Proof. It suffices to prove the $t.\dim(M) < \infty$. Suppose that M does not have finite type dimension and $N_1 \oplus N_2 \oplus N_3 \oplus \cdots$ be a direct sum of pairwise orthogonal non-zero submodules $N_i (i \geq 1)$ of M. Let $N = N_2 \oplus N_3 \oplus \cdots$. Since $N \perp N_1$, $N \cap N_1 = 0$, so N is non-essential submodule of M and by hypothesis, N is finitely generated. Therefore, $N \subseteq N_2 \oplus N_3 \oplus \cdots \oplus N_k$ for some integer $k \geq 2$. Thus, $N_{k+1} = 0$, which is contradiction. □

Next, we give our definition of type critical modules, which is similar to concept of critical modules.

Definition 3.22. *An R-module M is called α-$t.$critical if $tk\text{-}\dim(M) = \alpha$ and for every type submodule A of M, $tk\text{-}\dim(\frac{M}{A}) < \alpha$. Also, M is called $t.$critical, if it is an α-type critical (briefly, α-$t.$critical) for some α.*

Proposition 3.23. *Every non-zero type submodule of an R-module M with type Krull dimension is a $t.$critical submodule.*

Proof. Let $0 \neq B \leq_t M$ with the least type Krull dimension α. Clearly, for every $0 \neq C \leq_t B$, $tk\text{-}\dim(C) = \alpha$. If B is an α-$t.$critical, we are done. Otherwise there exists $0 \neq B_1 \leq_t B$ such that $tk\text{-}\dim(\frac{B}{B_1}) \nless \alpha$ and so, $tk\text{-}\dim(\frac{B}{B_1}) = \alpha$. If B_1 is an α-$t.$critical, we are done (note, $tk\text{-}\dim(B_1) = \alpha$). Now, if B_1 is not α-$t.$critical, there exists $0 \neq B_2 \leq_t B_1$ such that $tk\text{-}\dim(\frac{B_1}{B_2}) = \alpha$. By minimality of α, $tk\text{-}\dim(B_2) = \alpha$. Continuing in this manner, we obtain a descending chain $B_1 \supseteq B_2 \supseteq \cdots$ of type submodules of B such that $tk\text{-}\dim(\frac{B_{i-1}}{B_i}) = \alpha$, for each i, which is a contradiction with $tk\text{-}\dim(B) = \alpha$. □

Remark 3.24. Almost the dual of all results in this section are also true. The proofs are just a minor variant of arguments about type Krull dimension and for all of them the proofs follow mutatis mutandis *tk*-dim() by *tn*-dim() and *k*-dim() by *n*-dim().

Theorem 3.25. *Let M be an R-module. Then M has type Krull dimension if and only if M has type Noetherian dimension.*

Proof. The proof is by transfinite induction on $tn\text{-dim}\,(M) = \alpha$. If $\alpha = 0$, by Proposition 2.7, M satisfies ACC on its type submodules and so M satisfies DCC on its type submodules, hence $tk\text{-dim}\,(M) = 0$. Now, suppose that every module with type Noetherian dimension less than α has type Krull dimension and $tn\text{-dim}\,(M) = \alpha$. If M does not have type Krull dimension, then there exists $A_1 \subseteq_t M$ such that A_1 and $\frac{M}{A_1}$ do not have type Krull dimension, by Theorem 3.8. Since $\frac{M}{A_1}$ does not have type Krull dimension, by Theorem 3.8, there exists $\frac{A_2}{A_1} \subseteq_t \frac{M}{A_1}$ such that $\frac{A_2}{A_1}$ and $\frac{M}{A_2}$ do not have type Krull dimension. Continuing in this manner, we obtain an ascending chain $A_1 \subseteq A_2 \subseteq \cdots$ of type submodules of M. Since M has type Noetherian dimension, there exists $k \in \mathbb{N}$ such that for any $i \geq k$, $tn\text{-dim}\,(\frac{A_{i+1}}{A_i}) < \alpha$. Thus, according to the hypothesis induction, $\frac{A_{i+1}}{A_i}$ has type Krull dimension, which is contradiction. Therefore, M has type Krull dimension. The converse is similar. □

Theorem 3.26. *Let M be an R-module. Then $tn\text{-dim}\,(M) = \alpha$ if and only if $tn\text{-dim}\,(M) \nless \alpha$ and every non-empty set $\mathcal{F}$ of type submodules of M contains an element as A such that for every $A \subseteq B \in \mathcal{F}$, we have $tn\text{-dim}\,(\frac{B}{A}) < \alpha$.*

Proof. Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of type submodules of M and $\mathcal{F} = \{M_1, M_2, \cdots\}$ be a non-empty set of type submodules of M. So there exists $M_i \in \mathcal{F}$ such that $tn\text{-dim}\,(\frac{M_j}{M_i}) < \alpha$, for every $i \leq j$, hence $tn\text{-dim}\,(\frac{M_{j+1}}{M_j}) < \alpha$, for every $i \leq j$. Therefore, $tn\text{-dim}\,(M) = \alpha$.

Conversely, let $tn\text{-dim}\,(M) = \alpha$ and $\mathcal{F}$ be a non-empty set of type submodules of M. We choose an arbitrary element $A \in \mathcal{F}$, so either $tn\text{-dim}\,(\frac{B}{A}) < \alpha$ or $tn\text{-dim}\,(\frac{B}{A}) \nless \alpha$, for every $A \subseteq B \in \mathcal{F}$. If $tn\text{-dim}\,(\frac{B}{A}) < \alpha$, then the proof is complete. Otherwise, $tn\text{-dim}\,(\frac{B}{A}) \geq \alpha$ and since

$$tn\text{-dim}\,(\tfrac{B}{A}) \leq tn\text{-dim}\,(\tfrac{M}{A}) \leq tn\text{-dim}\,(M) = \alpha,\ tn\text{-dim}\,(\tfrac{B}{A}) = \alpha.$$

Thus, there exists $\frac{B_1}{A} \subseteq \frac{B}{A}$ such that $\frac{B_1}{A} \in \mathcal{F}_1$ (nonempty set of type submodules of $\frac{B}{A}$) and for every $\frac{B_1}{A} \subseteq \frac{B_2}{A} \in \mathcal{F}_1$ which $A \subseteq B_1 \subseteq B_2$, we have either $tn\text{-dim}\,(\frac{B_2}{B_1}) < \alpha$ or $tn\text{-dim}\,(\frac{B_2}{B_1}) \nless \alpha$. If $tn\text{-dim}\,(\frac{B_2}{B_1}) < \alpha$, then we are done. But, if $tn\text{-dim}\,(\frac{B_2}{B_1}) \nless \alpha$, that is, $tn\text{-dim}\,(\frac{B_2}{B_1}) \geq \alpha$ and since

$$tn\text{-dim}\,(\tfrac{B_2}{B_1}) \leq tn\text{-dim}\,(\tfrac{B_2}{A}) \leq tn\text{-dim}\,(\tfrac{B}{A}) = \alpha,\ tn\text{-dim}\,(\tfrac{B_2}{B_1}) = \alpha.$$

Continuing in this manner, we obtain a chain $B_1 \subsetneq B_2 \subsetneq \cdots$ of type submodules of M, by Lemmas 2.3(2),(3) and 2.4, such that $tn\text{-dim}\,(\frac{B_{i+1}}{B_i}) = \alpha$. This is contradiction with $tn\text{-dim}\,(M) = \alpha$. □

4. *t*-Artinian and *t*-Noetherian modules

In this section, we introduce and study the concept of *t-Artinian* and *t-Noetherian* modules.

Next, we give our definition.

Definition 4.1. *Let M be an R-module. We say that M is* t-Artinian *if for every descending chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of submodules of M, there exists an index $k \geq 1$ such that $M_i \subseteq_t M$, for every $i \geq k$. Dually, we say that M is* t-Noetherian *if, for every ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M, there exists an index k such that $M_i \subseteq_t M$ for every $i \geq k$. We say that a ring R is right* t-Artinian *(resp.,* t-Noetherian*) if R as an R-module is* t-Artinian *(resp.,* t-Noetherian*).*

By above definition, every submodule of a *t-Artinian* (resp., *t-Noetherian*) module is *t-Artinian* (resp., *t-Noetherian*).

Definition 4.2. *A module M is called type atomic (briefly, t-*atomic*), if its all submodules are type submodules.*

It follows from Lemmas 2.3(2) and 2.4 that, every submodule and quotient module of a t-atomic module is t-atomic. Also, every t-atomic module is *t-Artinian* and *t-Noetherian*.

Lemma 4.3. *If M is a* t-Artinian *module, then so does $\frac{M}{A}$, for every type submodule A of M.*

Proof. Let $\frac{M_0}{A} \supseteq \frac{M_1}{A} \supseteq \frac{M_2}{A} \supseteq \cdots$ be a chain of submodules of $\frac{M}{A}$. So $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a chain of submodules of M. Since M is *t-Artinian*, there exists $k \in \mathbb{N}$ such that for every $i \geq k$, $M_i \subseteq_t M$ and Lemma 2.4 implies that $\frac{M_i}{A} \subseteq_t \frac{M}{A}$. Therefore, $\frac{M}{A}$ is *t-Artinian*. □

Lemma 4.4. *A module M is* t-Artinian *if and only if for every non-empty set $\mathcal{F}$ of submodules of M, there exists $N \in \mathcal{F}$ such that for every $0 \neq N' \subseteq N$, if $N' \in \mathcal{F}$, then $N' \subseteq_t M$. Equivalently, M is* t-Artinian *if and only if for every non-empty chain $\mathcal{C}$ of submodules of M, there exists $N \in \mathcal{C}$ such that for every $N' \subseteq N$, if $N' \in \mathcal{C}$, then $N' \subseteq_t M$.*

Proof. Let $M_0 \supsetneq M_1 \supsetneq \dots$ be a descending chain of submodules of M and $\mathcal{F} = \{M_0, M_1, \dots\}$ be a non-empty set of submodules of M. So there exists $M_i \in \mathcal{F}$ which for every $0 \neq M_j \subseteq M_i$ such that $M_j \in \mathcal{F}$, implies that $M_j \subseteq_t M$. This shows that M is t-Artinian.

Conversely, suppose that M is t-Artinian and $\mathcal{F}$ is a non-empty set of submodules of M. Let M_i be an arbitrary element of $\mathcal{F}$ and for every $M_j \subseteq M_i, M_j \in \mathcal{F}$ either $M_j \subseteq_t M$ or $M_j \nsubseteq_t M$. So we have two cases:

Case1. If $M_j \subseteq_t M$, as needed.

Case2. If $M_j \nsubseteq_t M$, then there exists $M_j^{(1)} \subseteq M$ such that $M_j \subsetneq M_j^{(1)}$ and $M_j \| M_j^{(1)}$, hence $M_j \nsubseteq_t M_j^{(1)}$. Using a similar way, there exists $M_j^{(2)} \subseteq M_j^{(1)}$ such that $M_j \subsetneq M_j^{(2)}$ and $M_j \| M_j^{(2)}$, hence $M_j \nsubseteq_t M_j^{(2)}$. Continuing in this manner, we obtain a chain $M_j^{(1)} \supseteq M_j^{(2)} \supseteq \dots$ of parallel submodules, which is a contradiction. □

Remark 4.5. We see that if M is a *t-Artinian* module, by Lemma 4.4, it contains a non-zero submodule as A which is t-atomic.

We know that, every R-module with Krull dimension, has type Krull dimension. Moreover, we have shown that there exists a module with type Krull dimension such that it has no Krull dimension, see Remark 3.3. Now, we will find a condition in which its converse also holds.

Theorem 4.6. *Let M be a* t-Artinian *module with type Krull dimension. Then M has Krull dimension and tk-dim$\,(M) = k$-dim$\,(M)$.*

Proof. It suffices to prove the k-dim$\,(M) \leq tk$-dim$\,(M)$. We proceed by transfinite induction on tk-dim$\,(M) = \alpha$. Let $\alpha = 0$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a chain of submodules of M. Since M is *t-Artinian*, there exists integer k such that for any $i \geq k$, $M_i \subseteq_t M$ and so Lemma 2.3(2) implies that $M_k \supseteq M_{k+1} \supseteq \cdots$ is a chain of type submodules of M_k. Since by Lemma 3.4, tk-dim$\,(M_k) \leq tk$-dim$\,(M) = 0$, this chain will be stopped. That is the chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is stopped, thus k-dim$\,(M) = 0$. Let $\alpha > 0$ and that is true for every ordinal $\beta < \alpha$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a chain of submodules of M. Since M is *t-Artinian*, there exists an integer k such that for any $i \geq k$, $M_i \subseteq_t M$, so $M_k \supseteq M_{k+1} \supseteq \cdots$ is a chain of type submodules of M_k and then tk-dim$\,(\frac{M_i}{M_{i+1}}) < tk$-dim$\,(M_k) \leq \alpha$, for all $i \geq k$. Induction hypothesis gives that k-dim$\,(\frac{M_i}{M_{i+1}}) \leq tk$-dim$\,(\frac{M_i}{M_{i+1}}) < \alpha$, hence k-dim$\,(\frac{M_i}{M_{i+1}}) < \alpha$ and so k-dim$\,(M) \leq \alpha = tk$-dim$\,(M)$, completing the proof. □

Corollary 4.7. *Let M be a type atomic module. Then M has Krull dimension if and only if it has type Krull dimension. Moreover tk-dim $(M) = k$-dim (M).*

5. On α-DICCT modules

In this section, we apply the method of [9] to study α-DICCT modules. We introduce the next definition.

Definition 5.1. *An R-module M is said to be α-DICCT, if α is the least ordinal such that for any double infinite chain of type submodules of M, $\cdots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ there exists an integer k such that either tn-dim $(\frac{M_{i+1}}{M_i}) < \alpha$, for all $i \geq k$ or tk-dim $(\frac{M_{i+1}}{M_i}) < \alpha$, for all $i \leq k$. Also, R is called an α-DICCT ring, if it is α-DICCT as R-module.*

0-DICCT modules are called DICCT module.

Remark 5.2. It is well known that every α-DICC module has Krull dimension and so is a *DICCT* module, see [9, Corollary 1.1], [7, Proposition 1.4] and Remark 2.8.

Remark 5.3. We note that, if M is an α-DICCT module, then for every type submodule N of M, N is a β-DICCT module and $\frac{M}{N}$ is a γ-DICCT module, for some $\beta, \gamma \leq \alpha$, by virtue of Lemmas 2.3(3) and 2.4.

Lemma 5.4. *Let M be an R-module, then the following are equivalent:*

1. *M is an α-DICCT for some α.*
2. *For every type submodule A of M, either tk-dim $(A) \leq \alpha$ or tn-dim $(\frac{M}{A}) \leq \alpha$ and α is the least ordinal with this property.*

Proof. (1)⇒(2) Let M be an α-DICCT R-module and A is a type submodule of M. If tk-dim $(A) \nleq \alpha$, then there exists strictly descending chain $\cdots \subset A_{-2} \subset A_{-1} \subset A$ of type submodules of A such that tk-dim $(\frac{A_{i+1}}{A_i}) \geq \alpha$. Now, let $0 = \frac{A}{A} \subseteq \frac{A_1}{A} \subseteq \frac{A_2}{A} \subseteq \cdots$ be an ascending chain of type submodules of $\frac{M}{A}$. So, $\cdots \subset A_{-2} \subset A_{-1} \subset A \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a double infinite chain of type submodules of M, by Lemma 2.4. By (1), there exists $n \in \mathbb{N}$ such that tn-dim $(\frac{A_{i+1}}{A_i}) < \alpha$, for every $i \geq n$ and so tn-dim $(\frac{M}{A}) \leq \alpha$.

(2)⇒(1) Let $\cdots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be a double infinite chain of type submodules of M. By (2), tk-dim $(M_n) \leq \alpha$ or tn-dim $(\frac{M}{M_n}) \leq \alpha$, for all n. If tk-dim $(M_n) \leq \alpha$, then tk-dim $(\frac{M_{i+1}}{M_i}) < tk$-dim $(M_n) \leq \alpha$, for every $i \leq n$. Hence, tk-dim $(\frac{M_{i+1}}{M_i}) < \alpha$, for every $i \leq n$. If tn-dim $(\frac{M}{M_n}) \leq \alpha$, then tn-dim $(\frac{M_{i+1}/M_n}{M_i/M_n})$=$tn$-dim $(\frac{M_{i+1}}{M_i}) < tn$-dim $(\frac{M}{M_n}) \leq \alpha$, for every $i \geq n$.

Hence, $tn\text{-dim}\,(\frac{M_{i+1}}{M_i}) < \alpha$, for every $i \geq n$. Therefore, M is an α-DICCT R-module. $\square$

Corollary 5.5. *Let M be an R-module. Then M is DICCT if and only if for every type submodule A of M, $tk\text{-dim}\,(A) = 0$ or $tn\text{-dim}\,(\frac{M}{A}) = 0$. So, it can be said that M is DICCT if and only if for every type submodule A of M, $t.\text{dim}\,(A) < \infty$ or $t.\text{dim}\,(\frac{M}{A}) < \infty$, by Proposition 2.7.*

Theorem 5.6. *Let M be an α-DlCCT module. Then:*

(1) *For every infinite descending chain $M_1 \supseteq M_2 \supseteq \cdots$ of type submodules of M, we have either $tn\text{-dim}\,(\frac{M_i}{M_{i+1}}) \leq \alpha$, for all i, or there exists an integer k such that $tk\text{-dim}\,(\frac{M_i}{M_{i+1}}) < \alpha$, for all $i \geq k$.*

(2) *For every infinite ascending chain $N_1 \subseteq N_2 \subseteq \cdots$ of type submodules of M, we have either $tk\text{-dim}\,(\frac{N_{i+1}}{N_i}) \leq \alpha$, for all i, or there exists an integer k such that $tn\text{-dim}\,(\frac{N_{i+1}}{N_i}) < \alpha$, for all $i \geq k$.*

Proof. (1) Let $tn\text{-dim}\,(\frac{M_t}{M_{t+1}}) \not\leq \alpha$ for some t, then there exists an infinite chain $\frac{M'_1}{M_{t+1}} \subseteq \frac{M'_2}{M_{t+1}} \subseteq \cdots$ of type submodules of $\frac{M_t}{M_{t+1}}$ such that $tn\text{-dim}\,(\frac{M'_{j+1}}{M'_j}) \not< \alpha$ for all j. Hence, we have a double infinite $\cdots \subseteq M_{t+2} \subseteq M_{t+1} \subseteq M'_1 \subseteq M'_2 \subseteq \cdots$ of type submodules of M, by Lemmas 2.3(3) and 2.4 and definition of α-DICCT, there exists an integer k such that $tk\text{-dim}\,(\frac{M_m}{M_{m+1}}) < \alpha$, for every $m \geq k$.

(2) The proof is similar to part (1). $\square$

Corollary 5.7. *If M is a DlCCT module, then given any infinite descending chain of type submodules $N_1 \supseteq N_2 \supseteq \cdots$, $\frac{N_i}{N_{i+1}}$ satisfies ACC on its type submodules, for all $i \geq 1$ and given any infinite ascending chain of type submodules $P_1 \subseteq P_2 \subseteq \cdots$, $\frac{P_{i+1}}{P_i}$ satisfies DCC on its type submodules, for all $i \geq 1$.*

More generally, the following result holds.

Theorem 5.8. *Let M be an R-module. Then the following assertions are equivalent:*

(1) *M is α-DlCCT, for some ordinal α.*
(2) *M has type Krull dimension.*
(3) *M has type Noetherian dimension.*

Proof. We need only to show that (1)$\Leftrightarrow$(2), because, by Theorem 3.25, (2) $\Leftrightarrow$ (3) holds. We first assume that M is α-DlCCT and $M_1 \supseteq M_2 \supseteq \cdots$

is any infinite descending chain of type submodules of M. By the previous lemma, either $\frac{M_i}{M_{i+1}}$ has type Noetherian dimension for all $i \geq 1$ or there exists an integer m such that $\frac{M_i}{M_{i+1}}$ has type Krull dimension for all $i \geq m$. Therefore, in any case $\frac{M_i}{M_{i+1}}$ has type Krull dimension for all $i \geq m$. Now, since each chain of type submodules is a set and the collection of all chains is also a set, it would be clear that M has type Krull dimension.

Conversely, let *tk*-dim$\,(M) = \beta$ and *tn*-dim$\,(M) = \gamma$, then for any double infinite chain of type submodules

$$\cdots \subseteq M_{-n} \subseteq \cdots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

there exists k such that *tn*-dim$\,(\frac{M_{i+1}}{M_i}) < \gamma$ for all $i \geq k$ and *tk*-dim$\,(\frac{M_{i+1}}{M_i}) < \beta$, for all $i \leq k$, hence M is α-DICCT, where $\alpha \leq \min\{\beta, \gamma\}$. □

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ON TYPE KRULL DIMENSION OF MODULES

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بررسی بعد کرول همسان مدول‌ها

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در این مقاله، مفهوم بعد کرول همسان یک مدول معرفی شده و برخی از خواص مرتبط با آن مطالعه شده است. با استفاده از این مفهوم، ما برخی از نتایج ابتدایی در رابطه با مدول‌های دارای بعد کرول را تعمیم می‌دهیم. ثابت شده است که هر مدول با بعد کرول همسان همگن دارای بعد کرول همسان صفر است. هم‌چنین، نشان داده شده است که R-مدول M دارای بعد کرول همسان است اگر و تنها اگر دارای بعد نوتری همسان باشد. مشاهده می‌کنیم که هر مدول با بعد کرول، دارای بعد کرول همسان است، اما عکس آن در حالت کلی درست نیست. به علاوه، ما مدول‌های t-آرتینی (به ترتیب، t-نوتری) را معرفی می‌کنیم و نشان داده‌ایم که اگر M یک مدول t-آرتینی با بعد کرول همسان باشد، آن‌گاه دارای بعد کرول است و این دو بعد برای مدول M با هم برابرند. در پایان، مفهوم مدول‌های α-DICCT را معرفی می‌کنیم و ثابت می‌کنیم که R-مدول M یک مدول α-DICCT است اگر و تنها اگر بعد کرول (به ترتیب، نوتری) همسان داشته باشد.

کلمات کلیدی: زیرمدول همسان، بعد کرول همسان، بعد کرول، مدول‌های t-آرتینی.