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r-IDEAL IN A FRAME

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Abstract. Recently, the concept of r-ideal was introduced in a commutative ring and also in a commutative semigroup. Here, we provide a similar definition for r-ideal in a frame and investigate some of its properties. Some cases confirm that the properties of r-ideal in frames do not coincide with properties of r-ideal in commutative rings (or in commutative semigroups), necessarily. We find some characterization of r-ideal in a frame. Specially, we show that any proper r-ideal in a frame is an intersection of minimal prime ideals in this frame. Also, we establish a condition in which each ideal in a frame be an r-ideal as a characterization for boolean algebras.

1. Introduction

Distributive lattices are one of the most extensive and most satisfying chapters of lattice theory, which have provided the motivation for many results in this theory. Therefore, a thorough understanding of distributive lattice is indispensable for working the theory. This type of lattices has also been used quite extensively for applicable studies in computer science and engineering, especially distributed computing (vector clocks and global predicate detection), in concurrency theory (pomsets and occurrence nets), in programming language semantics (fixed-point semantics), and data mining (concept analysis) (see [4]).

A frame is a complete distributive lattice that meet is distributable to arbitrary joint. From its definition, a frame has a commutative multiplicative semigroup structure. So, many studies related to this type of semigroups can be considered for frames. Among these, the ideal structure of semigroups (and also commutative rings) is a highly developed area of research, as attested to by the numerous research articles and books. As a significant class, prime ideals and their generalizations have an important role in understanding the structure of semigroup (or ring). There are many similarities between the ideal theory of semigroups and that of rings. Many definitions and theorems from commutative ring theory have natural analogs in semigroup theory.

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With this motivation, we consider the concept of r-ideal in a commutative ring, introduced in 2015 by Mohammadian [6], as a generalization of the prime ideal. An ideal I of a commutative ring with identity R is called r-ideal if whenever $ab \in I$ and $\mathrm{Ann}(a) = (0)$ imply that $b \in I$. Prime ideals and r-ideals are not comparable in general, but it is verified that every maximal r-ideal in a ring is a prime ideal, while every minimal prime ideal is an r-ideal (see [6]). He compared and contrasted r-ideals with some well-known ideals and got a characterization for topological spaces based on r-ideals of the ring of their continuous functions. One year later, by introducing r-ideals in commutative semigroups in a similar way, several properties and characterizations of this class of ideals were determined (see [3]). Here, we define r-ideal in a frame: An ideal I is said to be an r-ideal if, for each element a and b of L, $a \wedge b \in I$ with $a^* = \perp$ implies $b \in I$. The concept of r-ideals in semigroup has some differences relative to r-ideals of rings; for instance, in any ring R, the condition $rR \cap I = rI$, for all $r \in R$ with $\mathrm{Ann}(r) = (0)$ can characterize an r-ideal, I, while some non-r-ideals in a commutative semigroup and yet all of proper ideals in a frame satisfy the (semigroup or) frame counterpart of this condition. So this is not a characterization in those structures. Maybe examples, like this, can justify the necessity of parallel study of these types of ideals in frame structure (see Example 2.3).

The plan of this paper is as follows: In Section 2, we define the concept of r-ideal in a frame in a way similar to [3] for an r-ideal in a commutative semigroup, and we prove some of the properties of this ideal. To study the lattice of all r-ideals in a frame (with inclusion order), we investigate these ideals under some construction, such as join and meet operator, in Section 3. As the main results of this paper, we characterize an r-ideal in a frame to be any intersection of minimal prime ideals of a frame and characterize boolean algebra to be a frame in which every ideal is r-ideal in the two last chapters of this paper. Also, related to these results, we prove a type of cancellative, which uniquely is true on boolean algebras.

2. r-IDEALS

In this section, we define an r-ideal in a frame and investigate some of its properties. Some of the results of this section are inspired the work in ring theory and semigroups, and many of our results are similar to ones in [6, 3]. However, the nature of our situation enables many of these results to arrive at a more satisfactory state.

For a general theory of frames, we refer to [7]. A frame is a complete lattice L in which the infinite distributive law $x \wedge \bigvee S = \bigvee_{s \in S}(x \wedge s)$ holds for all $x \in L$ and $S \subseteq L$.

Definition 2.1. An ideal I of a frame L is said to be an *r-ideal* if, for every element a and b of L, $a \wedge b \in I$ with $a^* = \perp$ implies $b \in I$.

For every frame L, it is evident that the zero ideal and L are r-ideals of L. To provide immediate nontrivial examples from the definition, the following proposition can be useful.

Proposition 2.2. *For every $a \neq \perp$ in a frame L, the ideal $\downarrow a^*$ is a proper r-ideal, where $\downarrow x := \{z \in L : z \leq x\}$ for every $x \in L$.*

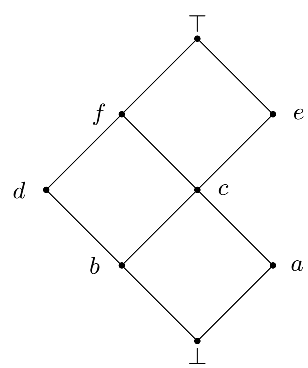
Proof. Assume that $a, x, y \in L$ with $x \wedge y \in \downarrow a^*$ and $x^* = \perp$. Then,

$$x \wedge y \leq a^* \Rightarrow x \wedge y \wedge a = \perp \Rightarrow y \wedge a \leq x^* = \perp \Rightarrow y \leq a^*.$$

Therefore, $\downarrow a^*$ is an r-ideal. □

A principal ideal in a frame needs not to be r-ideal, necessarily. A counterexample is provided in the frame illustrated below.

Example 2.3. Consider the frame $L := \{\perp, a, b, c, d, e, f, \top\}$ with the following Hasse diagram.



(1) The ideal $\downarrow f = \{\perp, a, b, c, d, f\}$ is not an r-ideal, since $c = e \wedge f \in \downarrow f$ and $f^* = \perp$, but e is not an element of $\downarrow f$.
(2) $\downarrow d^*$ and $\downarrow a^*$ are both r-ideals, but $\downarrow f = \downarrow d^* \vee \downarrow a^*$ is not an r-ideal.

Now, we would like to mention an r-ideal containing I, for an arbitrarily proper ideal I of frame L, and arrive at the smallest r-ideal containing ideal I.

Throughout this paper, for every frame L, we define

$$D(L) := \{a \in L : a^* = \perp\},$$

and we set

$$I_r := \{a \in L : a \wedge r \in I \ for\ some\, r \in D(L)\},$$

for every ideal I of L.

Proposition 2.4. *Let L be a frame. Then, the following statements are true:*

(1) *For every $a, b \in L$, $a \wedge b \in D(L)$ if and only if $a \in D(L)$ and $b \in D(L)$.*
(2) *For every ideal I of L, if $I \cap D(L) = \emptyset$, then I_r is the smallest r-ideal of L containing I.*
(3) *For every ideal I of L, $I \cap D(L) \neq \emptyset$ if and only if $I_r = L$; consequently, if $I \cap D(L) \neq \emptyset$, then I is not an ideal.*

Proof. (1). For any elements a and b of $D(L)$, we have

$$\begin{aligned}(a \wedge b)^* &= \bigvee \{t \in L : t \wedge a \wedge b = \perp\} \\ &= \bigvee \{t \in L : t \wedge a \leq b^* = \perp\} \\ &= \bigvee \{t \in L : t \leq a^* = \perp\} \\ &= \perp .\end{aligned}$$

Hence, $a \wedge b \in D(L)$. Now, suppose that $a, b \in L$ with $a \wedge b \in D(L)$. Then

$$a^* \wedge a \wedge b = \perp \Rightarrow a^* \leq (a \wedge b)^* = \perp \Rightarrow a \in D(L).$$

A similar argument shows that $b \in D(L)$.

(2). It is easy to check that I_r is an r-ideal. Let I be an ideal of L with $I \cap D(L) = \emptyset$. If J be another r-ideal containing I, then it is a straightforward result of the definition of r-ideal that $I_r \subseteq J$ and therefore, I_r is the smallest r-ideal of L containing I.

(3). Let $b \in I \cap D(L)$. Then, $c \wedge b \in {\downarrow} b \subseteq I$, for every element c in L, and so $L = I_r$. □

Remark 2.5. If I is an ideal of a frame L satisfying in part (2) of Proposition 2.4, then it is not necessarily an r-ideal of L. For example, consider $I = {\downarrow} b$, in Example 2.3. Although $I \cap D(L) = \emptyset$, but the ideal I is not an r-ideal.

Corollary 2.6. *Let I be a proper r-ideal in a frame L. Then, x^* is not an element in I for all element x in I.*

Proof. For every x in L, the element $x \vee x^*$ is in $D(L)$. Let a be an element in I, with $a^* \in I$ on the contrary. Then, $a \vee a^*$ is in $I \cap D(L)$, and this contradicts part (3) of Proposition 2.4. □

Remark 2.7. The converse of Corollary 2.6 is not necessarily true. The condition x^* is not an element in I, for all element x in I is true for every ideal I with $I \cap D(L) = \emptyset$, which is not necessarily an r-ideal (see Remark 2.5).

Moreover, we find a characterization of r-ideal in a frame with respect to the membership of x^{**} in the ideal for all the own elements in Theorem 4.5. Proposition 2.2 and its corollary determine some principal r-ideals in frame L. Moreover, it can be possible to specify some other elements a in L with $\downarrow a$ is not r-ideal, by the following proposition.

Proposition 2.8. *Let L be a frame and let $(a, b) \in D(L) \times L$. Then, the following statements are true:*

(1) $\left(\downarrow b\right)_r = \left(\downarrow (a \wedge b)\right)_r$.
(2) *If $\downarrow b$ is an r-ideal in a frame L and $a \wedge b \neq b$, then $\downarrow (a \wedge b)$ is not an r-ideal.*

Proof. The proof is straightforward. □

If A and B are nonempty subsets of a frame L, then we set

$$[A : B] := \left\{x \in L \colon x \wedge b \in A \text{ for every } b \in B\right\}.$$

If $B = \left\{x\right\}$, then we shall write $[A : x]$ in place of $[A : B]$.

Proposition 2.4 provides some results to study some behaviors of r-ideals. There are some characterizations to determine prime ideals that can be applied to r-ideals as well, in a similar way. For example, a proper ideal P in a frame L is prime if and only if $P = (P : a)$, for every $a \in L \setminus D(L)$, is an r-ideal as follows.

Proposition 2.9. *Let I be a proper ideal of L. Then, the following statements are true for every frame L:*

(1) *The ideal I is an r-ideal of L if and only if $I = [I : x]$ for every $x \in D(L) \setminus I$.*
(2) *The ideal I is an r-ideal of L if and only if for every nonempty subset J of L that is not included in I, $[I : J]$ is an r-ideal of L.*

Proof. (1). The proof is straightforward.

(2). *Necessity.* Assume that J is a nonempty subset of L with $J \not\subseteq I$. Let $x, y \in L$ with $x \wedge y \in [I : J]$ and let $x^* = \bot$ be given. Then $x \wedge y \wedge z \in I$ for every $z \in J$, which implies from I is an r-ideal of L that $y \in [I : J]$. Hence, $[I : J]$ is an r-ideal of L.

Sufficiency. Let $x, y \in L$ with $x \wedge y \in I$ and let $x^* = \bot$ be given. If $y \notin I$, then $J := \{y\} \nsubseteq I$, which implies that $x \in [I : J] \cap D(L) \neq \emptyset$. Thus, by Proposition 2.4, $[I : J]_r = L$, and this is a contradiction with the fact that $[I : J]$ is an r-ideal. □

3. r-ideals under some constructions

The investigation of r-ideals under various contexts of constructions is a useful step to study the behavior of them and compare them with other classical ideals.
It is evident that the intersection of any family of r-ideals is an r-ideal, as the following proposition shows.

Proposition 3.1. *If I and J are two ideals in a frame L, then $(I \wedge J)_r$ coincides with $I_r \wedge J_r$.*

Proof. It is clear that $I \wedge J \subseteq I_r \wedge J_r$ and so, $(I \wedge J)_r \subseteq I_r \wedge J_r$. Conversely, let x be in $I_r \wedge J_r$; then there exist a in I_r and b in J_r such that $x = a \wedge b$ and there are s_1 and s_2 in $D(L)$ such that $a \wedge s_1 \in I$ and $b \wedge s_2 \in J$. On the other hand, $s_1 \wedge s_2$ is in $D(L)$ and $x \wedge s_1 \wedge s_2 = (a \wedge s_1 \wedge s_2) \wedge (b \wedge s_1 \wedge s_2)$ is in $I \wedge J$, and so, $x \in (I \wedge J)_r$ which completes the proof. □

Indeed by Example 2.3, the ideal $I \vee J$ may not be r-ideal, for two r-ideals I and J (even if both of them are principal ideals). The following two propositions investigate some related conditions to this issue.

Proposition 3.2. *Let L be a frame and let $a, b \in L$ with $a \vee b = \top$. Then $I := {\downarrow}a^* \vee {\downarrow}b^*$ is an r-ideal.*

Proof. In the case when $I = L$, the proof is clear. Let $x, y \in L$ with $x \wedge y \in I \neq L$ and $x^* = \bot$ be given. Then, there exist $v \in {\downarrow}a^*$ and $w \in {\downarrow}b^*$ such that $x \wedge y = v \vee w$, which implies that

$$x \wedge y \wedge a \wedge b = \bot \Rightarrow y \wedge a \wedge b \leq x^* = \bot \Rightarrow y \wedge a \leq b^* \text{ and } y \wedge b \leq a^*.$$
$$\Rightarrow y = y \wedge (a \vee b) \in I.$$

Hence, I is an r-ideal. □

Proposition 3.2 can be generalized, for an arbitrary family $\{a_\lambda\}_{\lambda \in \Lambda}$ of elements of L having similar condition.

Proposition 3.3. *Let L be a frame and let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq L$ with $a_\lambda \vee a_\lambda^* = \top$, for every $\lambda \in \Lambda$, be given. Then $I := \bigvee_{\lambda \in \Lambda} {\downarrow}a_\lambda$ is an r-ideal.*

Proof. In the case when $I = L$, the proof is clear. Let $x, y \in L$ with $x \wedge y \in I \subsetneq L$ and $x^* = \bot$ be given. Then, there exists an element $(v_{\lambda_1}, \dots, v_{\lambda_n})$ in ${\downarrow}a_{\lambda_1} \times \cdots \times {\downarrow}a_{\lambda_n}$ such that $x \wedge y \leq \bigvee_{i=1}^n v_{\lambda_i}$. We set $z := \bigwedge_{i=1}^n a^*_{\lambda_i}$, which implies that $x \wedge y \wedge z = \bot$ and $z \vee z^* = \top$. Hence, $y = y \wedge (z \vee z^*) = y \wedge z^* \in I$. Therefore, I is an r-ideal. □

Proposition 3.4. *Let I and J be two proper ideals in a frame L such that $I \cap J = \{\bot\}$. If $I \vee J$ is an r-ideal, then I and J are also r-ideals.*

Proof. Let $x, y \in L$ with $x \wedge y \in I$ and $x^* = \bot$. Then $y \in I \vee J$, which implies that there exists an element (a, b) in $I \times J$ such that $y = a \vee b$. Thus,

$$(x \wedge a) \vee (x \wedge b) = x \wedge y \in I \Rightarrow x \wedge b \in I \cap T \Rightarrow x \wedge b = \bot \Rightarrow b \leq x^* = \bot$$
$$\Rightarrow y = a \in I.$$

Therefore, I is an r-ideal. □

The set of all r-ideals of a frame L will be denoted by $r - \mathrm{Id}(L)$. As shown in Example 2.3, the ideal $I \vee J$ may not be r-ideal, for two r-ideals I and J. However, it is possible to study $r - \mathrm{Id}(L)$, as a complete distributive lattice under set inclusion.

Proposition 3.5. *For every frame L, $(r - \mathrm{Id}(L), \subseteq)$ is a complete distributive lattice such that for every subset $\{I_\lambda\}_{\lambda \in \Lambda}$ of $r - \mathrm{Id}(L)$,*

$$\bigwedge_{\lambda \in \Lambda}^{r-\mathrm{Id}(L)} I_\lambda = \bigcap_{\lambda \in \Lambda} I_\lambda \text{ and } \bigvee_{\lambda \in \Lambda}^{r-\mathrm{Id}(L)} I_\lambda = \left(\bigvee_{\lambda \in \Lambda}^{\mathrm{Id}(L)} I_\lambda \right)_r .$$

.

Proof. It is clear that $r - \mathrm{Id}(L)$ is a partially ordered set with respect to set-inclusion. Consider $\{I_\lambda\}_{\lambda \in \Lambda}$, as a family of r-ideals in L. Since $\bigwedge_{\lambda \in \Lambda}^{r-\mathrm{Id}(L)} I_\lambda := \bigcap_{\lambda \in \Lambda} I_\lambda$, we infer that $I := \bigvee_{\lambda \in \Lambda}^{r-\mathrm{Id}(L)} I_\lambda$ exists. Thus, $I_\lambda \subseteq I$, for any $\lambda \in \Lambda$ and so, in the frame $Id(L)$, I is an upper bound for the family $\{I_\lambda\}_{\lambda \in \Lambda}$. Therefore, $\overset{Id(L)}{\bigvee}_{\lambda \in \Lambda} I_\lambda \subseteq I$. The fact that the ideal I is an r-ideal implies that $\overset{Id(L)}{\bigvee} I_\lambda \subseteq \left(\overset{\mathrm{Id}(L)}{\bigvee}_{\lambda \in \Lambda} I_\lambda \right)_r \subseteq I$. On the other hand, $I_\lambda \subseteq \left(\overset{\mathrm{Id}(L)}{\bigvee}_{\lambda \in \Lambda} I_\lambda \right)_r \in r - \mathrm{Id}(L)$, for any $\lambda \in \Lambda$. Thus, $I = \overset{r-\mathrm{Id}(L)}{\bigvee}_{\lambda \in \Lambda}$

$I_\lambda \subseteq \left(\bigvee^{\mathrm{Id}(L)}_{\lambda\in\Lambda} I_\lambda\right)_r$. Therefore, $\bigvee^{r-\mathrm{Id}(L)}_{\lambda\in\Lambda} I_\lambda = \left(\bigvee^{\mathrm{Id}(L)}_{\lambda\in\Lambda} I_\lambda\right)_r$. Hence, $(r-\mathrm{Id}(L),\subseteq)$ is a complete lattice. By Proposition 3.1, the distributivity of $r-\mathrm{Id}(L)$ can be proved straightforward. □

An ideal I of a commutative ring R with identity is called a cancellation ideal if whenever $IB = IC$ for ideals B and C in R, then $B = C$. In [1], it was shown that an ideal I is a cancellation ideal if and only if I is locally a regular principal ideal. Here, we try to find some facts about the frame counterpart of a cancellation ideal.

Proposition 3.6. *Let I be a proper ideal of frame L. The following statements are true:*

(1) *The ideal I is an r-ideal if and only if for every two ideals J and K in the frame L with $J\cap D(L)\neq\emptyset$, the fact that $J\cap K\subseteq I$ implies $K\subseteq I$.*
(2) *If J and K are two proper r-ideals in the frame L and $I\bigcap D(L)\neq\emptyset$, then $I\bigcap J = I\bigcap K$ implies $J = K$.*

Proof. (1). *Necessity.* Let I be an r-ideal of a frame L. Suppose that $J, K \in \mathrm{Id}(L)$ with $J\cap D(L)\neq\emptyset$. Then there exists an element x in J such that $x^* = \bot$. If $J\cap K\subseteq I$ and $y\in K$, then $x\wedge y\in J\cap K\subseteq I$, which implies that $y\in I$ because I is an r-ideal. Hence, $K\subseteq I$.

Sufficiency. Let $x, y\in L$ with $x\wedge y\in I$ and $x^* = \bot$ be given. From $\downarrow x\cap\downarrow y\subseteq I$ and $\downarrow x\cap D(L)\neq\emptyset$, we infer that $y\in\downarrow y\subseteq I$. Hence, I is an r-ideal.

(2). The proof is straightforward: $I\bigcap J = I\bigcap K\subseteq K$. So, by using part (1), $J\subseteq K$ and also $K\subseteq J$. □

Now, we may give a proposition for r-ideal in the Cartesian product of an arbitrary family of frames. It is well known that if $\{(L_i,\leq_i)\}_{i\in\alpha}$ is a family of frames, then the Cartesian product $\prod_{i\in\alpha} L_i$ with the structure of frame coordinatewise (which is the same as defining the order by

$$(a_i)_{i\in\alpha}\leq (b_i)_{i\in\alpha}\Leftrightarrow \text{ for all } i\in\alpha(a_i\leq_i b_i)$$

is a frame.

Proposition 3.7. *Let $\{L_i\}_{i\in\alpha}$ be an arbitrary family of frames and let $I := \prod_{i\in\alpha} I_i$ be a proper ideal of $L := \prod_{i\in\alpha} L_i$. Then the ideal I is an r-ideal of L if and only if there exists a nonempty subset β of α such that I_i is a proper r-ideal of L_i for every $i\in\beta$, and $I_i = L_i$ for every $i\in\alpha\setminus\beta$.*

Proof. Necessity. We set

$$\beta := \{i \in \alpha \colon I_i \neq L_i\}.$$

By our hypothesis, β is a nonempty subset of α. Let $j \in \beta$ be given. Suppose that $(x, y) \in D(L_j) \times L_j$ with $x \wedge y \in I_j$. We put $(a_i, b_i) = (\top, \bot)$ for all $i \in \alpha \setminus \{j\}$, and $(a_j, b_j) = (x, y)$. Then $((a_i)_{i\in\alpha}, (b_i)_{i\in\alpha}) \in D(L) \times L$ and $(a_i)_{i\in\alpha} \wedge (b_i)_{i\in\alpha} \in I$, which implies from our hypothesis that $(b_i)_{i\in\alpha} \in I$, and so $y = b_j \in I_j$. Hence I_j is a proper r-ideal of L_j for every $j \in \beta$.

Sufficiency. Let $((a_i)_{i\in\alpha}, (b_i)_{i\in\alpha}) \in D(L) \times L$ with $(a_i)_{i\in\alpha} \wedge (b_i)_{i\in\alpha} \in I$ be given. By the fact that for all value of $i \in \beta$, the ideal I_i is an r-ideal in the frame L_i, this implies $b_i \in I_i$, and so $(b_i)_{i\in\alpha} \in I$. □

4. r-IDEALS VS. PRIME IDEALS

The notions of prime ideals and r-ideals are different. Every prime ideal in a frame needs not to be an r-ideal. To illustrate this, consider $\downarrow f$ in Example 2.3, which is a prime ideal but not an r-ideal since $f \in D(L)$. We try to provide some conditions in which a prime ideal is an r-ideal, as in Proposition 4.6.

Every prime ideal contains a minimal prime ideal P, that is, a prime ideal P such that $Q \subset P$, for no prime ideal Q. For every ideal I of L, a prime ideal P is called a minimal prime ideal over I if $I \subseteq P$ and for any prime ideal Q of L, $I \subseteq Q \subseteq P$ implies that $P = Q$. The set of all minimal prime ideals over an ideal I will be denoted by $\mathrm{Min}(I)$. We recall that every ideal I of L is the intersection of all prime ideals containing it; $I = \bigcap \mathrm{Min}(I)$ (see [5, Page 64, Corollary 18]).

Proposition 4.1. *Let I be an ideal of a frame L. Then, the following statements are true:*

1. *If $P \in \mathrm{Min}(I)$ and $x \in P \setminus I$, then there exists an element $y \in L \setminus P$ such that $x \wedge y \in I$.*
2. *A prime ideal P of L is a minimal prime ideal of L if and only if for all $a \in P$, $\downarrow a^* \nsubseteq P$.*

Proof. (1). Suppose that $y \wedge x \notin I$ for every $y \in L \setminus P$. Let F be a filter generated by $(L \setminus P) \cup \{x\}$. From $F \cap I = \emptyset$, we conclude from [5, Theorem15] that there exists a prime ideal Q of L such that $I \subseteq Q$ and $Q \cap F = \emptyset$, which implies that $x \in P = Q$, and this is a contradiction. Hence,

$$\text{for all } x \in P \setminus I \ \text{ there exists } y \in L \setminus P (y \wedge x \in I).$$

(2). *Necessity.* The proof is straightforward by using part (1).

Sufficiency. Let Q be a prime ideal of L and let $Q \subsetneq P$. If $a \in P \setminus Q$, then ${\downarrow}a^* \subseteq Q \subseteq P$. This contradicts the fact that P is a minimal prime ideal of L. ☐

Now, we have provided a place where we can prove a suitable generalization of Proposition 3.2, as follows.

Proposition 4.2. *Let L be a frame and let $a \in L$ with $a \vee a^* = \top$ be given. If P is a minimal prime ideal of L, then $I := P \vee {\downarrow}a^*$ is an r-ideal of L.*

Proof. Let $x, y \in L$ with $x \wedge y \in I$ and $x^* = \bot$ be given. Then, there exist $v \in {\downarrow}a^*$ and $w \in P$ such that $x \wedge y = v \vee w$. By Proposition 4.1, there exists an element z in $L \setminus P$ such that $z \wedge w = \bot$. Therefore, we have

$$\begin{aligned} x \wedge y \wedge a \wedge z = \bot \Rightarrow y \wedge a \wedge z \leq x^* = \bot \Rightarrow y \wedge a \in P \\ \Rightarrow y = y \wedge (a \vee a^*) = (y \wedge a) \vee (y \wedge a^*) \in I. \end{aligned}$$

Hence, I is an r-ideal of L. ☐

Proposition 4.3. *Let L be a frame. Then, the following statements are true:*

1. *If I is a proper r-ideal of L and $P \in \mathrm{Min}(I)$, then P is a proper r-ideal of L; especially, every minimal prime ideal of L is a proper r-ideal.*
2. *If P is an r-ideal of L for every $P \in \mathrm{Min}(I)$, then I is an r-ideal of L.*
3. *If P is a prime ideal of L such that $P \cap D(L) = \emptyset$, then it is an r-ideal.*
4. *For every ideal I of L, $\mathrm{Min}(I_r) \subseteq \mathrm{Min}(I)$.*

Proof. (1). Let $x, y \in L$ with $x \wedge y \in P$ and $x^* = \bot$ be given. Thus, by Proposition 4.1, there exists an element $z \in L \setminus P$ such that $x \wedge y \wedge z \in I$, which implies that $y \wedge z \in I \subseteq P$, and so $y \in P$.

(2). Since $I = \bigcap \mathrm{Min}(I)$, it is an r-ideal.

(3). Let $x, y \in L$ with $x \wedge y \in P$ and $x^* = \bot$ be given. Since, by hypothesis, $x \notin P$, we conclude that $y \in P$.

(4). Let $P \in \mathrm{Min}(I_r)$ be given, and let Q be a prime ideal such that $I \subseteq Q \subseteq P$. Suppose that $x \in I_r$. Thus, there exists an element $y \in L$ such that $x \wedge y \in I$ and $y^* = \bot$. If $y \in Q$, then $y \wedge z \in P$, which implies from part (1) that $z \in P$ for every $z \in L$, and so $P = L$, which is a contradiction with the fact that $P \in \mathrm{Min}(I_r)$. From $x \wedge y \in Q$ and $y \notin Q$, we conclude that $x \in Q$. Hence, $I_r \subseteq Q \subseteq P \in \mathrm{Min}(I_r)$, which implies that $P = Q$. Therefore, $P \in \mathrm{Min}(I)$. ☐

We recall an old established theorem (see [2, Theorem 1.5] and prove a significant characterization of r-ideals in a frame L, with respect to minimal prime ideals in L.

Theorem 4.4. [2]. *Let I be an ideal in a distributive pseudo-complemented lattice $(L, \vee, \wedge, *, 0, 1)$. Then $x^{**} \in I$ for all x in I if and only if each minimal prime ideal over I is a minimal prime ideal of L if and only if I is an intersection of minimal prime ideals in the lattice L.*

Theorem 4.5. *Let I be a proper ideal in frame L. Then, the following statements are equivalent:*

(1) *The ideal I is an r-ideal of L.*
(2) *If x is an element in I, then x^{**} belongs to I.*
(3) *The ideal I is an intersection of minimal prime ideals of L.*

Proof. (1) $\Rightarrow$ (2). Let I be an r-ideal of L, and consider x in I. We show that x^{**} is also, an element in I. By the fact that $I = I_r$, there is $r \in D(L)$ such that $a := x \wedge r \in I$. It can be shown that $x^{**} \wedge a^* = x^{**} \wedge (x \wedge r)^* = \perp$. Let $y := x^{**}$. Then, $y^{**} \wedge a^* = \perp$ and so, $y^* \vee a \in D(L)$. The ideal I is an r-ideal and $y \wedge (y^* \vee a) = (y \wedge a) \in I$. Thus, $x^{**} = y$ is in I.

(2) $\Rightarrow$ (3). This is a part of Theorem 4.4.

(3) $\Rightarrow$ (1). By Proposition 4.6, every minimal prime ideal in L is an r-ideal. Since each intersection of r-ideals is also, r-ideal, the proof is complete. □

Proposition 4.6. *Let I and P be proper ideals of a frame L, where P is a prime ideal of L. Then, the following statements are true:*

(1) *If $I \cap P$ is an r-ideal, then I or P is an r-ideal.*
(2) *Let I and P be prime ideals that are not in a chain. If $I \cap P$ is an r-ideal, then I and P are r-ideals.*

Proof. (1). If $I \subseteq P$, that is $I \cap P = I$, then, by our hypothesis, I is an r-ideal. Now, suppose that $I \not\subseteq P$. Then, there exists an element i in $I \setminus P$. Let $(a, b) \in D(L) \times L$ with $a \wedge b \in P$ be given. Thus, $i \wedge a \wedge b \in I \cap P$, which implies that $i \wedge b \in I \cap P \subseteq P$, and we obtain that $b \in P$. Therefore, P is an r-ideal.

(2). The proof is similar to the proof of part (1). □

It is established, in Proposition 3.5 that the set of all r-ideals in a frame L can be considered as a complete distributive lattice $r-\mathrm{Id}(L)$. We characterize the prime elements of $r - \mathrm{Id}(L)$, as follows.

Proposition 4.7. *The following statements are true:*

(1) *An ideal P of a frame L is a prime element of $r - \mathrm{Id}(L)$ if and only if P is a prime r-ideal of L.*
(2) *An ideal P of a frame L is a minimal prime ideal of L if and only if P is a minimal prime element of $r - \mathrm{Id}(L)$.*

Proof. (1). *Necessity.* Let P be a prime element of $r - \mathrm{Id}(L)$. We show that P is a prime r-ideal in L. If a and b are two elements in L, with $a \wedge b \in P$ and b is in $D(L)$, then Proposition 3.1 implies $(\downarrow a)_r \wedge (\downarrow b)_r = (\downarrow(a \wedge b))_r \subseteq P_r = P$. Since $P \cap D(L) = \emptyset$, $(\downarrow b)_r$ is not contained in P and so, by the fact that P is a prime element of $r - \mathrm{Id}(L)$, the ideal $\downarrow(a)_r$ is contained in P. Therefore, $\downarrow(a) \subseteq P$ and a is in P.

Sufficiency. Let P be a prime r-ideal of L and let J and K be two r-ideals with $J \wedge K \subseteq P$ and $J \not\subseteq P$. Then, there exists an element j in $J \setminus P$ such that $j \wedge k \in P$ for all k in K. Thus, $k \in P$ for all k in K, which implies that $K \subseteq P$.

(2). *Necessity.* By Proposition 4.6 and part (1), P is a prime element of $r - \mathrm{Id}(L)$. Let Q be another prime element of $r - \mathrm{Id}(L)$, such that $Q \subseteq P$. Then, by part (1), Q is a prime r-ideal of L, which implies that $Q = P$. Hence, P is a minimal prime element of $r - \mathrm{Id}(L)$.

Sufficiency. By Proposition 4.5 and part (1), it is evident. □

5. A characterization of boolean algebras via r-ideals

In this section, we characterize frames where every ideal is an r-ideal. We start with the following definition.

Definition 5.1. A frame L is said to be r-meet-cancellative if for $(r, x, y) \in D(L) \times L \times L$, $r \wedge x = r \wedge y$ implies $x = y$.

Proposition 5.2. *Let L be a frame. Then, the following statements are equivalent:*

(1) *The frame L is a boolean algebra.*
(2) *The frame L is an r-meet-cancellative.*
(3) $D(L) = \{\top\}$.
(4) *If $I \in D(\mathrm{Id}(L))$ with $I \neq L$, then I is an r-ideal of L.*
(5) *Every principal ideal of L is an r-ideal of L.*
(6) *Every ideal of L is an r-ideal of L.*
(7) *For every ideal of L is an intersection of minimal prime ideals of L.*
(8) *Every prime ideal of L is an r-ideal of L.*
(9) *Every maximal ideal of L is an r-ideal of L.*

Proof. (1) $\Rightarrow$ (2). It is evident.

(2) $\Rightarrow$ (3). It is clear that $\{\top\} \subseteq D(L)$. If x belongs to $D(L)$, then $x \wedge x = x = x \wedge \top$, which implies from our hypothesis that $x = \top$. Hence, $D(L) = \{\top\}$.

(3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6). Let I be an arbitrary ideal of L. Then, by our hypothesis, $(I)_r = I$, and so I is an r-ideal of L. Hence, every ideal of L is an r-ideal.

(4) $\Rightarrow$ (5). Let $\top \neq a \in L$ be given. If $a \vee a^* = \top$, then, by Proposition 3.3, $\downarrow a$ is an r-ideal. Now, suppose that $a \vee a^* \neq \top$ and that $x, y \in L$ with $x \wedge y \in \downarrow a$ and $x^* = \bot$. If $y = \bot$, then $y \in \downarrow a$. Now, suppose that $y \neq \bot$. We set

$$\mathcal{A} := \big\{I \colon I \text{ is an ideal of } L \text{ such that } I \cap \downarrow a = \{\bot\}\big\}.$$

By Zorn's lemma, there exists a maximal element Q of $\mathcal{A}$ with respect to inclusion. From

$$\begin{aligned} Q \vee \downarrow a = L \Rightarrow &\text{ there exist } q \in Q\big(q \vee a = \top \text{ and } q \wedge a \in Q \cap \downarrow a = \{\bot\}\big) \\ &\Rightarrow a \vee a^* = \top, \end{aligned}$$

we conclude that $Q \vee \downarrow a$ is a proper essential ideal of L such that $Q \vee \downarrow a \in D\big(\mathrm{Id}(L)\big)$, which implies from our hypothesis that it is an r-ideal, and hence, $y \in Q \vee \downarrow a$. From

$$\begin{aligned} z \in Q \cap \downarrow y \Rightarrow &\big(x \wedge z \leq x \wedge y \in \downarrow a \text{ and } x \wedge z \in Q\big) \Rightarrow x \wedge z \in Q \cap \downarrow a = \{\bot\} \\ &\Rightarrow z = \bot, \end{aligned}$$

we conclude that $\downarrow y = (Q \vee \downarrow a) \wedge \downarrow y = (Q \wedge \downarrow y) \vee \downarrow (a \wedge y) = \downarrow (a \wedge y)$, which implies that $y = a \wedge y \in \downarrow a$. Therefore, every proper principal ideal of L is an r-ideal.

(5) $\Rightarrow$ (6). It is evident.

(6) $\Leftrightarrow$ (7). By Theorem 4.5, these two statements are equivalent:

(6) $\Rightarrow$ (8). It is evident.

(8) $\Rightarrow$ (2). Let $\top \neq a \in L$ be given. Now, suppose that $x, y \in L$ with $x \wedge y \in \downarrow a$ and $x^* = \bot$. If $y \notin \downarrow a$, then $\downarrow a \cap \uparrow y = \emptyset$, which implies that there exists a prime ideal P of L such that $\downarrow a \subseteq P$ and $P \cap \uparrow y = \emptyset$. Since, by hypothesis, P is an r-ideal and $\downarrow a \subseteq P$, we infer that $y \in P$, which is a contradiction. Therefore, every proper principal ideal of L is an r-ideal. Now, suppose that $r \wedge x = r \wedge y$ for some $(r, x, y) \in D(L) \times L \times L$. Let $x \neq \top$. Then $r \wedge y \in \downarrow x$ implies $y \leq x$, which yields that $y \neq \top$. A similar argument shows that $x \leq y$; that is, $x = y$.

(2) $\Rightarrow$ (1). Let a be an element in an r-meet-cancellative frame L. Thus, $a \vee a^* \in D\,(L)$, which implies that $(a \vee a^*) \wedge (a \vee a^*) = (a \vee a^*) \wedge \top$, and we deduce form our hypothesis that $a \vee a^* = \top$. Hence, L is a boolean algebra.

(7) $\Rightarrow$ (8). It is evident.

(8) $\Rightarrow$ (1). Let a be an element in a frame L, and it dose not have any complement on the contrary. Consider $x := a \vee a^* \neq \top$ and

$$\mathcal{A} = \big\{I \in Id(L) : x \in I \neq L\big\}.$$

If $\big\{I_\lambda\big\}_\lambda$ be a chain in the poset $(\mathcal{A}, \subseteq)$, then $\bigcup_{\lambda\in\Lambda} I_\lambda \neq L$ is an ideal containing x and the upper bound of the chain. Therefore, according to Zorn's lemma, $\mathcal{A}$ has a maximal element as M, which is also a maximal ideal of L. Since, by our hypothesis, $x \wedge \top = x$ is an element of the r-ideal M and $x^* = \perp$, we conclude that $\top \in M$, which is a contradiction. □

Let S be a subset of a frame L, which is closed under finite meets and without bottom element. Define a binary relation $\sim$ on L as follows:

$$x \sim y \;\Leftrightarrow\; x \wedge s = y \wedge s \text{ for some } s \in S. \tag{5.1}$$

Clearly this relation is an equivalence relation on L. This can be checked by a direct computation. The relation $\sim$ is a congruence on L in the sense that for all x and y in L, $x \sim y$ implies for all z in L, $z \vee x \sim z \vee y$ and $z \wedge x \sim z \wedge y$. The join and meet operations between the elements of $L/{\sim} := \big\{ [x]_\sim : x \in L\big\}$ as the quotient frame with respect to $\sim$ are naturally defined by

$$[x]_\sim \vee [y]_\sim := [x \vee y]_\sim$$
$$[x]_\sim \wedge [y]_\sim := [x \wedge y]_\sim .$$

One can easily confirm that $a \leq b$ if and only if $[a]_\sim \leq [b]_\sim$ for every $a, b \in L$. The notation $S^{-1}L$ will be used to show the quotient frame $(L/{\sim}, \vee, \wedge)$ and in the case of $S = D(L)$ is called *the classical frame of quotients of* L, which is denoted by $Q(L)$. One can easily check that

- $S^{-1}L$ is a frame.
- $[a]^*_\sim = [a^*]_\sim$ in $Q(L)$ for every $a \in L$.
- $[\perp]_\sim = \{ \perp \}$ in $Q(L)$.
- $D\big(Q(L)\big) = \big\{[\top]_\sim\big\} = \big\{D(L)\big\}$.

By Proposition 5.2 and the above topics, we have the next corollary.

Corollary 5.3. *The following statements is true for $Q(L)$, the classical frame of quotients of a frame L:*

(1) $|D(Q(L))| = 1$.
(2) $Q(L)$ *is an r-meet cancelative frame.*
(3) $Q(L)$ *is a boolean algebra.*

Let S be a subset of a frame L that is closed under finite meets and without bottom element. We have a frame map $f\colon L \to S^{-1}L$ defined by $f(x) = [x]_\sim$.

This is not in general injective. We define the extension I^e of an ideal I of L to be the ideal $(f(I))$ generated by $f(I)$ in $S^{-1}L$. Also, $f^{-1}(J)$ is called the contraction J^c of an ideal J of $S^{-1}L$.

Proposition 5.4. *For every proper ideal I of a frame L, I is an r-ideal if and only if $I = J^c$ for some ideal J in $Q(L)$.*

Proof. Necessity. Let I be an r-ideal of a frame L. it is well known that $I \subseteq I^{ec}$. Assume that $x \in L$ with $x \in I^{ec}$. Then we have $[x] \in I^e$, which implies that there exists an element $y \in I$ such that $[x] \leq [y]$, and thus

$$[x] = [x \wedge y] \Rightarrow \text{ there exists } z \in D(L)\big(z \wedge x = z \wedge (x \wedge y) \in I\big) \Rightarrow x \in I.$$

Hence, $I = I^{ec}$.

Sufficiency. Let J be an ideal of $Q(L)$. Let $x, y \in L$ with $x \wedge y \in J^c$ and $x^* = \bot$ be given. Then, $[x] \wedge [y] = [x \wedge y] \in J$, which implies that $[y] \in J$, and thus $y \in J^c$. □

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r-IDEAL IN A FRAME

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$r-$ ایده‌آل در قاب

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اخیراً، مفهوم $r-$ایده‌آل در حلقه‌ی جابه‌جایی و همچنین در نیم‌گروه جابه‌جایی تعریف شده است. در اینجا، تعریف مشابهی برای $r-$ایده‌آل در قاب ارائه می‌دهیم و برخی از ویژگی‌های آن را بررسی می‌کنیم. برخی از مثال‌ها تایید می‌کنند که ویژگی‌های $r-$ایده‌آل در قاب‌ها لزوماً با ویژگی‌های $r-$ایده‌آل در حلقه‌های جابه‌جایی (یا نیم‌گروه های جابه‌جایی) منطبق نیست. چند (ویژگی) مشخصه برای $r-$ایده‌آل در قاب می‌یابیم. به طور خاص، نشان می‌دهیم هر $r-$ایده‌آل سره در قاب، اشتراکی از ایده‌آل‌های اوّل در آن قاب است. همچنین، جبر بول‌ها را تحت شرط اینکه هر ایده‌آل قاب، $r-$ایده‌آل باشد، مشخص می‌کنیم.

کلمات کلیدی: قاب، $r-$ایده‌آل، ایده‌آل اول مینیمال، جبر بول.