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GENERALIZATIONS OF RADICAL IDEALS IN NONCOMMUTATIVE RINGS

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Abstract. In this study, we present the generalization of the concept of radical-ideals in noncommutative rings with nonzero identity. Let R be a noncommutative ring with $1 \neq 0$ and $S(R)$ be the set of all ideals of R. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and let ρ be a special radical. A proper ideal I of R is said to be a ϕ-ρ-ideal of R if whenever $a, b \in R$ with $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$ and $a \notin \rho(R)$, then $b \in I$. Many of the results on n-ideals, J-ideals and their generalizations like weakly n-ideals and weakly J-ideals will follow as special cases from results proved for ϕ-ρ-ideals in this paper.

1. Introduction

In [13] the notion of an n-ideal was introduced. Later, following this, in [10] the notion of a $\mathcal{J}$-ideal was introduced. The $\mathcal{J}$-ideal is connected to the Jacobson radical and the n-radical is connected to the prime radical. Lately, in [8] we extended these notions to noncommutative rings and show that these notions are special cases of a general type of ideal, connected to a special radical.

In [11] the notion of a weakly $\mathcal{J}$-ideal was introduced for a commutative ring as a new generalization of $\mathcal{J}$-ideals. In [9] the notion of a weakly $\mathcal{J}$-ideal was extended to noncommutative rings and it was also shown that it is a special case of a more general type of ideal connected to a special radical In [1] the notion of an almost prime ideal in noncommutative rings was introduced. In [7] the notion of an almost ρ-ideal for a special radical ρ was introduced. Let R be a noncommutative ring with $0 \neq 1$ and $S(R)$ be the lattice of all ideals of R. Suppose that $\phi : S(R) \to S(R) \cup \{\emptyset\}$ is a function. In this study we generalize the concept of ρ-ideal in a noncommutative ring via a function $\phi : S(R) \to S(R) \cup \{\emptyset\}$. We show that this covers all the previous definitions in a noncommutative ring.

For the following definitions of special radicals and related results we refer the reader to [4].

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A class ρ of rings forms a radical class in the sense of Amitsur-Kurosh, if ρ has the following three properties:

(1) The class ρ is closed under homomorphism, that is, if $R \in \rho$, then $R/I \in \rho$ for every $I \lhd R$.
(2) Let R be any ring. If we define $\rho(R) = \sum\{I \lhd R : I \in \rho\}$, then $\rho(R) \in \rho$.
(3) For any ring R the factor ring $R/\rho(R)$ has no nonzero ideal in ρ i.e. $\rho(R/\rho(R)) = 0$.

A class $\mathcal{M}$ of rings is a **special class** if it is hereditary, consists of prime rings and satisfies the following condition $(*)$ if $0 \neq I \lhd R$, $I \in \mathcal{M}$ and R a prime ring, then $R \in \mathcal{M}$.

Let $\mathcal{M}$ be any special class of rings. The class

$$\mathcal{U}(\mathcal{M}) = \{R : R \text{ has no nonzero homomorphic image in } \mathcal{M}\}$$

of rings forms a radical class of rings and the upper radical class $\mathcal{U}(\mathcal{M})$ is called a special radical class.

Let ρ be a special radical with special class $\mathcal{M}$ i.e. $\rho = \mathcal{U}(\mathcal{M})$. Now let $\mathcal{S}_\rho = \{R : \rho(R) = 0\}$. If $\mathcal{P}$ denotes the class of prime rings, then for the special radical ρ it follows from [4] that $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$. For a ring R we have $\rho(R) = \cap\{I \lhd R : R/I \in \mathcal{P} \cap \mathcal{S}_\rho\}$ i.e. ρ has the intersection property relative to the class $\mathcal{P} \cap \mathcal{S}_\rho$.

Let $I \lhd R$. Then $\rho(R/I) = \rho^*(I)/I$ for some uniquely determined ideal $\rho^*(I)$ of R with $\rho(I) \subseteq I \subseteq \rho^*(I)$ and $\rho^*(I)$ is called the radical of the ideal I while $\rho(I)$ is the radical of the ring I.

We also have $\rho^*(I) = \rho(R)$ if and only if $I \subseteq \rho(R)$.

In what follows, let ρ be a special radical with special class $\mathcal{M}$. Hence $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$.

The following are some of the well known special radicals which are defined in [4], prime radical $\mathcal{P}$, Levitski radical $\mathcal{L}$, Kőthe's nil radical $\mathcal{N}$, Jacobson radical $\mathcal{J}$ and the Brown McCoy radical $\mathcal{G}$.

Recall [8], that if $\mathcal{P}(R)$ is the prime radical of the ring R, then an ideal P of R is a $\mathcal{P}$-ideal if for $a, b \in R$ and $aRb \subseteq P$, then $a \in \mathcal{P}(R)$ or $b \in P$. This notion was first introduced by [13] for commutative rings and defined as an n-ideal as follows. If P is an ideal of the commutative ring R and $a, b \in R$ such that $ab \in P$, then $a \in \sqrt{\{0\}}$ or $b \in P$ where $\sqrt{\{0\}}$ is the nil radical of R. In [8] it was shown that if the ring is commutative these two notions are the same. Also recall that if $\mathcal{J}(R)$ is the Jacobson radical of the ring R, then the ideal P of the ring is a $\mathcal{J}$-ideal of R if for $a, b \in R$ and $aRb \subseteq P$, then $a \in \mathcal{J}(R)$ or $b \in P$. This notion was first introduced by [10] for commutative rings and defined as an J-ideal as follows. If P is an ideal of the commutative ring R and $a, b \in R$ such that $ab \in P$, then $a \in J(R)$ or $b \in P$ where $J(R)$ is the Jacobson radical of R. In [8] it was again shown that if the ring is commutative these two notions are the same. In [8] it was shown that both these notions are included in the more general notion of a ρ-ideal. In [8] the notion of *ρ-ideal* was defined as follows. If ρ is a special radical, then an ideal P of the ring R is called a ρ-ideal if $a, b \in R$ and $aRb \subseteq P$, then $a \in \rho(R)$ or $b \in P$. In a very recent paper, in [11], the notion of weakly J-ideals in commutative rings was presented. A weakly J-ideal of the commutative ring R is a proper ideal with the property that $a, b \in R$, $0 \neq ab \in I$ and $a \notin J(R)$ implies $b \in I$. In [9] it is shown that this notion is again part of a more general notion of a weakly ρ-ideal for some special radical. A proper ideal with the property that $a, b \in R$, $\{0\} \neq aRb \subseteq I$ and $a \notin \rho(R)$ implies $b \in I$ for a special radical ρ is called a *weakly ρ-ideal* . In [7] the notion of an almost ρ-ideal was introduced. For a special radical ρ an ideal P of a noncommutative ring is an *almost ρ-ideal* if for $a, b \in R$ $aRb \subseteq P$ and $aRb \nsubseteq P^2$ implies that $a \in \rho(R)$ or $b \in P$. As is a natural step ahead, we show that the notions of ρ-ideal, weakly ρ-ideal and almost ρ-ideal form part of an even more general situation. Let R be a noncommutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. We call I a *ϕ-ρ-idea*l of R if whenever $a, b \in R$ and $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$, then either $a \in \rho(R)$ or $b \in I$.

2. Definitions and general results

In what follows, the rings are noncommutative, but not necessarily assumed to have an identity element unless indicated. We note that for an element $a \in R$,

$$\langle a \rangle = \left\{ \sum_{i=1}^{n} r_i a s_i + ra + as + ma : n \in \mathbb{N},\ m \in \mathbb{Z},\ r_i, s_i, r, s \in R \right\}.$$

Clearly if R is a ring with identity element, then

$$\langle a \rangle = \left\{ \sum_{i=1}^{n} r_i a s_i : n \in \mathbb{N},\ r_i, s_i \in R \right\}.$$

Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. For two elements a and b of a ring R, the following statements are equivalent:

(1) $\langle a \rangle \langle b \rangle \subseteq \phi(I)$;
(2) $a \langle b \rangle \subseteq \phi(I)$;
(3) $\langle a \rangle b \subseteq \phi(I)$;
(4) $aRb \subseteq \phi(I)$ and $ab \in \phi(I)$.

Definition 2.1. Let R be a noncommutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. We call I a *ϕ-ρ-ideal* of R if whenever $a, b \in R$ and $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$, then either $a \in \rho(R)$ or $b \in I$.

Definition 2.2. Let R be a commutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. We call I a *ϕ-ρ-ideal* of R if whenever $a, b \in R$ and $ab \in I - \phi(I)$, then either $a \in \rho(R)$ or $b \in I$.

For a commutative ring with identity and a special radical ρ we show in the following proposition how to construct ϕ-ρ-ideals.

Proposition 2.3. *Let ρ be a special radical and let R be a commutative ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. For any ring R and an ideal $P \nsubseteq R$ the following are equivalent:*

(1) *P is a ϕ-ρ-ideal;*
(2) *$(P : a) = P \cup (\phi(R) : a)$ for every $a \notin \rho(R)$;*
(3) *$(P : a) = P$ or $(P : a) = (\phi(R) : a)$ for every $a \notin \rho(R)$;*
(4) *For ideals A and B of R $AB \subseteq P$ and $AB \nsubseteq \phi(P)$ implies $A \subseteq \rho(R)$ or $B \subseteq P$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R - \rho(R)$. Let $b \in (P : a)$ so $ab \in P$.
If $ab \in P - \phi(P)$, then $b \in P$. If $ab \in \phi(P)$, then $b \in (\phi(P) : a)$ so $(P : a) \subseteq P \cup (\phi(R) : a)$. The other containment always holds (remember we are assuming $\phi(P) \subseteq P$).

(2) $\Rightarrow$ (3) If an ideal is a union of two ideals, it is equal to one of them.

(3) $\Rightarrow$ (4) Let A and B be ideals of R with $AB \subseteq P$. Suppose that $A \nsubseteq \rho(R)$ and $B \nsubseteq P$. We show that $AB \subseteq \phi(P)$. Let $a \in A$. First, suppose that $a \notin \rho(R)$, then $aB \subseteq P$ gives $B \subseteq (P : a)$. Now $B \nsubseteq P$, so $(P : a) = (\phi(P) : a)$. Hence $aB \subseteq \phi(P)$. Next, let $a \in \rho(R) \cap P$. Choose $a^{'} \in A - \rho(R)$. Then $a + a^{'} \in A - \rho(R)$ and by the first case, $a^{'}B, (a + a^{'})B \subseteq \phi(P)$. Let $b \in B$. Then $ab = (a + a')b - a^{'}b \in \phi(P)$. So $aB \subseteq \phi(P)$. Thus $AB \subseteq \phi(P)$.

(4) $\Rightarrow$ (1) Let $ab \in P - \phi(P)$. Then $\langle a \rangle \langle b \rangle \subseteq P$, but $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$. So $\langle a \rangle \subseteq \rho(R)$ or $\langle b \rangle \subseteq P$, i.e. $a \in \rho(R)$ or $b \in P$. $\square$

Remark 2.4. If R is a commutative ring, the above definitions are equivalent. Definition 2.1 implies Definition 2.2: Let $a, b \in R$ such that $ab \in P - \phi(P)$. Hence $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. From Definition 2.1 $a \in \rho(R)$ or $b \in P$. For the other direction, suppose Definition 2.2 is satisfied. Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(R)$. Hence $\langle a\rangle \langle b\rangle \subseteq P$, but $\langle a\rangle \langle b\rangle \nsubseteq \phi(P)$. From Proposition 2.3 $a \in \langle a\rangle \subseteq \rho(R)$ or $b \in \langle b\rangle \subseteq P$ and we are done.

Remark 2.5. Let R be a noncommutative ring with identity and ρ a special radical. Let I be a ϕ-ρ-ideal of R. Consider the following functions:

(1) If $\phi(I) = \{\emptyset\}$ for every $I \in S(R)$, then we say that $\phi = \phi_\emptyset$ and I is called a $\phi_\emptyset$-ρ-ideal ideal of R. Hence I is a ρ-ideal of R. See [8] for results on ρ-ideals and if ρ is the prime radical or the Jacobson radical, see [13] and [10] for results on n-ideals and J-ideals in commutative rings.

(2) If $\phi(I) = \{0\}$ for every $I \in S(R)$, then we say that $\phi = \phi_0$ and I is called a ϕ_0-ρ-ideal of R, and hence I is a weakly ρ-ideal of R. See[9] for results on weakly ρ-ideals and if ρ is the Jacobson radical see [11] for results on weakly J-ideals in commutative rings.

(3) If $\phi(I) = I$ for every $I \in S(R)$, then we say that $\phi = \phi_1$ and I is called a ϕ_1-ρ-ideal of R.

(4) If $n \geq 2$ and $\phi(I) = I^n$ for every $I \in S(R)$, then we say that $\phi = \phi_n$ and I is called a ϕ_n-ρ-ideal of R. In particular, if $n = 2$ and $\phi(I) = I^2$ for every $I \in S(R)$, then we say that I is an almost-ρ-ideal of R. See [7] for results on almost-ρ-ideals.

(5) If $\phi(I) = \bigcap_{n=1}^{\infty} I^n$ for every $I \in S(R)$, then we say that $\phi = \phi_\omega$ and I is called a ϕ_ω-ρ-ideal (ω-ρ-ideal) of R.

Since $I - \phi(I) = I - (I \cap \phi(I))$, without loss of generality, we may assume that $\phi(I) \subseteq I$. Given two functions $\psi_1, \psi_2 : S(R) \to S(R) \cup \{\emptyset\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in S(R)$.

Definition 2.6. Let R be a noncommutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. We call I a *ϕ-ρ-primary ideal* if whenever $a, b \in R$ such that $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$, then $a \in \rho^*(I)$ or $b \in I$.

Example 2.7. Let ρ be the prime radical. For every ring the zero ideal is a ϕ_k-ρ-ideal of R for all $k \geq 0$. However, it may not be a $\mathcal{P}$-ideal. Consider the ring $M_2(\mathbb{Z}pq)$ for p and q distinct prime integers. Consider the ring $M_2(\mathbb{Z}_{pq})$ and the ideal $I = \{0\}$.

Now $\begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} M_2(\mathbb{Z}_{pq}) \begin{bmatrix} \overline{q} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \subseteq \left\langle \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \right\rangle$ but $\begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin \mathcal{P}(M_2(\mathbb{Z}_{pq}))$ and $\begin{bmatrix} \overline{q} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin \left\langle \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \right\rangle$. So $\left\langle \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \right\rangle$ is a not a $\mathcal{P}$-ideal ($\phi_\emptyset$-$\mathcal{P}$-ideal) of $M_2(\mathbb{Z}_{pq})$.

Example 2.8. Consider the ring $M_2(\mathbb{Z}p^2q)$ for p and q distinct prime integers and consider the ideal $P = M_2\left(p^2\mathbb{Z}_{p^2q}\right)$ of R. Clearly, P is a ϕ_2-$\mathcal{P}$-ideal of R since $P^2 = P$. Indeed, P is not a $\mathcal{P}-$ideal of R since

$$\begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} M_2(\mathbb{Z}p^2q) \begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \subseteq P$$

and $\begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin \mathcal{P}(M_2(\mathbb{Z}p^2q))$ and $\begin{bmatrix} \overline{p} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin P$.

Lemma 2.9. *Let ρ be a special radical and let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is an ideal of the ring R such that $I \subseteq \rho(R)$, then I is a ϕ-ρ-ideal if and only if I is a ϕ-ρ-primary ideal.*

Proof. Suppose I is a ϕ-ρ-ideal of R and $a, b \in R$ such that $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$. Since $I \subseteq \rho(R)$, we have $\rho^*(I) = \rho(R)$ and because I is a ϕ-ρ-ideal of R, we have $a \in \rho(R) = \rho^*(I)$ or $b \in I$. Hence I is a ϕ-ρ-primary ideal. Now, again using the fact that $\rho(R) = \rho^*(I)$, we have that I is a ϕ-ρ-ideal of R if it is a ϕ-ρ-primary ideal. □

Let I be a proper ideal of R. Then

$$\sqrt{I} = \sum \{V \lhd R | V^n \subseteq I \text{ for some positive integer } n\}$$

denotes the radical ideal of R. Recall from [3] a proper ideal I of R is a *ϕ-principally right primary ideal* of R if whenever $a, b \in R$ with $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$ implies $a \in I$ or $b \in \sqrt{I}$ and a *ϕ-2-absorbing principally right primary ideal* of R if whenever $a, b, c \in R$ with $aRbRc \subseteq I$ and $aRbRc \nsubseteq \phi(I)$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Theorem 2.10. *Let ρ be any special radical. For any $I \in S(R)$, the following statements hold:*

1. *Let $\psi_1, \psi_2 : S(R) \to S(R) \cup \{\emptyset\}$ be two functions such that $\psi_1 \leq \psi_2$. If I is a ψ_1-ρ-ideal, then I is a ψ_2-ρ-ideal;*
2. *I is a ρ-ideal $\Rightarrow$ I is a weakly-ρ-ideal $\Rightarrow$ I is ϕ_ω-ρ-ideal $\Rightarrow$ I is a ϕ_{k+1}-ρ-ideal for every $k \geq 2$ $\Rightarrow$ I is a ϕ_k-ρ-ideal for every $k \geq 2$ $\Rightarrow$ I is an almost-ρ-ideal;*
3. *I is a ϕ-ρ-ideal $\Rightarrow$ I is a ϕ-principally right primary ideal $\Rightarrow$ I is ϕ-2-absorbing principally right primary ideal;*
4. *I is an idempotent ideal of R $\Rightarrow$ I is a ϕ_k-ρ-ideal for every $k \geq 1$;*
5. *I is a ϕ_k-ρ-ideal for every $k \geq 2$ $\Rightarrow$ I is a ϕ_ω-ρ-ideal.*

Proof. (1) It is straightforward.

(2) It is clear that there is a linear ordering

$$\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{k+1} \leq \phi_k \leq \cdots \leq \phi_2 \leq \phi_1.$$

So we obtain the result.

(3) It is clear.

(4) Since I is idempotent $I^k = I^2 = I$ for all $k \geq 2$. Hence $\phi_k(I) = I$ for all $k \geq 1$. We are done.

(5) It is clear by 2. □

Theorem 2.11. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Let I be a proper ideal of R.*

1. *Let $\phi(I) \subseteq \rho(R)$. If I is a ϕ-ρ-ideal of R, then $I/\phi(I)$ is a ρ-ideal of $R/\phi(I)$.*
2. *Let $\phi(I)$ be a ρ-ideal of R. If $I/\phi(I)$ is a ρ-ideal of $R/\phi(I)$, then I is a ϕ-ρ-ideal of R*

Proof. (1) Suppose that I is a ϕ-ρ-ideal of R. Let $a + \phi(I)$, $b + \phi(I) \in R/\phi(I)$ such that $(a + \phi(I))R/\phi(I)(b + \phi(I)) \subseteq I/\phi(I)$ and $a + \phi(I) \notin \rho(R/\phi(I))$. Hence $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$. Since $\rho(R)/\phi(I) \subseteq \rho(R/\phi(I))$ and $a + \phi(I) \notin \rho(R/\phi(I))$, we have $a \notin \rho(R)$. Thus, as I is a ϕ-ρ-ideal of R, we obtain $b \in I$. This implies that $b + \phi(I) \in I/\phi(I)$ and we are done.

(2) Let $I/\phi(I)$ be a ρ-ideal of $R/\phi(I)$. Choose $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$ and $a \notin \rho(R)$. Then $(a+\phi(I))R/\phi(I)(b+\phi(I)) \subseteq I/\phi(I)$ and since $a \notin \rho(R)$ we also have $a + \phi(I) \notin \rho(R/\phi(I))$. Now, since $I/\phi(I)$ is a ρ-ideal of $R/\phi(I)$, we have $(b + \phi(I)) \in I/\phi(I)$. Hence $b \in I$ and we are done. □

Theorem 2.12. *Let ρ be a special radical with R a noncommutative ring and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function. If I is a proper ideal of R such that $I = \rho(R)$, then the following are equivalent:*

1. *I is a ϕ-ρ-ideal;*
2. *I is ϕ-prime;*
3. *I is ϕ-principally right primary.*

Proof. (1) ⇒ (2) Let $r, s \in R$ such that $rRs \subseteq I$ and $rRs \nsubseteq \phi(I)$. Suppose $r \notin I = \rho(R)$. Since I is a ϕ-ρ-ideal, we have $s \in I$ and hence I is ϕ-prime.

(2) ⇒ (1) This is clear since $I = \rho(R)$.

(2) ⇒ (3) Let $a, b \in R$ such that $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$. Since I is ϕ-prime, $a \in I$ or $b \in I$. Hence $a \in I$ or $\langle b \rangle^n \subseteq I$ for every $n \in \mathbb{N}$. By [5, Corollary 3.4] we have I is ϕ-principally right primary.

(3) ⇒ (2) Let $a, b \in R$ such that $aRb \subseteq I$ and $aRb \nsubseteq \phi(I)$. If $b \notin I$, then $\langle b \rangle^n \nsubseteq I$ and since I is ϕ-principally right primary, we have $a \in I$. Hence I is ϕ-prime. □

Proposition 2.13. *Let ρ be the prime radical $\mathcal{P}$. For R a noncommutative ring and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function and I a proper ideal of R, such that $I = \mathcal{P}(R)$, then if I is ϕ-principally right primary, it is ϕ-principally right 2-absorbing primary.*

Proof. Follows as in [6, Proposition 2.2] since $\sqrt{I} \subseteq \mathcal{P}(I) = \mathcal{P}(R)$. □

Proposition 2.14. *Let ρ be the prime radical $\mathcal{P}$. If R is a noncommutative ring and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function and I a proper ideal of R such that $I = \mathcal{P}(R)$, then I is ϕ-principally right 2-absorbing primary if and only if it I is ϕ-2-absorbing.*

Proof. Since $\sqrt{I} \subseteq \mathcal{P}(I) = \mathcal{P}(R) = I$ this is clear. □

Proposition 2.15. *Let R be a noncommutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. If I is a ϕ-ρ-ideal, then $I - \phi(I) \subseteq \rho(R)$.*

Proof. Suppose I is a ϕ-ρ-ideal and suppose $I - \phi(I) \nsubseteq \rho(R)$. Hence there exists $a \in I - \phi(I)$ with $a \notin \rho(R)$. Hence $aR1 \subseteq I$ and $aR1 \nsubseteq \phi(I)$. Since I is a ϕ-ρ-ideal, we have $1 \in I$, a contradiction. Hence $I - \phi(I) \subseteq \rho(R)$. □

Theorem 2.16. *Let R be a noncommutative ring and ρ a special radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. If $R/(I - \phi(I)) \in \mathcal{S}_\rho \cap \mathcal{P}$ and I is a ϕ-ρ-ideal, then $\rho(R) = I - \phi(I)$.*

Proof. Since ρ is a special radical, we have

$$\rho(R) = \cap \{A \lhd R : R/A \in \mathcal{S}_\rho \cap \mathcal{P}\} .$$

Now, since $R/(I - \phi(I)) \in \mathcal{S}_\rho \cap \mathcal{P}$, we have $\rho(R) \subseteq I - \phi(I)$. From Proposition 2.15 we have $I - \phi(I) \subseteq \rho(R)$ and we get $\rho(R) = I - \phi(I)$. □

The following proposition is well known.

Proposition 2.17. *For the right ideals A, B, P of any ring R, if $P \subseteq A \cup B$, then either $P \subseteq A$ or $P \subseteq B$. In particular, if $P = A \cup B$, then either $P = A$ or $P = B$.*

For a noncommutative ring with identity and a special radical ρ, we show in the following proposition how to construct ϕ-ρ-ideals.

Theorem 2.18. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. For any ring R and an ideal $P \nsubseteq R$ the following are equivalent:*

1. *P is a ϕ-ρ-ideal;*
2. *$(P : aR) = P \cup (\phi(P) : aR)$ for every $a \in R - \rho(R)$;*
3. *$(P : aR) = P$ or $(P : aR) = (\phi(P) : aR)$ for every $a \in R - \rho(R)$;*
4. *If $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$, then $\langle a \rangle \subseteq \rho(R)$ or $\langle b \rangle \subseteq P$;*
5. *If $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$, then $\langle a \rangle \subseteq \rho(R)$ or $b \in P$;*
6. *If A and B are ideals of R such that $AB \subseteq P$ and $AB \nsubseteq \phi(P)$, then $A \subseteq \rho(R)$ or $B \subseteq P$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R - \rho(R)$ and let $x \in P \cup (\phi(P) : aR)$. If $x \in P$, then $aRx \subseteq P$ and $x \in (P : aR)$. Now, if $x \in (\phi(P) : aR)$, then $aRx \subseteq \phi(P) \subseteq P$ and again we get $x \in (P : aR)$ and we have $P \cup (\phi(P) : aR) \subseteq (P : aR)$. Now, let $x \in (P : aR)$. So $aRx \subseteq P$. If $aRx \nsubseteq \phi(P)$, then $x \in P$ since P is a ϕ-ρ-ideal. If $aRx \subseteq \phi(P)$, then $x \in (\phi(P) : aR)$. Hence

$$(P : aR) = P \cup (\phi(P) : aR).$$

(3) $\Rightarrow$ (1) Let $a, b \in R$ and $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. If $a \notin \rho(R)$, then from (3) we have $(P : aR) = P$ or $(P : aR) = (\phi(P) : aR)$. Since $aRb \nsubseteq \phi(P)$, we have $(P : aR) = P$ and therefore $b \in P$ and hence P is a ϕ-ρ-ideal.

(2) $\Rightarrow$ (3) This is clear from Proposition 2.17.

(1) $\Rightarrow$ (4) Let $a, b \in R$ be such that $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$. Now $aRb \subseteq \langle a \rangle \langle b \rangle \subseteq P$ and since $\langle a \rangle \langle b \rangle \subseteq \phi(P)$ if and only if $aRb \subseteq \phi(P)$ and $ab \in \phi(P)$, we have $aRb \nsubseteq \phi(P)$. From (1) it follows that $a \in \rho(R)$ or $b \in P$ and we are done.

(4) $\Rightarrow$ (5) Let $a, b \in R$ such that $\langle a \rangle b \subseteq P$ and $\langle a \rangle b \nsubseteq \phi(P)$. Hence $\langle a \rangle RbR = \langle a \rangle \langle b \rangle \subseteq P$. Now $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$ for if $\langle a \rangle \langle b \rangle \subseteq \phi(P)$, then $\langle a \rangle b \subseteq \langle a \rangle \langle b \rangle \subseteq \phi(P)$ a contradiction. Hence from (3) it follows that $\langle a \rangle \subseteq \rho(R)$ or $b \in \langle b \rangle \subseteq P$.

(1) $\Rightarrow$ (6) Let A and B be ideals of R with $AB \subseteq P$. Suppose that $A \nsubseteq \rho(R)$ and $B \nsubseteq P$. We show that $AB \subseteq \phi(P)$. Let $a \in A$. First, suppose that $a \notin \rho(R)$. Then $aRB \subseteq P$ gives $B \subseteq (P : Ra)$. Now $B \nsubseteq P$; so $(P : Ra) = (\phi(P) : Ra)$. Hence $aB \subseteq \phi(P)$. Next, let $a \in \rho(R) \cap P$. Choose $a^{'} \in A - \rho(R)$. Then $a + a^{'} \in A - \rho(R)$. So by the first case,

$$a^{'}B, (a + a^{'})B \subseteq \phi(P).$$

Let $b \in B$. Then $ab = (a + a^{'})b - a^{'}b \in \phi(I)$. So $aB \subseteq \phi(P)$. Thus $AB \subseteq \phi(P)$.

(6) $\Rightarrow$ (1) Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. Now $(RaR)(RbR) \subseteq P$ and $(RaR)(RbR) \nsubseteq \phi(P)$. From (6) we have

$$a \in RaR \subseteq \rho(R) \text{ or } b \in RbR \subseteq P.$$

Hence P is ϕ-ρ-ideal.

(5) $\Rightarrow$ (1) Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. Now $\langle a \rangle \langle b \rangle \subseteq P$. If $\langle a \rangle \langle b \rangle \subseteq \phi(P)$, then $aRb \subseteq \langle a \rangle \langle b \rangle \subseteq \phi(P)$, a contradiction. Hence $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$. From our assumption we have $RaR \subseteq \rho(R)$ or $RbR \subseteq P$. Hence $a \in \rho(R)$ or $b \in RbR \subseteq P$ and we are done.

(4) $\Rightarrow$ (1) Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. Now $\langle a \rangle \langle b \rangle = RaRRbR \subseteq P$. If $\langle a \rangle \langle b \rangle \subseteq \phi(P)$, then $aRb \subseteq \langle a \rangle \langle b \rangle \subseteq \phi(P)$ a contradiction. Hence $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$. From (4) we have $a \in \langle a \rangle \subseteq \rho(R)$ or $b \in \langle b \rangle \subseteq P$ and we are done. ☐

Proposition 2.19. *Let (R, M) be a local ring and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Let ρ be a special radical and suppose P is an ideal of R such that $P \cap M^2 \subseteq \phi(P)$. Then P is a ϕ-ρ-ideal.*

Proof. Let A and B be ideals of R such that $AB \subseteq P$. Since $A \subseteq M$ and $B \subseteq M$ we have $AB \subseteq P \cap M^2 \subseteq \phi(P)$. Hence, since $AB \subseteq P$ and $AB \subseteq \phi(P)$, it follows from Proposition 2.18 that P is a ϕ-ρ-ideal. ☐

Theorem 2.20. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. For any ring R and any ideal $I \neq R \in S(R)$ the following statements hold:*

1. *If I is a ϕ-ρ-ideal of R, then $I/\phi(I)$ is a weakly-ρ-ideal of $R/\phi(I)$;*
2. *If $I/\phi(I)$ is a weakly-ρ-ideal of $R/\phi(I)$ and $\phi(I) \subseteq \rho(R)$, then I is a ϕ-ρ-ideal of R.*

Proof. (1) Let $\{\overline{0}\} \neq (r + \phi(I))R/\phi(I)(s + \phi(I)) \subseteq I/\phi(I)$ and let

$$(r + \phi(I)) \notin \rho(R/\phi(I)).$$

Since ρ is a special radical $\rho(R)/\phi(I) \subseteq \rho(R/\phi(I))$. Hence $r \notin \rho(R)$. Since $rRs \subseteq I$ and $rRs \nsubseteq \phi(I)$ and I is a ϕ-ρ-ideal of R, we have $s \in I$ and hence $(s + \phi(I)) \in I/\phi(I)$ and we are done.

(2) Let $r, s \in R$ such that $rRs \subseteq I$ and $rRs \nsubseteq \phi(I)$ and $r \notin \rho(R)$. Since $\phi(I) \subseteq \rho(R)$ and $\rho(R)/\phi(I) \subseteq \rho(R/\phi(I))$ as in [8, Theorem 1.10], we have $(r + \phi(I)) \notin \rho(R/\phi(I))$. Hence, since

$$\{\overline{0}\} \neq (r+\phi(I))R/\phi(I)(s+\phi(I)) \subseteq I/\phi(I)$$

and $I/\phi(I)$ is a weakly-ρ-ideal of $R/\phi(I)$, we have $(s+\phi(I)) \in I/\phi(I)$. Thus, $s \in I$ and we are done. □

Theorem 2.21. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If $\rho(R) = \{0\}$ and $R \notin \mathcal{P}$, then R has no proper ϕ-ρ-ideals for $\phi \neq \phi_1$.*

Proof. Assume, on the contrary, that Q is a ϕ-ρ-ideal of R. From Proposition 2.15 we have $Q - \phi(Q) \subseteq \rho(R) = \{0\}$. Thus $\phi(Q) = Q$, and so $\phi = \phi_1$. Thus R has no proper ϕ-ρ-ideals for $\phi \neq \phi_1$. □

Corollary 2.22. *Let ρ be a special radical and let $R \in S\rho$ be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function with $\phi \neq \phi_1$. Then the following are equivalent:*

(1) *R is a prime ring;*
(2) *$\{0\}$ is a ϕ-ρ-ideal.*

Proof. (1) ⇒ (2) Since $R \in \mathcal{P}$, we have $\rho(R) = \{0\}$ is a prime ideal. Now, if $a, b \in R$ such that $a \notin \rho(R) = \{0\}$ with $aRb = \{0\}$ and $aRb \nsubseteq \phi(0)$, then $b = 0$ since $\{0\}$ is a prime ideal. Hence $\{0\}$ is a ϕ-ρ-ideal.

(2) ⇒ (1) This is clear by Theorem 2.21. □

The next two theorems give conditions for when a ϕ-ρ-ideal is a ρ-ideal.

Theorem 2.23. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-ρ-ideal such that $\rho(R) \subseteq I$ and $\rho(R)I \nsubseteq \phi(I)$, then I is a ρ-ideal.*

Proof. Let $r, s \in R$ such that $rRs \subseteq I$. If $rRs \nsubseteq \phi(I)$, then $r \in \rho(R)$ or $s \in I$ and I is a ρ-ideal. So suppose $rRs \subseteq \phi(I)$. Here there are three cases:

Case 1. Let $rI \nsubseteq \phi(I)$. Then there exists $x \in I$ such that $rRx \nsubseteq \phi(I)$. Since $rR(x+s) = (rRx+rRs) \subseteq I$ and $rR(x+s) \nsubseteq \phi(I)$ and I is a ϕ-ρ-ideal, we have $r \in \rho(R)$ or $(x+s) \in I$. Hence $r \in \rho(R)$ or $s \in I$.

Case 2. Let $\rho(R)s \nsubseteq \phi(I)$. Then there exists $y \in \rho(R) \subseteq I$ such that $yRs \nsubseteq \phi(I)$. Since $(y+r)Rs = (yRs+rRs)$ we have $(y+r)Rs \subseteq I$. Now $(y+r)Rs \nsubseteq \phi(I)$ since $rRs \subseteq \phi(I)$ and $yRs \nsubseteq \phi(I)$. Since I is a ϕ-ρ-ideal, we have $(y+r) \in \rho(R)$ or $s \in I$. Hence $r \in \rho(R)$ or $s \in I$.

Case 3. Let $rI \subseteq \phi(I)$ and $\rho(R)s \subseteq \phi(I)$. Since $\rho(R)I \nsubseteq \phi(I)$, there are $z \in \rho(R) \subseteq I$ and $i \in I$ such that $zRi \nsubseteq \phi(I)$. Hence
$(z+r)R(s+i) = (zRs + zRi + rRs + rRi) \subseteq I$ and $(z+r)R(s+i) \nsubseteq \phi(I)$.

Since I is a ϕ-ρ-ideal, we have $(z+r) \in \rho(R)$ or $(s+i) \in I$. Hence $r \in \rho(R)$ or $s \in I$ and we have I is a ρ-ideal. □

Corollary 2.24. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-ρ-ideal such that $\rho(R) \subseteq I$ and $\rho(R)I \nsubseteq \phi(I)$, then I is a prime ideal.*

Proof. From Theorem 2.23 I is a ρ-ideal and also $I \subseteq \rho(R)$ and thus we have $I = \rho(R)$. From [8, Corollary 1.14 (1)] it follows that I is a prime ideal. □

Corollary 2.25. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-ρ-ideal that is not a ρ-ideal, then $\rho(R)I \subseteq \phi(I)$.*

Theorem 2.26. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and $\phi(I)$ a ρ-ideal. Now I is a ϕ-ρ-ideal if and only if I is a ρ-ideal.*

Proof. $\Rightarrow$Suppose $r, s \in R$ such that $rRs \subseteq I$ and $r \notin \rho(R)$.
If $rRs \subseteq \phi(I)$, then $s \in \phi(I) \subseteq I$ since $\phi(I)$ is a ρ-ideal. If $rRs \nsubseteq \phi(I)$, then $s \in I$ since I is a ϕ-ρ-ideal. Hence I is a ρ-ideal.
$\Leftarrow$This is obvious. □

Remark 2.27. If A, B are ideals of R, then $(A : B)^* = \{a \in R : aB \subseteq A\}$ is an ideal of R.

Theorem 2.28. *Let ρ be a special radical and let $R \in \mathcal{S}\rho$ be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If P is an ideal such that $(\phi(P) : P)^* \subseteq \rho(R)$, then the following statements are equivalent:*

(1) P is a ρ-ideal;
(2) P is a ϕ-ρ-ideal.

Proof. (1) $\Rightarrow$ (2) This is clear from the definition.

(2) $\Rightarrow$ (1) Suppose that the ϕ-ρ-ideal P is not a ρ-ideal. Then there exists $a, b \in R$ such that $aRb \subseteq P$ and $a \notin \rho(R)$ and $b \notin P$. Since P is a ϕ-ρ-ideal we have that $aRb \subseteq \phi(P)$.
Now, $aR(b+P) = aRb + aRP \subseteq P$. If $aR(b+P) \subseteq \phi(P)$, then $aRP \subseteq \phi(P)$. Hence $a \in (\phi(P) : P)^* \subseteq \rho(R)$ a contradiction. If $aR(b+P) \nsubseteq \phi(P)$, then since P is a ϕ-ρ-ideal we have from Proposition 2.18 that $a \in \rho(R)$ or $b \in P$ a contradiction.

Proposition 2.29. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function which preserves order. If I is a*

ϕ-ρ-ideal of R with $x \notin I$ and $(\phi(I) : Rx) \subseteq \phi(I : Rx)$, then $(I : Rx)$ is also a ϕ-ρ-ideal.

Proof. Let I be a ϕ-ρ-ideal of R with $x \notin I$ and let $a, b \in R$ such that $aRb \subseteq (I : Rx)$ and $aRb \nsubseteq \phi((I : Rx))$ with $a \notin \rho(R)$. Now, $aRbRx \subseteq I$ and $aRbRx \nsubseteq \phi(I)$ since $(\phi(I) : Rx) \subseteq \phi(I : Rx)$. From Proposition 2.18 we have $bRx \subseteq I$ and hence $b \in (I : Rx)$. Thus $(I : Rx)$ is a ϕ-ρ-ideal of R. □

Theorem 2.30. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If $I \subseteq \rho(R)$ and I is a ϕ-ρ-ideal I of R that is not a ρ-ideal, then $I^2 = \phi(I)$.*

Proof. Suppose $I^2 \nsubseteq \phi(I)$. Let $r, s \in R$ and $rRs \subseteq I$. If $rRs \nsubseteq \phi(I)$, then since I is a ϕ-ρ-ideal of R we are done. So suppose $rRs \subseteq \phi(I)$. Here there are three cases:

Case 1. Let $rI \nsubseteq \phi(I)$. Then there exists $x \in I$ such that $rx \notin \phi(I)$. Hence $rRx \nsubseteq \phi(I)$. Now, $rR(x+s) \nsubseteq \phi(I)$ for if $rR(x+s) = rRx+rRs \subseteq \phi(I)$, then $rRx \subseteq \phi(I)$ since $rRs \subseteq \phi(I)$, a contradiction. Hence we have $rR(x+s) \subseteq I$ and $rR(x + s) \nsubseteq \phi(I)$. Since I is a ϕ-ρ-ideal I of R, we have $r \in \rho(R)$ or $(x + s) \in I$, i.e. $r \in \rho(R)$ or $s \in I$.

Case 2. Let $Is \nsubseteq \phi(I)$. Then there exists $y \in I$ such that $ys \notin \phi(I)$. Now, since $yRs \subseteq I$ and $rRs \subseteq I$ we have
$(y + r)Rs = yRs + rRs \subseteq I$. Since $rRs \subseteq \phi(I)$ and $yRs \nsubseteq \phi(I)$, we have $(y + r)Rs \nsubseteq \phi(I)$. Hence, since I is a ϕ-ρ-ideal, we have $(y + r) \in \rho(R)$ or $s \in I$ i.e. $r \in \rho(R)$ or $s \in I$.

Case 3. Let $rI \subseteq \phi(I)$ and $Is \subseteq \phi(I)$. Since $I^2 \nsubseteq \phi(I)$, there exist $c \in I \subseteq \rho(R)$ and $d \in I$ such that $cd \notin \phi(I)$. Now we have

$$(c + r)R(d + s) = cRd + cRs + rRd + rRs \subseteq I$$

and $(c+r)R(d+s) \nsubseteq \phi(I)$. Now, since I is a ϕ-ρ-ideal, we have $(c+r) \in \rho(R)$ or $(d+s) \in I$ i.e. $r \in \rho(R)$ or $s \in I$. Thus I is a ρ-ideal and we are done. □

Corollary 2.31. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If $I \subseteq \rho(R)$ and $I^2 \nsubseteq \phi(I)$, then I is a ρ-ideal of R if and only if it is a ϕ-ρ-ideal.*

Theorem 2.32. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-ρ-ideal of R such that $\phi(I)$ is a ρ-ideal, then I is a ρ-ideal.*

Proof. Suppose $a, b \in R$ such that $aRb \subseteq I$ and $a \notin \rho(R)$. If $aBb \subseteq \phi(I)$, then $b \in \phi(I) \subseteq I$ since $\phi(I)$ is a ρ-ideal. If $\phi(I) \nsubseteq I$, then $b \in I$ since I is a ϕ-ρ-ideal. □

Let J be an ideal and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function. Define the function $\phi_J : S(R/J) \to S(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal I of R with $J \in S(R)$ and $I \subseteq J$ with $\phi_J(I/J) = \{\emptyset\}$ if $\phi(I) = \{\emptyset\}$. Observe that $\phi_J(I/J) \subseteq I/J$ and $(\phi_\alpha)_J = \phi_\alpha$ for $\alpha \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$. From [8, Corollary 1.11] we have that if $J \in S(R)$ and I is a ρ-ideal of R containing J, then I/J is also a ρ-ideal of R/J.

The next theorem shows that if I is a ϕ-ρ-ideal of R, then I/J is also a ϕ_J-ρ-ideal of R/J.

Theorem 2.33. *Let ρ be a special radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If $P \subseteq I \in S(R)$ is a ϕ-ρ-ideal of R, then P/I is a ϕ_J-ρ-ideal of R/I.*

Proof. Suppose $\overline{A}\ \overline{B} \subseteq \overline{P} = P/I$ and $\overline{A}\ \overline{B} \nsubseteq \phi_I(\overline{P})$. Assume that $\overline{A} = A/I$ and $\overline{B} = B/I$ for ideals $A \supseteq I$ and $B \supseteq I$. Suppose that

$$\overline{A} = (A + I)/I \nsubseteq \rho(R/I) \subseteq \rho(R)/I.$$

Now $(AB + I) \subseteq P/I$ and $(AB + I) \nsubseteq \phi_I(\overline{P}) = (\phi(P) + I)/I$. Hence $AB \subseteq P$ and $AB \nsubseteq \phi(P)$. Also, since $(A + I)/I \nsubseteq \rho(R)/I$ we have $A \nsubseteq \rho(R)$. From Proposition 2.18 we have $B \subseteq P$. Hence $\overline{B} \subseteq \overline{P}$ and it follows that P/I is a ϕ_J-ρ-ideal of R/I. □

Proposition 2.34. *Let ρ be a special radical and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function which preserves order. Let I be a ρ-ideal of R contained in the ideal J. If J/I is a ρ-ideal of R/I, then J is a ϕ-ρ-ideal of R.*

Proof. Choose $x, y \in R$ such that $xRy \subseteq J$ and $xRy \nsubseteq \phi(J)$ with $x \notin \rho(R)$. Then we have 2 cases:

Case 1. Let $xRy \subseteq I$. Then, as I is a ρ-ideal of R and $xRy \nsubseteq \phi(I)$ since ϕ preserves order, we get $y \in I \subseteq J$, as desired.

Case 2. Let $xRy \subseteq J$ and $xRy \nsubseteq I$. This implies that

$$xRy + I = (x + I)R/I(y + I) \subseteq J/I.$$

Also, as $x \notin \rho(R)$, it follows as in [8, Theorem 1.10] that $(x + I) \notin \rho(R/I)$. Therefore, since J/I is a ρ-ideal of R/I and $(x + I) \notin \rho(R/I)$, we have $y + I \in J/I$, so $y \in J$ and we are done. □

Let R be a noncommutative ring with identity and M an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r, m)(s, n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ via the mapping $(r, m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

If ρ is a special radical, it follows from [14] that if R is any ring, then $\rho(R \boxplus M) = \rho(R) \boxplus M$ for all $R - R$-bimodules M.

Theorem 2.35. *Let R be a noncommutative ring with identity and M an $R - R$-bimodule. Let ρ be a special radical. Let $\phi_1 : S(R) \to S(R) \cup \{\emptyset\}$ and $\phi_2 : S(R \boxplus M) \to S(R \boxplus M) \cup \{\emptyset\}$ be functions satisfying $\phi_2(I \boxplus M) = \phi_1(I) \boxplus M$ for some proper ideal I of R. If $I \boxplus M$ is a ϕ_2-ρ-ideal of $R \boxplus M$, then I is a ϕ_1-ρ-ideal of R.*

Proof. Let $r, s \in R$ such that $rRs \subseteq I$ and $rRs \nsubseteq \phi_1(I)$ with $r \notin \rho(R)$. Now $(r, 0)R \boxplus M(s, 0) \subseteq I \boxplus M$ and $(r, 0) \notin \rho(R \boxplus M) = \rho(R) \boxplus M$, since $r \notin \rho(R)$. We also have $\phi_2(I \boxplus M) = \phi_1(I) \boxplus M$ and since $rRs \nsubseteq \phi_1(I)$, we have $(r, 0)R \boxplus M(s, 0) \nsubseteq \phi_2(I \boxplus M)$. Hence, since $I \boxplus M$ is a ϕ_2-ρ-ideal of $R \boxplus M$, we have $(s, 0) \in I \boxplus M$ and so $s \in I$. Thus I is a ϕ_1-ρ-ideal of R. ☐

Remark 2.36. Let R_1 and R_2 be two nonzero noncommutative rings with nonzero identity and $R = R_1 \times R_2$. Let $\phi_1 : S(R_1) \to S(R_1) \cup \{\emptyset\}$ and $\phi_2 : S(R_2) \to S(R_2) \cup \{\emptyset\}$ be two functions and $\psi = \phi_1 \times \phi_2$. For a special radical ρ the ring R has no proper ψ-ρ-ideal where $\psi \neq \psi_1$. Indeed, assume that $I = I_1 \times I_2$ is a nonzero ψ-ρ-ideal for some $\psi \neq \psi_1$. Since $I \neq \psi(I)$, there is an element $(a, b) \in I - \psi(I)$. On the other hand, $(a, 1)R(1, b) \subseteq I$ and $(a, 1)R(1, b) \nsubseteq \psi(I)$ and also $(1, b)R(a, 1) \subseteq I$ and $(1, b)R(a, 1) \nsubseteq \psi(I)$ but neither $(1, b)$ nor $(a, 1)$ is an element of $\rho(R)$. It implies that $(a, 1), (1, b) \in I$. Thus we conclude $1 \in I_2$ and $1 \in I_1$, and so $I = I_1 \times I_2 = R$, which is a contradiction.

3. Generalization of n-ideals

The results in the previous sections are valid for any special radical and any noncommutative ring with identity. As an example in this section the special radical will be the prime radical. In [2] E. Y. Çelikel introduced the notion of generalizations of $n-$ideals for commutative rings with identity element. The author investigated many properties of n-ideals with properties similar to that of prime ideals. We show that for the prime radical many of the results proved by Ece Yetkin Çelikel are also true for noncommutative rings.

In what follows for the noncommutative ring R, $\mathcal{P}(R)$ will denote the prime radical of the ring R.

Definition 3.1. [2, Definition 2.1] Let R be a noncommutative ring and I a proper ideal of R. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function. We call I a *ϕ-n-ideal* of R if whenever $a, b \in R$ and $aRb \subseteq I$ and $aRb \nsubseteq I$, then either $a \in \mathcal{P}(R)$ or $b \in I$.

Let I be a ϕ-n-ideal of R. Then define:

(1) If $\phi(J) = \emptyset$ for all $J \in \mathcal{S}(R)$, then we say that $\phi = \phi_\emptyset$ and I is called a *$\phi_\emptyset$-n-ideal*, and hence is an *n-ideal* of R.
(2) If $\phi(J) = 0$ for all $J \in \mathcal{S}(R)$, then we say that $\phi = \phi_0$ and I is called a *ϕ_0-n-ideal*, and hence is a (*weakly n-ideal*) of R.
(3) If $\phi(J) = J$ for all $J \in \mathcal{S}(R)$, then we say that $\phi = \phi_1$ and I is called a *ϕ_1-n-ideal*, (any ideal) of R.
(4) If $k \geq 2$ and if $\phi(J) = J^k$ for all $J \in \mathcal{S}(R)$, then we say that $\phi = \phi_k$ and I is called a *ϕ_k-n-ideal* (*k-almost n-ideal*) of R. In special, if $k = 2$, then we call I an *almost n-ideal* of R.
(5) If $\phi(J) = \bigcap_{i=1}^{\infty} J^i$ for all $J \in \mathcal{S}(R)$ then we say that $\phi = \phi_\varpi$ and I is called a *ϕ_ϖ-n-ideal*, (*ϖ-n-ideal) of* R.

Lemma 3.2. *(See* [2, Lemma 2.4]*)Let $\mathcal{P}$ be the prime radical and let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is an ideal of the ring R such that $I \subseteq \mathcal{P}(R)$, then I is a ϕ-n-ideal if and only if I is a ϕ-n-primary ideal.*

Proof. The proof follows from Lemma 2.9, by taking ρ to be the prime radical. □

Theorem 3.3. *(See* [2, Theorem 2.3)]*)Let $\mathcal{P}$ be the prime radical. For any $I \in S(R)$, the following statements hold:*

(1) *Let $\psi_1, \psi_2 : S(R) \to S(R) \cup \{\emptyset\}$ be two functions such $\psi_1 \leq \psi_2$. If I is a ψ_1-n-ideal, then I is a ψ_2-n-ideal;*
(2) *I is an n-ideal $\Rightarrow$ I is a weakly-n-ideal $\Rightarrow$ I is ϕ_ϖ-n-ideal $\Rightarrow$ I is a ϕ_{k+1}-n-ideal for every $k \geq 2$ $\Rightarrow$ I is a ϕ_k-n-ideal for every $k \geq 2$ $\Rightarrow$ I is an almost-n-ideal;*
(3) *I is a ϕ-n-ideal $\Rightarrow$ I is a ϕ-principally right primary ideal $\Rightarrow$ I is ϕ-2-absorbing principally right primary ideal;*
(4) *I is an idempotent ideal of $R \Rightarrow I$ is a ϕ_k-n-ideal for every $k \geq 1$;*
(5) *I is a ϕ_k-n-ideal for every $k \geq 2 \Rightarrow I$ is a ϕ_ω-n-ideal.*

Proof. The proof follows from Theorem 2.10, by taking ρ to be the prime radical. $\square$

Theorem 3.4. *(See [2, Theorem 2.5]) Let $\mathcal{P}$ be the prime radical and R a noncommutative ring and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function. If I is a proper ideal of R such that $I = \mathcal{P}(R)$, then the following are equivalent:*

1. *I is a ϕ-n-ideal;*
2. *I is ϕ-prime;*
3. *I is ϕ-principally right primary.*

Proof. The proof follows from Theorem 2.12, by taking ρ to be the prime radical. $\square$

Theorem 3.5. *(See [2, Theorem 2.10])Let R be a noncommutative ring and $\mathcal{P}$ the prime radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. If I is ϕ-n-ideal, then $I - \phi(I) \subseteq \mathcal{P}(R)$.*

Proof. The proof follows from Proposition 2.18, by taking ρ to be the prime radical. $\square$

Theorem 3.6. *(See [2, Theorem 2.14]) Let R be a noncommutative ring and $\mathcal{P}$ be the prime radical. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R such that $I - \phi(I)$ is a prime ideal. If $I \lhd R$ is a ϕ-n-ideal, then $\mathcal{P}(R) = I - \phi(I)$.*

Proof. The proof follows from Theorem 2.16, by taking ρ to be the prime radical. $\square$

Theorem 3.7. *(See [2, Theorem 2.7]) Let $\mathcal{P}$ be the prime radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. For any ring R and an ideal $P \nsubseteq R$ the following are equivalent:*

1. *P is a ϕ-n-ideal;*
2. *$(P : aR) = P \cup (\phi(P) : aR)$ for every $a \in R - \mathcal{P}(R)$;*
3. *$(P : aR) = P$ or $(P : aR) = (\phi(P) : aR)$ for every $a \in R - \mathcal{P}(R)$;*
4. *If $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$, then $\langle a \rangle \subseteq \mathcal{P}(R)$ or $\langle b \rangle \subseteq P$;*
5. *If $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq P$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(P)$, then $\langle a \rangle \subseteq \mathcal{P}(R)$ or $b \in P$;*
6. *If A and B are ideals of R such that $AB \subseteq P$ and $AB \nsubseteq \phi(P)$, then $A \subseteq \mathcal{P}(R)$ or $B \subseteq P$.*

Proof. The proof follows from Theorem 2.18, by taking ρ to be the prime radical. □

Theorem 3.8. *(See* [2, Theorem 2.6] *Let ρ the prime radical $\mathcal{P}$ and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. For any ring R and any ideal $R \neq P \in S(R)$ the following statements hold:*

(1) *If I is a ϕ-n-ideal of R, then $I/\phi(I)$ is a weakly-n-ideal of $R/\phi(I)$;*
(2) *If $I/\phi(I)$ is a weakly-n-ideal of $R/\phi(I)$ and $\phi(I) \subseteq \mathcal{P}(R)$, then I is a ϕ-n-ideal of R.*

Proof. The proof follows from Theorem 2.20, by taking ρ to be the prime radical. □

Theorem 3.9. *(See* [2, Theorem 2.13]*) Let ρ be the prime radical $\mathcal{P}$ and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If R is a semi-prime ring that is not a prime ring, then R has no proper ϕ-n-ideals for $\phi \neq \phi_1$.*

Proof. The proof follows from Theorem 2.21, by taking ρ to be the prime radical. □

Theorem 3.10. *(See* [2, Theorem 2.16]*) Let ρ be the prime radical $\mathcal{P}$ and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Let I be a ϕ-n-ideal such that $\mathcal{P}(R) \subseteq I$. If $\mathcal{P}(R)I \nsubseteq \phi(I)$, then I is a prime ideal.*

Proof. The proof follows from Theorem 2.23, by taking ρ to be the prime radical. □

Theorem 3.11. *(See* [2, Theorem 2.18]*) Let ρ be the prime radical $\mathcal{P}$ and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If I is a ϕ-n-ideal I of R such that $\phi(I)$ is an n-ideal, then I is an n-ideal.*

Proof. The proof follows from Theorem 2.32, by taking ρ to be the prime radical. □

Proposition 3.12. *Let ρ be the prime radical and let R be a ring with identity. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. If $P \subseteq I \in S(R)$ is a ϕ-n-ideal of R, then P/I is a ϕ_J-n-ideal of R/I.*

Proof. The proof follows from Theorem 2.33, by taking ρ to be the prime radical. □

Theorem 3.13. *(See* [2, Theorem 2.24]*) Let R be a noncommutative ring with identity and M an $R-R$-bimodule. Let ρ be the prime radical $\mathcal{P}$. Let*

$\phi_1 : S(R) \to S(R) \cup \{\emptyset\}$ and $\phi_2 : S(R \boxplus M) \to S(R \boxplus M) \cup \{\emptyset\}$ be functions satisfying $\phi_2(I \boxplus M) = \phi_1(I) \boxplus M$ for some proper ideal I of R. If $I \boxplus M$ is a ϕ_2-n-ideal of $R \boxplus M$, then I is a ϕ_1-n-ideal of R.

Proof. The proof follows from Theorem 2.35, by taking ρ to be the prime radical. □

Remark 3.14. Let R_1 and R_2 be two nonzero noncommutative rings with nonzero identity and $R = R_1 \times R_2$. Let $\phi_1 : S(R_1) \to S(R_1) \cup \{\emptyset\}$ and $\phi_2 : S(R_2) \to S(R_2) \cup \{\emptyset\}$ be two functions and $\psi = \phi_1 \times \phi_2$. For the prime radical $\mathcal{P}$ the ring R has no proper ψ-n-ideal where $\psi \neq \psi_1$.

Proof. The proof follows from Remark 2.36, by taking ρ to be the prime radical. □

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GENERALIZATIONS OF RADICAL IDEALS IN NONCOMMUTATIVE RINGS

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تعمیم‌هایی از ایده‌آل‌های رادیکالی در حلقه‌های ناجابه‌جایی

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دانشکده ریاضی، دانشگاه نلسون ماندلا، گکبرها، آفریقای جنوبی

در این مقاله، تعمیم مفهوم ایده‌آل‌های رادیکالی در حلقه‌های ناجابه‌جایی با عنصر یک ناصفر را ارائه می‌دهیم. فرض کنید R یک حلقه‌ی ناجابه‌جایی با $0 \neq 1$ باشد و $S(R)$ مجموعه‌ی همه‌ی ایده‌آل‌های R باشد. همچنین فرض کنید $\phi : S(R) \to S(R) \cup \emptyset$ یک تابع و ρ یک رادیکال ویژه باشد. ایده‌آل صحیح I از R را ρ-ϕ-ایده‌آل می‌نامند هرگاه برای $a, b \in R$ از $aRb \subseteq I$ و $aRb \nsubseteq \phi(I)$ و $a \notin \rho(R)$، نتیجه شود $b \in I$. بسیاری از نتایج مربوط به n-ایده‌آل‌ها، J-ایده‌آل‌ها و تعمیم‌های آن‌ها مانند n-ایده‌آل‌های ضعیف و J-ایده‌آل‌های ضعیف را می‌توان به عنوان حالت‌های خاصی از نتایج اثبات‌شده برای ρ-ϕ-ایده‌آل‌ها در این مقاله مشاهده کرد.

کلمات کلیدی: رادیکال ویژه ρ، ρ-ایده‌آل، ϕ-ایده‌آل اول، ϕ-ρ-ایده‌آل.