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A STUDY OF (α_1, α_2)-FUZZY SUBRINGS AND (α_1, α_2)-FUZZY IDEALS OF A RING

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Abstract. As an extension to Liu's definition of fuzzy subring and fuzzy ideal, a new notion of (α_1, α_2)-fuzzy subring and (α_1, α_2)-fuzzy ideal of a ring is introduced. We have provided examples and analyzed their properties. Additionally, we have defined (α_1, α_2)-fuzzy coset of a (α_1, α_2)-fuzzy ideal of a ring and studied some of its properties.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [11] in 1965. Then in 1971, Rosenfeld [9] used this concept to define a fuzzy subgroupoid and a fuzzy subgroup. In 1982, Liu [3] studied fuzzy invariant subgroups, fuzzy ideals and proved some properties. In 2013, Sharma [10] introduced the concept of an α-fuzzy subgroup. Nimbhorkar and Khubchandani in [8] introduce and studied the concept of fuzzy weakly irreducible ideals of a ring, in [5] gave characterizations of L-fuzzy hollow modules and L-fuzzy multiplication modules and in [6] prove some properties of fuzzy semi-essential submodules and fuzzy semi-closed submodules. These contribute to the significant development in fuzzy algebraic theory. Also, Khubchandani and Khubchandani [2] studied fuzzy α-modularity in fuzzy α-lattices. In this paper, we introduce the concept of (α_1, α_2)-fuzzy subring and (α_1, α_2)-fuzzy ideal of a ring. This motivation is from the work done by Nimbhorkar and Khubchandani [7] in which they have define (a, b)-fuzzy subrings and (a, b)-fuzzy ideals of a ring.

2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity. We recall some definitions and results.

Definition 2.1. [11] Let S be a nonempty set. A mapping $\xi : S \to [0, 1]$ is called a fuzzy subset of S.

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Remark 2.2. [11] If ξ and σ are two fuzzy subsets of a non-empty set S, then
(i) $\xi \subseteq \sigma$ if and only if $\xi(x) \leq \sigma(x)$;
(ii) $(\xi \cup \sigma)(x) = \max\{\xi(x), \sigma(x)\} = \xi(x) \vee \sigma(x)$;
(iii) $(\xi \cap \sigma)(x) = \min\{\xi(x), \sigma(x)\} = \xi(x) \wedge \sigma(x)$; for all $x \in S$.

Remark 2.3. [11] Let μ, ξ and σ be fuzzy sets in the universe of discourse U, then
(i) Associative Property:
$\mu \cup (\xi \cup \sigma) = (\mu \cup \xi) \cup \sigma$ and $\mu \cap (\xi \cap \sigma) = (\mu \cap \xi) \cap \sigma$;
(ii)Distributive Property:
$\sigma \cap (\mu \cup \xi) = (\sigma \cap \mu) \cup (\sigma \cap \xi)$ and $\sigma \cup (\mu \cap \xi) = (\sigma \cup \mu) \cap (\sigma \cup \xi)$.

Definition 2.4. [4] Let X and Y be two nonempty sets and $g : X \to Y$ be a mapping. Let $\xi \in [0,1]^X$ and $\sigma \in [0,1]^Y$. Then the image $g(\xi) \in [0,1]^Y$ and the inverse image $g^{-1}(\sigma) \in [0,1]^X$ are defined as follows:
for all $y \in Y$,

$$g(\xi)(y) = \begin{cases} \vee\{\xi(x) \mid x \in X, g(x) = y\}, & \text{if } g^{-1}(y) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

and $g^{-1}(\sigma)(x) = \sigma(g(x))$ for all $x \in X$.

Definition 2.5. [11] Let ξ be a fuzzy subset of a set S and let $t \in [0,1]$. The set $\xi_t = \{x \in R \mid \xi(x) \geqslant t\}$ is called a level subset of ξ.

Clearly, $\xi_t \subseteq \xi_s$ whenever $t > s$.

Definition 2.6. [3] A fuzzy subset ξ of R is called a fuzzy subring, if for all $x, y \in R$, the following conditions hold:
(i) $\xi(x - y) \geq \min(\xi(x), \xi(y))$;
(ii) $\xi(xy) \geq \min(\xi(x), \xi(y))$.

Definition 2.7. [3] A fuzzy subset ξ of R is called a fuzzy ideal, if for all $x,\ y \in R$, the following conditions are satisfied:
(i) $\xi(x - y) \geq \min(\xi(x), \xi(y))$;
(ii) $\xi(xy) \geq \max(\xi(x), \xi(y))$.

Definition 2.8. [1] Let A be a fuzzy ideal of R. For any $x \in R$ define a map $\hat{A}_x : R \to [0,1]$ by $\hat{A}_x(r) = A(r - x)$, for all $r \in R$. $\hat{A}_x$ is called the fuzzy coset of R determine by x and A.

3. (α_1, α_2)-FUZZY SUBSETS AND THEIR PROPERTIES

In this section, we define (α_1, α_2)-fuzzy subset with an example and studied some of its properties.

Definition 3.1. Let ξ be a fuzzy subset of a non-empty set S. Let $0 \leq \alpha_2 < \alpha_1 \leq 1$. Then the fuzzy set $\xi_{\alpha_2}^{\alpha_1}$ of S defined by

$$\xi_{\alpha_2}^{\alpha_1}(x) = \min\{\xi(x), \alpha_1 - \alpha_2\},$$

for all $x \in S$, is called as the (α_1, α_2)-fuzzy subset of S with respect to the fuzzy set ξ.

Example 3.2. Let ξ be the fuzzy subset of the set $S = \mathbb{Z}_{10}$ defined as follows:

$$\xi(x) = \begin{cases} 1, & \text{if } x = \{0, 5\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

Let $\alpha_1 = 0.8$, $\alpha_2 = 0.5$. Then

$$\xi_{0.5}^{0.8}(x) = \begin{cases} 0.3, & \text{if } x = \{0, 5\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

Hence, ξ is an $(0.8, 0.5)$-fuzzy subset of $\mathbb{Z}_{10}$ with respect to the fuzzy set ξ.

Lemma 3.3. *(i) Let ξ and η be two fuzzy subsets of X. Then* $(\xi \cap \eta)_{\alpha_2}^{\alpha_1} = \xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1}$.
(ii) Let $g : X \longrightarrow Y$ be an onto mapping and η be a fuzzy subset of Y. Define $\eta \circ g : X \to [0, 1]$ by $(\eta \circ g)(x) = \eta(g(x))$. Then $\eta_{\alpha_2}^{\alpha_1} \circ g = (\eta \circ g)_{\alpha_2}^{\alpha_1}$.
(iii) Let $g : X \longrightarrow Y$ be a onto mapping and η be two fuzzy subsets of Y. Then $g^{-1}(\eta_{\alpha_2}^{\alpha_1}) = (g^{-1}(\eta))_{\alpha_2}^{\alpha_1}$.

Proof. (i): For all $x \in X$ we have

$$\begin{aligned} (\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x) &= \min\{(\xi \cap \eta)(x), \alpha_1 - \alpha_2\} \\ &= \min\{\min\{\xi(x), \eta(x)\}, \alpha_1 - \alpha_2\} \\ &= \min\{\min\{\xi(x), \alpha_1 - \alpha_2\}, \min\{\eta(x), \alpha_1 - \alpha_2\}\} \\ &= \min\{\xi_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(x)\} \\ &= \xi_{\alpha_2}^{\alpha_1}(x) \cap \eta_{\alpha_2}^{\alpha_1}(x) \\ &= (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x). \end{aligned}$$

Hence, $(\xi \cap \eta)_{\alpha_2}^{\alpha_1} = \xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1}$.

(ii): For all $x \in X$, we have

$$\begin{aligned} (\eta_{\alpha_2}^{\alpha_1} \circ g)(x) &= \eta_{\alpha_2}^{\alpha_1}(g(x)) \\ &= \min\{\eta(g(x)), \alpha_1 - \alpha_2\} \\ &= \min\{(\eta \circ g)(x), \alpha_1 - \alpha_2\} \\ &= (\eta \circ g)_{\alpha_2}^{\alpha_1}(x). \end{aligned}$$

Hence, $\eta^{\alpha_1}_{\alpha_2} \circ g = (\eta \circ g)^{\alpha_1}_{\alpha_2}$.

(iii): Consider

$$\begin{aligned} g^{-1}(\eta^{\alpha_1}_{\alpha_2})(x) &= \eta^{\alpha_1}_{\alpha_2}(g(x)) \\ &= \min\{\eta(g(x)), \alpha_1 - \alpha_2\} \\ &= \min\{g^{-1}(\eta(x)), \alpha_1 - \alpha_2\} \\ &= (g^{-1}(\eta))^{\alpha_1}_{\alpha_2}(x), \quad \text{for all } x \in X. \end{aligned}$$

Hence, $g^{-1}(\eta^{\alpha_1}_{\alpha_2}) = (g^{-1}(\eta))^{\alpha_1}_{\alpha_2}$.

□

4. (α_1, α_2)-FUZZY SUBRINGS

In this section, we introduce the concept of (α_1, α_2)-fuzzy subrings and prove some results.

Definition 4.1. Let ξ be a fuzzy subset of a non-empty set R. Let $0 \leq \alpha_2 < \alpha_1 \leq 1$. Then ξ is called an (α_1, α_2)-fuzzy subring of R if $\xi^{\alpha_1}_{\alpha_2}$ is a fuzzy subring of R, that is, if the following conditions hold:
(i) $\xi^{\alpha_1}_{\alpha_2}(x - y) \geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\}$;
(ii) $\xi^{\alpha_1}_{\alpha_2}(xy) \geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\}$, for all $x, y \in R$.

Example 4.2. Define a fuzzy subset ξ of the ring $R = \mathbb{Z}$ as follows:

$$\xi(x) = \begin{cases} 1, & \text{if } x \in <11^3>, \\ 0.9, & \text{if } x \in <11^2> \sim <11^3>, \\ 0.8, & \text{if } x \in <11> \sim <11^2>, \\ 0.7, & \text{if } x \in \mathbb{Z} \sim <11>. \end{cases}$$

Now, if we take $\alpha_1 = 0.5$, $\alpha_2 = 0.3$, $\alpha_1 - \alpha_2 = 0.2$ and so
$\xi(x) > \alpha_1 - \alpha_2$, for all $x \in \mathbb{Z}$. Hence

$$\xi^{0.5}_{0.3}(x) = \min\{\xi(x), \alpha_1 - \alpha_2\} = \alpha_1 - \alpha_2, \text{ for all } x \in R.$$

Therefore, $\xi^{0.5}_{0.3}(x - y) \geq \min\{\xi^{0.5}_{0.3}(x), \xi^{0.5}_{0.3}(y)\}$ and

$$\xi^{0.5}_{0.3}(xy) \geq \min\{\xi^{0.5}_{0.3}(x), \xi^{0.5}_{0.3}(y)\}.$$

Hence, ξ is an $(0.5, 0.3)$-fuzzy subring of $\mathbb{Z}$.

Proposition 4.3. *If ξ is a fuzzy subring of R, then ξ is also (α_1, α_2)-fuzzy subring of R.*

Proof. For $x, y \in R$ we have

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(x-y) &= \min\{\xi(x-y), \alpha_1-\alpha_2\} \\
&\geq \min\{\min\{\xi(x), \xi(y)\}, \alpha_1-\alpha_2\}, \\
&\quad \text{(since } \xi \text{ is a fuzzy subring of } R) \\
&= \min\{\min\{\xi(x), \alpha_1-\alpha_2\}, \min\{\xi(y), \alpha_1-\alpha_2\}\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}. \qquad (4.1)
\end{aligned}$$

Also,

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(xy) &= \min\{\xi(xy), \alpha_1-\alpha_2\} \\
&\geq \min\{\min\{\xi(x), \xi(y)\}, \alpha_1-\alpha_2\}, \\
&\quad \text{(since } \xi \text{ is a fuzzy subring of } R) \\
&= \min\{\min\{\xi(x), \alpha_1-\alpha_2\}, \min\{\xi(y), \alpha_1-\alpha_2\}\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}. \qquad (4.2)
\end{aligned}$$

It follows from (4.1) and (4.2), that ξ is (α_1, α_2)-fuzzy subring of R. □

The following example shows that the converse of Proposition 4.3 need not hold.

Example 4.4. Let ξ be the fuzzy subset of the ring $R = \mathbb{Z}_{12}$ defined as follows:

$$\xi(x) = \begin{cases} 0.3, & \text{if } x = \{0, 6\}, \\ 0.9, & \text{otherwise.} \end{cases}$$

As for $x = 9$, $y = 3$, $\xi(9) = \xi(3) = 0.9$ and $\xi(x-y) = \xi(9-3) = \xi(6) = 0.3$. Thus, $\xi(x-y) \ngeq \min\{\xi(x), \xi(y)\}$. Hence, ξ is not a fuzzy subring of R.
We note that if $\alpha_1 = 0.3$, $\alpha_2 = 0.1$, then $\alpha_1 - \alpha_2 = 0.2$ and so $\xi(x) > \alpha_1 - \alpha_2$ for all $x \in R$. Hence

$$\xi_{0.1}^{0.3}(x) = \min\{\xi(x), \alpha_1 - \alpha_2\} = \alpha_1 - \alpha_2, \text{ for all } x \in R.$$

Therefore, $\xi_{0.1}^{0.3}(x-y) \geq \min\{\xi_{0.1}^{0.3}(x), \xi_{0.1}^{0.3}(y)\}$ and

$$\xi_{0.1}^{0.3}(xy) \geq \min\{\xi_{0.1}^{0.3}(x), \xi_{0.1}^{0.3}(y)\}.$$

Hence, ξ is an $(0.3, 0.1)$-fuzzy subring of $\mathbb{Z}_{12}$.

Proposition 4.5. *The intersection of two (α_1, α_2)-fuzzy subrings of a ring R is again an (α_1, α_2)-fuzzy subring of R.*

Proof. Let ξ and η be two (α_1, α_2)-fuzzy subrings of a ring R. For $x, y \in R$, we have

$$\begin{aligned}(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x-y) &= (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x-y), \text{ by Lemma 3.3} \\ &= \min\{\xi_{\alpha_2}^{\alpha_1}(x-y), \eta_{\alpha_2}^{\alpha_1}(x-y)\}\} \\ &\geq \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}, \min\{\eta_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\ &= \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(x)\}, \min\{\xi_{\alpha_2}^{\alpha_1}(y), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\ &= \min\{(\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x), (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(y)\} \\ &= \min\{(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x)), (\xi \cap \eta)_{\alpha_2}^{\alpha_1}(y))\}. \end{aligned} \tag{4.3}$$

Also,

$$\begin{aligned}(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(xy) &= (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(xy), \text{ by Lemma 3.3} \\ &= \min\{\xi_{\alpha_2}^{\alpha_1}(xy), \eta_{\alpha_2}^{\alpha_1}(xy)\} \\ &\geq \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}, \min\{\eta_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\ &= \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(x)\}, \min\{\xi_{\alpha_2}^{\alpha_1}(y), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\ &= \min\{(\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x), (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(y)\} \\ &= \min\{(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x)), (\xi \cap \eta)_{\alpha_2}^{\alpha_1}(y))\}. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4), that $\xi \cap \eta$ is an (α_1, α_2)-fuzzy subring of R. ☐

The following example shows that the union of two (α_1, α_2)-fuzzy subrings of a ring R need not be an (α_1, α_2)-fuzzy subring of R.

Example 4.6. Define fuzzy subsets ξ and η of the ring $R = \mathbb{Z}$ as follows:

$$\xi(x) = \begin{cases} 0.75, & \text{if } x \in 8\mathbb{Z}, \\ 0.20, & \text{otherwise.} \end{cases}$$

$$\eta(x) = \begin{cases} 0.50, & \text{if } x \in 11\mathbb{Z}, \\ 0.02, & \text{otherwise.} \end{cases}$$

Let $\alpha_1 = 0.9, \alpha_2 = 0.1$. Then $\alpha_1 - \alpha_2 = 0.8$. We note that ξ and η are $(0.9, 0.1)$-fuzzy subrings of $\mathbb{Z}$. We know that, $(\xi \cup \eta)(x) = \max\{\xi(x), \eta(x)\}$. Therefore,

$$(\xi \cup \eta)(x) = \begin{cases} 0.75, & \text{if } x \in 8\mathbb{Z}, \\ 0.50, & \text{if } x \in 11\mathbb{Z}, \\ 0.02, & \text{if } x \notin 8\mathbb{Z} \cup 11\mathbb{Z}. \end{cases}$$

Let $x = 22, y = 8$ and $x - y = 14$. Then $(\xi \cup \eta)(x) = 0.5$, $(\xi \cup \eta)(y) = 0.75$ and $(\xi \cup \eta)(x - y) = 0.02$. Also,

$$(\xi \cup \eta)_{0.1}^{0.9}(x) = \min\{(\xi \cup \eta)(x), 0.8\} = \min\{0.5, 0.8\} = 0.5.$$
$$(\xi \cup \eta)_{0.1}^{0.9}(y) = \min\{(\xi \cup \eta)(y), 0.8\} = \min\{0.75, 0.8\} = 0.75.$$
$$(\xi \cup \eta)_{0.1}^{0.9}(x - y) = \min\{(\xi \cup \eta)(x - y), 0.8\} = \min\{0.02, 0.8\} = 0.02.$$

Thus,

$$(\xi \cup \eta)_{0.1}^{0.9}(x - y) \ngeq \min\{(\xi \cup \eta)_{0.1}^{0.9}(x), (\xi \cup \eta)_{0.1}^{0.9}(y)\}.$$

Hence, $\xi \cup \eta$ is not a $(0.9, 0.1)$-fuzzy subring of R.

Theorem 4.7. *Let g be a homomorphism from a ring R onto a ring R'. If ξ is an (α_1, α_2)-fuzzy subring of R', then $g^{-1}(\xi)$ is an (α_1, α_2)-fuzzy subring of R.*

Proof. Let $x, y \in R$. We have

$$\begin{aligned}
(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(x - y) &= g^{-1}(\xi_{\alpha_2}^{\alpha_1})(x - y), \text{ by Lemma 3.3} \\
&= \xi_{\alpha_2}^{\alpha_1}((g(x - y))) \\
&= \xi_{\alpha_2}^{\alpha_1}(g(x) - g(y)) \\
&\geq \min\{\xi_{\alpha_2}^{\alpha_1}(g(x)), \xi_{\alpha_2}^{\alpha_1}(g(y))\}, \\
&\quad (\text{since } \xi \text{ is an } (\alpha_1, \alpha_2)\text{-fuzzy subring of } R') \\
&= \min\{g^{-1}(\xi_{\alpha_2}^{\alpha_1}(x)), g^{-1}(\xi_{\alpha_2}^{\alpha_1}(y))\} \\
&= \min\{(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(x), (g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(y)\}. \qquad (4.5)
\end{aligned}$$

We have

$$\begin{aligned}
(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(xy) &= g^{-1}(\xi_{\alpha_2}^{\alpha_1})(xy), \text{ by Lemma 3.3} \\
&= \xi_{\alpha_2}^{\alpha_1}((g(xy))) \\
&= \xi_{\alpha_2}^{\alpha_1}(g(x)g(y)) \\
&\geq \min\{\xi_{\alpha_2}^{\alpha_1}(g(x)), \xi_{\alpha_2}^{\alpha_1}(g(y))\}, \\
&\quad (\text{since } \xi \text{ is an } (\alpha_1, \alpha_2)\text{-fuzzy subring of } R') \\
&= \min\{g^{-1}(\xi_{\alpha_2}^{\alpha_1})(x), g^{-1}(\xi_{\alpha_2}^{\alpha_1})(y)\} \\
&= \min\{(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(x), (g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(y)\}, \text{ by Lemma 3.3}. \qquad (4.6)
\end{aligned}$$

From (4.5) and (4.6), it follows that $g^{-1}(\xi)$ is an (α_1, α_2)-fuzzy subring of R. □

Definition 4.8. Let ξ be a fuzzy subset of R. For $t \in [0,1]$, the (α_1, α_2)-level subset of ξ is denoted by $(\xi_{\alpha_2}^{\alpha_1})_t$ and is defined as $(\xi_{\alpha_2}^{\alpha_1})_t = \{x \in R \mid \xi_{\alpha_2}^{\alpha_1}(x) \geq t\}$.

Example 4.9. Let $\xi : \mathbb{Z}_{10} \to [0,1]$ be as follows:

$$\xi(x) = \begin{cases} 0.9, & if\ x = \{0,2,4,6,8\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

Let $\alpha_1 = 0.8, b = 0.1$ and $t = 0.2$. We have $\alpha_1 - \alpha_2 = 0.7$. Then

$$\xi_{\alpha_2}^{\alpha_1}(x) = \xi_{0.1}^{0.8}(x) = \begin{cases} 0.7, & \text{if } x = \{0,2,4,6,8\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

and $(\xi_{0.1}^{0.8})_{0.2} = \{x \in \mathbb{Z}_9 \mid \xi_{0.1}^{0.8}(x) \geq 0.2\} = \mathbb{Z}_{10}$.

Theorem 4.10. *Let R be a ring, $t \in [0,1]$ and ξ be an (α_1, α_2)-fuzzy subring of R. If the (α_1, α_2)-level subset is nonempty, then $(\xi_{\alpha_2}^{\alpha_1})_t$ is a subring of R.*

Proof. We note that if $x, y \in (\xi_{\alpha_2}^{\alpha_1})_t$, then $(\xi_{\alpha_2}^{\alpha_1})(x) \geq t$ and $(\xi_{\alpha_2}^{\alpha_1})(y) \geq t$. We have $(\xi_{\alpha_2}^{\alpha_1})(x-y) \geq \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\} = \min\{t,t\} = t$. This implies that

$$x - y \in (\xi_{\alpha_2}^{\alpha_1})_t. \tag{4.7}$$

We have, $(\xi_{\alpha_2}^{\alpha_1})(xy) \geq \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\} = \min\{t,t\} = t$. This implies that

$$xy \in (\xi_{\alpha_2}^{\alpha_1})_t. \tag{4.8}$$

From (4.7) and (4.8), we conclude that $(\xi_{\alpha_2}^{\alpha_1})_t$ is a subring of R. □

Theorem 4.11. *Let R be a ring and ξ be a fuzzy subset of R. Suppose that $(\xi_{\alpha_2}^{\alpha_1})_t$ is a subring of R, for all $t \in [0,1]$. Then ξ is an (α_1, α_2)-fuzzy subring of R.*

Proof. Let $x, y \in R$, $(\xi_{\alpha_2}^{\alpha_1})(x) = t_1$ and $(\xi_{\alpha_2}^{\alpha_1})(y) = t_2$ where $t_1, t_2 \in [0,1]$. Then $(\xi_{\alpha_2}^{\alpha_1})_{t_1}$ and $(\xi_{\alpha_2}^{\alpha_1})_{t_2}$ are subrings of R. Since, $t_1 \wedge t_2 \leq t_1$ and $t_1 \wedge t_2 \leq t_2$, we have $(\xi_{\alpha_2}^{\alpha_1})_{t_1} \subseteq (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$ and $(\xi_{\alpha_2}^{\alpha_1})_{t_2} \subseteq (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$. Hence, $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$ and $y \in (\xi_{\alpha_2}^{\alpha_1})_{t_2}$ implies $x, y \in (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$. Then $x - y$ and $xy \in (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$, since $(\xi_{\alpha_2}^{\alpha_1})_t$ is a subring of R, for all $t \in [0,1]$. This implies

$$(\xi_{\alpha_2}^{\alpha_1})(x-y) \geq t_1 \wedge t_2 = \min\{(\xi_{\alpha_2}^{\alpha_1})(x), (\xi_{\alpha_2}^{\alpha_1})(y)\}$$

and $(\xi_{\alpha_2}^{\alpha_1})(xy) \geq t_1 \wedge t_2 = \min\{(\xi_{\alpha_2}^{\alpha_1})(x), (\xi_{\alpha_2}^{\alpha_1})(y)\}$. This proves that ξ is an (α_1, α_2)-fuzzy subring of R. □

Definition 4.12. Let ξ be an (α_1, α_2)-fuzzy subring of R and $t \in [0,1]$. Then the subring $(\xi_{\alpha_2}^{\alpha_1})_t$ is said to be an (α_1, α_2)-level subring of ξ.

Example 4.13. Let $R = \mathbb{Z}_9 \times \mathbb{Z}_9$. Define a fuzzy subset ξ as follows:

$$\xi(x) = \begin{cases} 1, & \text{if } x = \{(0,0),(0,3),(3,0),(3,3),(3,6), \\ & \qquad (6,3),(0,6),(6,0),(6,6)\}, \\ 0.7, & \text{otherwise.} \end{cases}$$

We note that for $\alpha_1 = 1, \alpha_2 = 0.2, \alpha_1 - \alpha_2 = 0.8$, then ξ is a $(1, 0.2)$-fuzzy subring of R. Also,

$$\xi^1_{0.2}(x) = \begin{cases} 0.8, & \text{if } x = \{(0,0),(0,3),(3,0),(3,3),(3,6), \\ & \qquad (6,3),(0,6),(6,0),(6,6)\}, \\ 0.7, & \text{otherwise.} \end{cases}$$

If $t = 0.8$, then
$(\xi^1_{0.2})_t = (\xi^1_{0.2})_{0.8} = \{(0,0),(0,3),(3,0),(3,3),(3,6),(6,3),(0,6),(6,0),(6,6)\}$ is a subring of R and $(1, 0.2)$-level subring of ξ.

Theorem 4.14. *Let ξ be an (α_1, α_2)-fuzzy subring of a ring R. Then two (α_1, α_2)-level subrings $(\xi^{\alpha_1}_{\alpha_2})_{t_1}$, $(\xi^{\alpha_1}_{\alpha_2})_{t_2}$ with $t_1 < t_2$ are equal if and only if there is no $x \in R$ such that $t_1 \leq \xi^{\alpha_1}_{\alpha_2}(x) < t_2$.*

Proof. Let $(\xi^{\alpha_1}_{\alpha_2})_{t_1} = (\xi^{\alpha_1}_{\alpha_2})_{t_2}$. If there exists $x \in R$ such that $t_1 \leq \xi^a_b(x) < t_2$, then $x \in (\xi^{\alpha_1}_{\alpha_2})_{t_1}$, but $x \notin (\xi^{\alpha_1}_{\alpha_2})_{t_2}$ which is a contradiction.

Conversely, suppose there is no $x \in R$ such that $t_1 \leq \xi^{\alpha_1}_{\alpha_2}(x) < t_2$. As $t_1 < t_2$ implies $(\xi^{\alpha_1}_{\alpha_2})_{t_2} \subseteq (\xi^{\alpha_1}_{\alpha_2})_{t_1}$. Now, if $x \in (\xi^{\alpha_1}_{\alpha_2})_{t_1}$, then $(\xi^{\alpha_1}_{\alpha_2})(x) \geq t_1$. Clearly, $\xi^{\alpha_1}_{\alpha_2}(x) \nleq t_2$. Since $\xi^{\alpha_1}_{\alpha_2}(x)$ and t_2 are real numbers, it follows that $\xi^{\alpha_1}_{\alpha_2}(x) \geq t_2$, i.e., $x \in (\xi^{\alpha_1}_{\alpha_2})_{t_2}$. Hence, $(\xi^{\alpha_1}_{\alpha_2})_{t_1} = (\xi^{\alpha_1}_{\alpha_2})_{t_2}$. □

5. (α_1, α_2)-FUZZY IDEALS

In this section, we introduce the concept of (α_1, α_2)-fuzzy ideals and prove some of its properties.

Definition 5.1. Let ξ be a fuzzy subset of R and $0 \leq \alpha_2 < \alpha_1 \leq 1$. Then ξ is called an (α_1, α_2)-fuzzy ideal of R if the following conditions hold:
(R_1) $\xi^{\alpha_1}_{\alpha_2}(x - y) \geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\}$;
(R_2) $\xi^{\alpha_1}_{\alpha_2}(xy) \geq \max\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\}$, for all $x, y \in R$.

Remark 5.2. Let ξ be an (α_1, α_2)-fuzzy subset of a commutative ring R. Then $\xi^{\alpha_1}_{\alpha_2}$ satisfies (R_2) if and only if $\xi^{\alpha_1}_{\alpha_2}(xy) \geq \xi^{\alpha_1}_{\alpha_2}(x)$, $\forall x, y \in R$.

Example 5.3. Let R be the ring of integers and O be the set of all odd integers. Define a fuzzy subset ξ of R as follows:

$$\xi(x) = \begin{cases} 1, & \text{if } x \in O, \\ 0.5, & \text{otherwise.} \end{cases}$$

For $\alpha_1 = 0.7$ and $\alpha_2 = 0.3$, ξ is an $(0.7, 0.3)$-fuzzy ideal of $\mathbb{Z}$.

Proposition 5.4. *If ξ is a fuzzy ideal of R, then ξ is also (α_1, α_2)-fuzzy ideal of R.*

Proof. For $x, y \in R$, we have

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(x-y) &= \min\{\xi(x-y), \alpha_1 - \alpha_2\} \\
&\geq \min\{\min\{\xi(x), \xi(y)\}, \alpha_1 - \alpha_2\}, \\
&\quad (\text{since } \xi \text{ is a fuzzy ideal of } R) \\
&= \min\{\min\{\xi(x), \alpha_1 - \alpha_2\}, \min\{\xi(y), \alpha_1 - \alpha_2\}\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}. \qquad (5.1)
\end{aligned}$$

Also,

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(xy) &= \min\{\xi(xy), \alpha_1 - \alpha_2\} \\
&\geq \min\{\max\{\xi(x), \xi(y)\}, \alpha_1 - \alpha_2\}, \\
&\quad (\text{since } \xi \text{ is a fuzzy ideal of } R) \\
&= \max\{\min\{\xi(x), \xi(y)\}, \alpha_1 - \alpha_2\} \\
&= \max\{\min\{\xi(x), \alpha_1 - \alpha_2\}, \min\{\xi(y), \alpha_1 - \alpha_2\}\} \\
&= \max\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}. \qquad (5.2)
\end{aligned}$$

It follows from (5.1) and (5.2), that ξ is a (α_1, α_2)-fuzzy ideal of ring R. □

The following example shows that the converse of Proposition 5.4 may not be true.

Example 5.5. Define a fuzzy subset ξ of the ring $R = \mathbb{Z}_{14}$ as follows:

$$\xi(x) = \begin{cases} 0.4, & \text{if } x = \{0, 7\}, \\ 0.8, & \text{otherwise.} \end{cases}$$

We note that for $x = 12$, $y = 7$, $\xi(x) = 0.8$, $\xi(y) = 0.4$, $xy = 84 = 0$, $\xi(xy) = 0.4$. Thus, $\xi(xy) \ngeq \max\{\xi(x), \xi(y)\}$. Hence ξ is not a fuzzy ideal of $\mathbb{Z}_{14}$. But ξ is a $(0.5, 0.2)$-fuzzy ideal of $\mathbb{Z}_{14}$.

Proposition 5.6. *If ξ is an (α_1, α_2)-fuzzy ideal of R, then*

$$\xi_{\alpha_2}^{\alpha_1}(0) \geq \xi_{\alpha_2}^{\alpha_1}(x) \geq \xi_{\alpha_2}^{\alpha_1}(1),$$

for all $x \in R$.

Proof. For any $x \in R$, we have

$$\xi_{\alpha_2}^{\alpha_1}(0) = \xi_{\alpha_2}^{\alpha_1}(x - x)$$

$\geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(x)\}$, since ξ is an (α_1,α_2)-fuzzy ideal of R.
$= \xi^{\alpha_1}_{\alpha_2}(x)$.
$= \xi^{\alpha_1}_{\alpha_2}(x.1)$
$\geq \xi^{\alpha_1}_{\alpha_2}(1)$.

Hence, $\xi^{\alpha_1}_{\alpha_2}(0) \geq \xi^{\alpha_1}_{\alpha_2}(x) \geq \xi^{\alpha_1}_{\alpha_2}(1)$, for all $x \in R$. □

Proposition 5.7. *If ξ is an (α_1,α_2)-fuzzy ideal of ring R with $\xi^{\alpha_1}_{\alpha_2}(x-y) = \xi^{\alpha_1}_{\alpha_2}(0)$, then $\xi^{\alpha_1}_{\alpha_2}(x) = \xi^{\alpha_1}_{\alpha_2}(y)$, for all $x, y \in R$.*

Proof. Since ξ is an (α_1,α_2)-fuzzy ideal of R,

$$\begin{aligned}
\xi^{\alpha_1}_{\alpha_2}(x) &= \xi^{\alpha_1}_{\alpha_2}(x-y+y) \\
&\geq \min\{\xi^{\alpha_1}_{\alpha_2}(x-y), \xi^{\alpha_1}_{\alpha_2}(y)\} \\
&= \min\{\xi^{\alpha_1}_{\alpha_2}(0), \xi^{\alpha_1}_{\alpha_2}(y)\} \\
&= \xi^{\alpha_1}_{\alpha_2}(y). \\
\xi^{\alpha_1}_{\alpha_2}(y) &= \xi^{\alpha_1}_{\alpha_2}(y-x+x) \\
&\geq \min\{\xi^{\alpha_1}_{\alpha_2}(y-x), \xi^{\alpha_1}_{\alpha_2}(x)\} \\
&= \min\{\xi^{\alpha_1}_{\alpha_2}(0), \xi^{\alpha_1}_{\alpha_2}(x)\} \\
&= \xi^{\alpha_1}_{\alpha_2}(x).
\end{aligned}$$

Hence, $\xi^{\alpha_1}_{\alpha_2}(x) = \xi^{\alpha_1}_{\alpha_2}(y)$, for all $x, y \in R$. □

Proposition 5.8. *Let ξ be an (α_1,α_2)-fuzzy ideal of R. If for some $t \in [0,1]$, the (α_1,α_2)-level subset $(\xi^{\alpha_1}_{\alpha_2})_t$, is nonempty, then it is an ideal of R where $(\xi^{\alpha_1}_{\alpha_2})_t = \{x \in R \mid \xi^{\alpha_1}_{\alpha_2}(x) \geq t\}$.*

Proof. Let $x, y \in (\xi^{\alpha_1}_{\alpha_2})_t$. Then $\xi^{\alpha_1}_{\alpha_2}(x) \geq t$ and $\xi^{\alpha_1}_{\alpha_2}(y) \geq t$. As ξ is an (α_1,α_2)-fuzzy ideal of R,

$$(\xi^{\alpha_1}_{\alpha_2})(x-y) \geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\} = \min\{t,t\} = t.$$

Hence,

$$x - y \in (\xi^{\alpha_1}_{\alpha_2})_t. \tag{5.3}$$

Let $r \in R$ be arbitrary and $x \in (\xi^{\alpha_1}_{\alpha_2})_t$, then $\xi^{\alpha_1}_{\alpha_2}(x) \geq t$.
Case(i): If $r \in (\xi^{\alpha_1}_{\alpha_2})_t$, then $(\xi^{\alpha_1}_{\alpha_2})(r) \geq t$. Then

$$\xi^{\alpha_1}_{\alpha_2}(rx) \geq \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(x)\} \geq t.$$

Implies $rx \in (\xi^{\alpha_1}_{\alpha_2})_t$.
Case(ii): If $r \notin (\xi^{\alpha_1}_{\alpha_2})_t$, then $(\xi^{\alpha_1}_{\alpha_2})(r) < t$. Then

$$\xi^{\alpha_1}_{\alpha_2}(rx) \geq \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(x)\} = \xi^{\alpha_1}_{\alpha_2}(x) \geq t, as\ \ \xi^{\alpha_1}_{\alpha_2}(r) < t \leq \xi^{\alpha_1}_{\alpha_2}(x).$$

Implies $rx \in (\xi_{\alpha_2}^{\alpha_1})_t$.
Hence, from case(i) and case(ii),

$$rx \in (\xi_{\alpha_2}^{\alpha_1})_t. \tag{5.4}$$

From (5.3) and (5.4), we conclude that $(\xi_{\alpha_2}^{\alpha_1})_t$ is an ideal of R. □

Proposition 5.9. *Let ξ be an (α_1, α_2)-fuzzy subset of R. Suppose that $(\xi_{\alpha_2}^{\alpha_1})_t$ is an ideal for all $t \in [0,1]$. Then ξ is (α_1, α_2)-fuzzy ideal of R.*

Proof. Let $x, y \in R$ and $\xi_{\alpha_2}^{\alpha_1}(x) = t_1$, $\xi_{\alpha_2}^{\alpha_1}(y) = t_2$, where $t_1, t_2 \in [0,1]$. Then $(\xi_{\alpha_2}^{\alpha_1})_{t_1}$ and $(\xi_{\alpha_2}^{\alpha_1})_{t_2}$ are ideals of R. Since, $t_1 \wedge t_2 \leq t_1$ and $t_1 \wedge t_2 \leq t_2$. This implies that $(\xi_{\alpha_2}^{\alpha_1})_{t_1} \subseteq (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$ and $(\xi_{\alpha_2}^{\alpha_1})_{t_2} \subseteq (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$. Hence, $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$ and $y \in (\xi_{\alpha_2}^{\alpha_1})_{t_2}$, which implies that $x, y \in (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$
and so $x - y \in (\xi_{\alpha_2}^{\alpha_1})_{t_1 \wedge t_2}$. Thus,

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(x-y) &\geq t_1 \wedge t_2 = \min\{t_1, t_2\}, \\
&\quad \text{as } t_1, t_2 \text{ are real numbers belonging to } [0,1] \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}.
\end{aligned} \tag{5.5}$$

For $x, y \in R$, if $\xi_{\alpha_2}^{\alpha_1}(x) = t_1$, then $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$. Therefore, $xy \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$ implies $\xi_{\alpha_2}^{\alpha_1}(xy) \geq t_1$. Hence,

$$\xi_{\alpha_2}^{\alpha_1}(xy) \geq \xi_{\alpha_2}^{\alpha_1}(x). \tag{5.6}$$

Similarly,

$$\xi_{\alpha_2}^{\alpha_1}(xy) \geq \xi_{\alpha_2}^{\alpha_1}(y). \tag{5.7}$$

Hence, from (5.6) and (5.7),

$$\xi_{\alpha_2}^{\alpha_1}(xy) \geq \max\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}. \tag{5.8}$$

Thus, from (5.5) and (5.8), we conclude that ξ is an (α_1, α_2)-fuzzy ideal of R. □

Corollary 5.10. *If ξ is an (α_1, α_2)-fuzzy ideal of R, then*

$$\{x \in R \mid \xi_{\alpha_2}^{\alpha_1}(x) = \xi_{\alpha_2}^{\alpha_1}(0)\}$$

is an ideal of R, where 0 is the additive identity of R.

Proof. Let $\varrho = \{x \in R \mid \xi_{\alpha_2}^{\alpha_1}(x) = \xi_{\alpha_2}^{\alpha_1}(0)\}$. Let $x, y \in \varrho$. Then $\xi_{\alpha_2}^{\alpha_1}(x) = \xi_{\alpha_2}^{\alpha_1}(0)$ and $\xi_{\alpha_2}^{\alpha_1}(y) = \xi_{\alpha_2}^{\alpha_1}(0)$. As ξ is an (α_1, α_2)-fuzzy ideal, we have

$$\begin{aligned}
\xi_{\alpha_2}^{\alpha_1}(x-y) &\geq \min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(0), \xi_{\alpha_2}^{\alpha_1}(0)\} \\
&= \xi_{\alpha_2}^{\alpha_1}(0).
\end{aligned}$$

By Proposition 5.6, we have $\xi^{\alpha_1}_{\alpha_2}(0) \geq \xi^{\alpha_1}_{\alpha_2}(x-y)$. Thus, $\xi^{\alpha_1}_{\alpha_2}(x-y) = \xi^{\alpha_1}_{\alpha_2}(0)$, which implies that $x - y \in \varrho$. Let $r \in R$ and $x \in \varrho$. Then $\xi^{\alpha_1}_{\alpha_2}(x) = \xi^{\alpha_1}_{\alpha_2}(0)$. Also,

$$\begin{aligned}\xi^{\alpha_1}_{\alpha_2}(rx) &\geq \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(x)\}\\ &= \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(0)\}\\ &= \xi^{\alpha_1}_{\alpha_2}(0), \text{ by Proposition 5.6.}\end{aligned}$$

Again by Proposition 5.6, $\xi^{\alpha_1}_{\alpha_2}(0) \geq \xi^{\alpha_1}_{\alpha_2}(rx)$ Thus $\xi^{\alpha_1}_{\alpha_2}(0) = \xi^{\alpha_1}_{\alpha_2}(rx)$ and so $rx \in \varrho$. Hence ϱ is an ideal of R. □

Proposition 5.11. *If ξ is an (α_1, α_2)-fuzzy ideal of R, then*

$$\{x \in R \mid \xi^{\alpha_1}_{\alpha_2}(x) > t\}$$

is an ideal of R for all $t \in [0, 1]$.

Proof. Let us write $(\xi^{\alpha_1}_{\alpha_2})_t = \{x \in R \mid \xi^{\alpha_1}_{\alpha_2}(x) > t\}$. Let $x, y \in (\xi^{\alpha_1}_{\alpha_2})_t$. Then $\xi^{\alpha_1}_{\alpha_2}(x) > t$ and $\xi^{\alpha_1}_{\alpha_2}(y) > t$. As ξ is an (α_1, α_2)-fuzzy ideal of R, we have

$$\xi^{\alpha_1}_{\alpha_2}(x-y) \geq \min\{\xi^{\alpha_1}_{\alpha_2}(x), \xi^{\alpha_1}_{\alpha_2}(y)\} > \min\{t, t\} = t.$$

Hence, $x - y \in (\xi^{\alpha_1}_{\alpha_2})_t$. Now let $x \in \xi^{\alpha_1}_{\alpha_2}(x)$ and $r \in R$.
Case(i): If $r \in (\xi^{\alpha_1}_{\alpha_2})_t$, then $(\xi^{\alpha_1}_{\alpha_2})(r) > t$. Then

$$\xi^{\alpha_1}_{\alpha_2}(rx) \geq \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(x)\} > t.$$

Implies $rx \in (\xi^{\alpha_1}_{\alpha_2})_t$.
Case(ii): If $r \notin (\xi^{\alpha_1}_{\alpha_2})_t$, then $(\xi^{\alpha_1}_{\alpha_2})(r) \leq t$. Then

$$\xi^{\alpha_1}_{\alpha_2}(rx) \geq \max\{\xi^{\alpha_1}_{\alpha_2}(r), \xi^{\alpha_1}_{\alpha_2}(x)\} = \xi^{\alpha_1}_{\alpha_2}(x) > t, as\ \ \xi^{\alpha_1}_{\alpha_2}(r) \leq t < \xi^{\alpha_1}_{\alpha_2}(x).$$

Implies $rx \in (\xi^{\alpha_1}_{\alpha_2})_t$.
Hence, from case(i) and case(ii), $rx \in (\xi^{\alpha_1}_{\alpha_2})_t$.
Thus, $\{x \in R \mid \xi^{\alpha_1}_{\alpha_2}(x) > t\}$ is an ideal of R for all $t \in [0, 1]$. □

Definition 5.12. Let ξ be an (α_1, α_2)-fuzzy ideal of R. Then the ideals $(\xi^{\alpha_1}_{\alpha_2})_t$, for $t \in [0, 1]$ are called (α_1, α_2)-level ideals of R.

Remark 5.13. Let ξ be an (α_1, α_2)-fuzzy ideal of R and $t_1, t_2 \in [0, 1]$ be such that $t_1 \leq t_2$. We note that if $x \in (\xi^{\alpha_1}_{\alpha_2})_{t_2}$, then $(\xi^{\alpha_1}_{\alpha_2})(x) \geq t_2 \geq t_1$. Hence, $x \in (\xi^{\alpha_1}_{\alpha_2})_{t_1}$. Thus $(\xi^{\alpha_1}_{\alpha_2})_{t_2} \subseteq (\xi^{\alpha_1}_{\alpha_2})_{t_1}$.

Proposition 5.14. *Let ξ be a (α_1, α_2)-fuzzy ideal of R. Two level ideals $(\xi^{\alpha_1}_{\alpha_2})_{t_1}$, $(\xi^{\alpha_1}_{\alpha_2})_{t_2}$ with $t_1 < t_2$ are equal if and only if there is no $x \in R$ such that $t_1 \leq \xi^{\alpha_1}_{\alpha_2}(x) < t_2$.*

Proof. Assume that $(\xi_{\alpha_2}^{\alpha_1})_{t_1} = (\xi_{\alpha_2}^{\alpha_1})_{t_2}$. If there exists $x \in R$ such that $t_1 \leq \xi_{\alpha_2}^{\alpha_1}(x) < t_2$, then $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$ but $x \notin (\xi_{\alpha_2}^{\alpha_1})_{t_2}$, a contradiction.

Conversely, suppose that there is no $x \in R$ such that $t_1 \leq \xi_{\alpha_2}^{\alpha_1}(x) < t_2$. Since, $t_1 < t_2$ we have $(\xi_{\alpha_2}^{\alpha_1})_{t_2} \subseteq (\xi_{\alpha_2}^{\alpha_1})_{t_1}$. Now if $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_1}$, then $t_1 \leq \xi_{\alpha_2}^{\alpha_1}(x)$. Hence, by the given condition it follows that $\xi_{\alpha_2}^{\alpha_1}(x) \nless t_2$. Since $\xi_{\alpha_2}^{\alpha_1}(x)$ and t_2 are real numbers belonging to $[0,1]$, this implies that $\xi_{\alpha_2}^{\alpha_1}(x) \geq t_2$. Hence $x \in (\xi_{\alpha_2}^{\alpha_1})_{t_2}$. Therefore, $(\xi_{\alpha_2}^{\alpha_1})_{t_1} = (\xi_{\alpha_2}^{\alpha_1})_{t_2}$. □

Proposition 5.15. *The intersection of two (α_1, α_2)-fuzzy ideals of R is again an (α_1, α_2)-fuzzy ideal.*

Proof. Let ξ and η be two (α_1, α_2)-fuzzy ideals of R. For $x, y \in R$, we have

$$\begin{aligned}
(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x-y) &= (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x-y), \text{ by Lemma 3.3} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x-y), \eta_{\alpha_2}^{\alpha_1}(x-y)\} \\
&\geq \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}, \min\{\eta_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\
&= \min\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(x)\}, \min\{\xi_{\alpha_2}^{\alpha_1}(y), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(x) \cap \eta_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y) \cap \eta_{\alpha_2}^{\alpha_1}(y)\} \\
&= \min\{(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x), (\xi \cap \eta)_{\alpha_2}^{\alpha_1}(y)\}. \qquad (5.9)
\end{aligned}$$

Also, we have

$$\begin{aligned}
(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(xy) &= (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(xy), \text{ by Lemma 3.3} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(xy), \xi_{\alpha_2}^{\alpha_1}(xy)\} \\
&\geq \min\{\max\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}, \max\{\eta_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(y)\}\}, \\
&\quad \text{as all the quantities involved belong to } [0,1] \\
&= \max\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y)\}, \min\{\eta_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\
&= \max\{\min\{\xi_{\alpha_2}^{\alpha_1}(x), \eta_{\alpha_2}^{\alpha_1}(x)\}, \min\{\xi_{\alpha_2}^{\alpha_1}(y), \eta_{\alpha_2}^{\alpha_1}(y)\}\} \\
&= \max\{(\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(x)), (\xi_{\alpha_2}^{\alpha_1} \cap \eta_{\alpha_2}^{\alpha_1})(y)\} \\
&= \max\{(\xi \cap \eta)_{\alpha_2}^{\alpha_1}(x), (\xi \cap \eta)_{\alpha_2}^{\alpha_1}(y)\}. \qquad (5.10)
\end{aligned}$$

It follows from (5.9) and (5.10), $\xi \cap \eta$ is an (α_1, α_2)-fuzzy ideal of R. □

The following example shows that the union of two (α_1, α_2)-fuzzy ideals may not be an (α_1, α_2)-fuzzy ideal.

Example 5.16. Let $R = \mathbb{Z}_{18}$. Define fuzzy subsets ξ and η as follows:

$$\xi(x) = \begin{cases} 0.8, & \text{if } x = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

$$\eta(x) = \begin{cases} 0.5, & \text{if } x = \{0, 3, 6, 9, 12, 15\}, \\ 0.1, & \text{otherwise.} \end{cases}$$

Let $\alpha_1 = 1$ and $\alpha_2 = 0.1$. Then ξ and η are $(1, 0.1)$-fuzzy ideals of $\mathbb{Z}_{18}$. We have

$$(\xi \cup \eta)(x) = \begin{cases} 0.8, & \text{if } x = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}, \\ 0.5, & \text{if } x = \{3, 9, 15\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

For $\alpha_1 = 1$ and $\alpha_2 = 0.1$, we have $\alpha_1 - \alpha_2 = 0.9$. If we take $x = 9, y = 4$, then $x - y = 5$. Also, $(\xi \cup \eta)(x) = 0.5$, $(\xi \cup \eta)(y) = 0.8$ and $(\xi \cup \eta)(x - y) = 0.3$. Now,

$$\begin{aligned} &(\xi \cup \eta)_{\alpha_2}^{\alpha_1}(x) = \min\{0.5, 0.9\} = 0.5, \\ &(\xi \cup \eta)_{\alpha_2}^{\alpha_1}(y) = \min\{0.8, 0.9\} = 0.8, \\ &(\xi \cup \eta)_{\alpha_2}^{\alpha_1}(x - y) = \min\{0.3, 0.9\} = 0.3. \\ &(\xi \cup \eta)_{\alpha_2}^{\alpha_1}(x - y) \ngeq \min\{(\xi \cup \eta)_{\alpha_2}^{\alpha_1}(x), (\xi \cup \eta)_{\alpha_2}^{\alpha_1}(y)\}. \end{aligned}$$

Thus, $\xi \cup \eta$ is not a $(1, 0.1)$-fuzzy ideal of $\mathbb{Z}_{18}$.

Proposition 5.17. *Let $g : R \to R'$ be an onto homomorphism of a ring R to a ring R'. If ξ is an (α_1, α_2)-fuzzy ideal of R', then $g^{-1}(\xi)$ is an (α_1, α_2)-fuzzy ideal of R which is constant on $\ker g$.*

Proof. For $x, y \in R.$ we have

$$\begin{aligned} &(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(x - y) \\ &= g^{-1}(\xi_{\alpha_2}^{\alpha_1})(x - y), \quad \text{by Lemma 3.3} \\ &= \xi_{\alpha_2}^{\alpha_1}(g(x - y)) \\ &= \xi_{\alpha_2}^{\alpha_1}(g(x) - g(y)) \\ &\geq \min\{\xi_{\alpha_2}^{\alpha_1}(g(x)), \xi_{\alpha_2}^{\alpha_1}(g(y))\}, \\ &\quad (\text{as } \xi \text{ is } (\alpha_1, \alpha_2)\text{-fuzzy ideal of } R') \\ &= \min\{g^{-1}(\xi_{\alpha_2}^{\alpha_1})(x), g^{-1}(\xi_{\alpha_2}^{\alpha_1})(y)\} \\ &= \min\{(g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(x), (g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(y)\}, \quad \text{by Lemma 3.3} \end{aligned} \tag{5.11}$$

Also, we have

$$\begin{aligned} (g^{-1}(\xi))_{\alpha_2}^{\alpha_1}(xy) &= g^{-1}(\xi_{\alpha_2}^{\alpha_1})(xy) \\ &= \xi_{\alpha_2}^{\alpha_1}(g(xy)) \end{aligned}$$

$$= \xi^{\alpha_1}_{\alpha_2}(g(x)g(y))$$
$$\geq \max\{\xi^{\alpha_1}_{\alpha_2}(g(x)), \xi^{\alpha_1}_{\alpha_2}(g(y))\},$$
$$\text{(as } \xi \text{ is } (\alpha_1, \alpha_2)\text{-fuzzy ideal of } R')$$
$$= \max\{g^{-1}(\xi^{\alpha_1}_{\alpha_2})(x), g^{-1}(\xi^{\alpha_1}_{\alpha_2})(y)\}$$
$$= \max\{(g^{-1}(\xi))^{\alpha_1}_{\alpha_2}(x), (g^{-1}(\xi))^{\alpha_1}_{\alpha_2}(y)\}, \text{ by Lemma 3.3.} \quad (5.12)$$

It follows from (5.11) and (5.12) that $g^{-1}(\xi)$ is an (α_1, α_2)-fuzzy ideal of R. Next if $p \in kerg$, then $g(p) = 0'$, where $0'$ is the additive identity of R'. Therefore, $(g^{-1}(\xi))^{\alpha_1}_{\alpha_2}(p) = \xi^{\alpha_1}_{\alpha_2}(g(p)) = \xi^{\alpha_1}_{\alpha_2}(0')$ and so $g^{-1}(\xi)$ is constant on $kerg$. □

6. (α_1, α_2)-FUZZY QUOTIENT RINGS

The motivation to define (α_1, α_2)-fuzzy coset of a (α_1, α_2)-fuzzy ideal of a ring was from Abou-Zaid [1] where he define the fuzzy coset in definition 2.8. In this section we have define two operations of addition and multiplication on (α_1, α_2)-fuzzy cosets of a (α_1, α_2)-fuzzy ideal of a ring and proved the collection of all (α_1, α_2)-fuzzy ideals forms a ring under these operations.

Definition 6.1. Let ξ be an (α_1, α_2)-fuzzy ideal of R. For any $x \in R$, define a fuzzy set $x + \xi^{\alpha_1}_{\alpha_2} : R \to [0, 1]$ by, $(x + \xi^{\alpha_1}_{\alpha_2})(y) = \min\{\xi(y - x), \alpha_1 - \alpha_2\}$, for all $y \in R$. Then the fuzzy set $x + \xi^{\alpha_1}_{\alpha_2}$ is called an (α_1, α_2)-fuzzy coset of the (α_1, α_2)-fuzzy ideal ξ of R.

Proposition 6.2. *If ξ is an (α_1, α_2)-fuzzy ideal of R, then*
(i) $0 + \xi^{\alpha_1}_{\alpha_2} = \xi^{\alpha_1}_{\alpha_2}$.
(ii) For any $t \in [0, 1]$, $(x + \xi^{\alpha_1}_{\alpha_2})_t = x + (\xi^{\alpha_1}_{\alpha_2})_t$.
(iii) $\xi^{\alpha_1}_{\alpha_2}(x) = \xi^{\alpha_1}_{\alpha_2}(0) \Leftrightarrow x + \xi^{\alpha_1}_{\alpha_2} = \xi^{\alpha_1}_{\alpha_2}$.

Proof. (i): We have

$$\begin{aligned}(0 + \xi^{\alpha_1}_{\alpha_2})(x) &= \min\{\xi(x - 0), \alpha_1 - \alpha_2\} \\ &= \min\{\xi(x), \alpha_1 - \alpha_2\} \\ &= \xi^{\alpha_1}_{\alpha_2}(x).\end{aligned}$$

Hence, $0 + \xi^{\alpha_1}_{\alpha_2} = \xi^{\alpha_1}_{\alpha_2}$.
(ii): Let $y \in R$. We have

$$\begin{aligned} y \in (x + \xi^{\alpha_1}_{\alpha_2})_t &\Leftrightarrow (x + \xi^{\alpha_1}_{\alpha_2})(y) \geq t \\ &\Leftrightarrow \min\{\xi(y - x), \alpha_1 - \alpha_2\} \geq t \\ &\Leftrightarrow \{\min\{\xi(y), \xi(x)\}, \alpha_1 - \alpha_2\} \geq t\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \{\min\{\xi(y), \alpha_1 - \alpha_2\}, \min\{\xi(x), \alpha_1 - \alpha_2\}\} \geq t \\
&\Leftrightarrow \min\{\xi_{\alpha_2}^{\alpha_1}(y), \xi_{\alpha_2}^{\alpha_1}(x)\} \geq t \\
&\Leftrightarrow \xi_{\alpha_2}^{\alpha_1}(y - x) \geq t \\
&\Leftrightarrow y - x \in (\xi_{\alpha_2}^{\alpha_1})_t \\
&\Leftrightarrow y \in x + (\xi_{\alpha_2}^{\alpha_1})_t.
\end{aligned}
$$

Hence, $(x + \xi_{\alpha_2}^{\alpha_1})_t = x + (\xi_{\alpha_2}^{\alpha_1})_t$.
(iii): Assume that

$$\xi_{\alpha_2}^{\alpha_1}(x) = \xi_{\alpha_2}^{\alpha_1}(0). \tag{6.1}$$

Then for $y \in R$, we have

$$
\begin{aligned}
(x + \xi_{\alpha_2}^{\alpha_1})(y) &= \min\{\xi(y - x), \alpha_1 - \alpha_2\} \\
&\geq \min\{\min\{\xi(y), \xi(x)\}, \alpha_1 - \alpha_2\} \\
&= \min\{\min\{\xi(y), \alpha_1 - \alpha_2\}, \min\{\xi(x), \alpha_1 - \alpha_2\}\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(y), \xi_{\alpha_2}^{\alpha_1}(x)\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(y), \xi_{\alpha_2}^{\alpha_1}(0)\}, \text{ from (6.1)} \\
&= \xi_{\alpha_2}^{\alpha_1}(y), \text{ by Proposition 5.6} \\
&= \xi_{\alpha_2}^{\alpha_1}(y - x + x) \\
&\geq \min\{\xi_{\alpha_2}^{\alpha_1}(y - x), \xi_{\alpha_2}^{\alpha_1}(x)\} \\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(y - x), \xi_{\alpha_2}^{\alpha_1}(0)\}, \text{ from (6.1)} \\
&= \xi_{\alpha_2}^{\alpha_1}(y - x), \text{ by Proposition 5.6} \\
&= \min\{\xi(y - x), \alpha_1 - \alpha_2\} \\
&= (x + \xi_{\alpha_2}^{\alpha_1})(y).
\end{aligned}
$$

Thus, $x + \xi_{\alpha_2}^{\alpha_1} = \xi_{\alpha_2}^{\alpha_1}$.

Conversely, assume that $x + \xi_{\alpha_2}^{\alpha_1} = \xi_{\alpha_2}^{\alpha_1}$

$$
\begin{aligned}
&\Rightarrow (x + \xi_{\alpha_2}^{\alpha_1})(0) = \xi_{\alpha_2}^{\alpha_1}(0) \\
&\Rightarrow \min\{\xi(0 - x), \alpha_1 - \alpha_2\} = \xi_{\alpha_2}^{\alpha_1}(0) \\
&\Rightarrow \min\{\xi(-x), \alpha_1 - \alpha_2\} = \xi_{\alpha_2}^{\alpha_1}(0) \\
&\Rightarrow \min\{\xi(x), \alpha_1 - \alpha_2\} = \xi_{\alpha_2}^{\alpha_1}(0) \\
&\Rightarrow \xi_{\alpha_2}^{\alpha_1}(x) = \xi_{\alpha_2}^{\alpha_1}(0).
\end{aligned}
$$

□

Theorem 6.3. *Let ξ be a (α_1, α_2)-fuzzy ideal of R and ϱ be the collection of all (α_1, α_2)-fuzzy cosets of ξ. Then ϱ is a ring under the operations,*

$$(x+\xi_{\alpha_2}^{\alpha_1})+(y+\xi_{\alpha_2}^{\alpha_1})=(x+y)+\xi_{\alpha_2}^{\alpha_1}$$

and

$$(x+\xi_{\alpha_2}^{\alpha_1})\cdot(y+\xi_{\alpha_2}^{\alpha_1})=(x\cdot y)+\xi_{\alpha_2}^{\alpha_1},$$

for all $x, y \in R$.

Proof. First we shall show that these two operations are well–defined. Let $x+\xi_{\alpha_2}^{\alpha_1}=x^{'}+\xi_{\alpha_2}^{\alpha_1}$ and $y+\xi_{\alpha_2}^{\alpha_1}=y^{'}+\xi_{\alpha_2}^{\alpha_1}$. Then for $x^{'}, y^{'} \in R$, $(x+\xi_{\alpha_2}^{\alpha_1})(x')=(x^{'}+\xi_{\alpha_2}^{\alpha_1})(x')$ and $(y+\xi_{\alpha_2}^{\alpha_1})(y')=(y^{'}+\xi_{\alpha_2}^{\alpha_1})(y')$. Then by definition 6.1, $\min\{\xi(x^{'}-x),\alpha_1-\alpha_2\}=\min\{\xi(x^{'}-x^{'}),\alpha_1-\alpha_2\}$ and $\min\{\xi(y^{'}-y),\alpha_1-\alpha_2\}=\min\{\xi(y^{'}-y^{'}),\alpha_1-\alpha_2\}$. Therefore,

$$\min\{\xi(x^{'}-x),\alpha_1-\alpha_2\}=\min\{\xi(0),\alpha_1-\alpha_2\}$$

and

$$\min\{\xi(y^{'}-y),\alpha_1-\alpha_2\}=\min\{\xi(0),\alpha_1-\alpha_2\}.$$

Therefore, $\xi_{\alpha_2}^{\alpha_1}(x^{'}-x)=\xi_{\alpha_2}^{\alpha_1}(0)$ and $\xi_{\alpha_2}^{\alpha_1}(y^{'}-y)=\xi_{\alpha_2}^{\alpha_1}(0)$, by Definition 3.1. Hence,

$$\xi_{\alpha_2}^{\alpha_1}(x^{'}-x)=\xi_{\alpha_2}^{\alpha_1}(0)\ \text{ and } \xi_{\alpha_2}^{\alpha_1}(y^{'}-y)=\xi_{\alpha_2}^{\alpha_1}(0). \tag{6.2}$$

For $z \in R$, we have

$$\begin{aligned}
&((x+y)+\xi_{\alpha_2}^{\alpha_1})(z)\\
&=\min\{\xi(z-(x+y)),\alpha_1-\alpha_2\}\\
&=\min\{\xi(z-x-y),\alpha_1-\alpha_2\}\\
&=\min\{\xi(z-x^{'}-y^{'}+x^{'}-x+y^{'}-y),\alpha_1-\alpha_2\}\\
&\geq \min\{\{\xi(z-x^{'}-y^{'}),\xi(x^{'}-x),\xi(y^{'}-y)\},\alpha_1-\alpha_2\},\\
&\quad \text{since } \xi \text{ is a fuzzy ideal of } R.\\
&=\min\{\min\{\xi(z-x^{'}-y^{'}),\alpha_1-\alpha_2\},\min\{\xi(x^{'}-x),\alpha_1-\alpha_2\},\\
&\qquad\quad \min\{\xi(y^{'}-y),\alpha_1-\alpha_2\}\}\\
&=\min\{\xi_{\alpha_2}^{\alpha_1}(z-x^{'}-y^{'}),\xi_{\alpha_2}^{\alpha_1}(x^{'}-x),\xi_{\alpha_2}^{\alpha_1}(y^{'}-y)\}\\
&=\min\{\xi_{\alpha_2}^{\alpha_1}(z-x^{'}-y^{'}),\xi_{\alpha_2}^{\alpha_1}(0),\xi_{\alpha_2}^{\alpha_1}(0)\},\ \text{from (6.2).}\\
&=\xi_{\alpha_2}^{\alpha_1}(z-x^{'}-y^{'}),\ \text{by Proposition 5.6}\\
&=\min\{\xi(z-x^{'}-y^{'}),\alpha_1-\alpha_2\}\\
&=((x^{'}+y^{'})+\xi_{\alpha_2}^{\alpha_1})(z).
\end{aligned}$$

Thus $((x+y)+\xi_{\alpha_2}^{\alpha_1})(z) \geq ((x^{'}+y^{'})+\xi_{\alpha_2}^{\alpha_1})(z)$. Similarly, we can show that $((x^{'}+y^{'})+\xi_{\alpha_2}^{\alpha_1})(z) \geq ((x+y)+\xi_{\alpha_2}^{\alpha_1})(z)$. Hence,

$$((x^{'}+y^{'})+\xi_{\alpha_2}^{\alpha_1})(z) = ((x+y)+\xi_{\alpha_2}^{\alpha_1})(z). \tag{6.3}$$

We have

$$\begin{aligned}
&(xy+\xi_{\alpha_2}^{\alpha_1})(z)\\
&= \min\{\xi(z-xy), \alpha_1-\alpha_2\}\\
&= \min\{\xi(z-x^{'}y^{'}+x^{'}y^{'}-xy), \alpha_1-\alpha_2\}\\
&\geq \min\{\min\{\xi(z-x^{'}y^{'}), \xi(x^{'}y^{'}-xy)\}, \alpha_1-\alpha_2\},\\
&\qquad \text{since } \xi \text{ is a fuzzy ideal of } R\\
&= \min\{\min\{\xi(z-x^{'}y^{'}), \alpha_1-\alpha_2\}, \min\{\xi(x^{'}y^{'}-xy), \alpha_1-\alpha_2\}\}\\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(z-x^{'}y^{'}), \xi_{\alpha_2}^{\alpha_1}(x^{'}y^{'}-xy)\}. \qquad (6.4)
\end{aligned}$$

We have

$$\begin{aligned}
&\xi_{\alpha_2}^{\alpha_1}(x^{'}y^{'}-xy)\\
&= \xi_{\alpha_2}^{\alpha_1}(x^{'}y^{'}-x^{'}y+x^{'}y-xy)\\
&= \xi_{\alpha_2}^{\alpha_1}(x^{'}(y^{'}-y)+(x^{'}-x)y)\\
&\geq \min\{\xi_{\alpha_2}^{\alpha_1}(x(y^{'}-y)), \xi_{\alpha_2}^{\alpha_1}((x^{'}-x)y)\}, \text{ by Proposition 5.4}\\
&\geq \min\{\max\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(y^{'}-y)\}, \max\{\xi_{\alpha_2}^{\alpha_1}(x^{'}-x), \xi_{\alpha_2}^{\alpha_1}(y)\}\}\\
&= \min\{\max\{\xi_{\alpha_2}^{\alpha_1}(x), \xi_{\alpha_2}^{\alpha_1}(0)\}, \max\{\xi_{\alpha_2}^{\alpha_1}(0), \xi_{\alpha_2}^{\alpha_1}(y)\}, \text{ from (6.2)}.\\
&= \min\{\xi_{\alpha_2}^{\alpha_1}(0), \xi_{\alpha_2}^{\alpha_1}(0)\}, \quad \text{by Proposition 5.6}\\
&= \xi_{\alpha_2}^{\alpha_1}(0). \qquad (6.5)
\end{aligned}$$

Now, (6.4) becomes

$$\begin{aligned}
(xy+\xi_{\alpha_2}^{\alpha_1})(z) &= \min\{\xi_{\alpha_2}^{\alpha_1}(z-x^{'}y^{'}), \xi_{\alpha_2}^{\alpha_1}(0)\}\\
&= \xi_{\alpha_2}^{\alpha_1}(z-x^{'}y^{'}), \quad \text{by Proposition 5.6}\\
&= \min\{\xi(z-x^{'}y^{'}), \alpha_1-\alpha_2\}\\
&= (x^{'}y^{'}+\xi_{\alpha_2}^{\alpha_1})(z).
\end{aligned}$$

Similarly, we can show that $(x^{'}y^{'}+\xi_{\alpha_2}^{\alpha_1})(z) \geq (xy+\xi_{\alpha_2}^{\alpha_1})(z)$.

Hence, $(xy+\xi_{\alpha_2}^{\alpha_1})(z) = (x^{'}y^{'}+\xi_{\alpha_2}^{\alpha_1})(z)$. Thus, the operations $+$ and $\cdot$ are well defined.

Further we have, $(x + \xi^{\alpha_1}_{\alpha_2}) + ((-x) + \xi^{\alpha_1}_{\alpha_2}) = (0 + \xi^{\alpha_1}_{\alpha_2}) = \xi^{\alpha_1}_{\alpha_2}$

$(x + \xi^{\alpha_1}_{\alpha_2}) \cdot (1 + \xi^{\alpha_1}_{\alpha_2}) = (1 + \xi^{\alpha_1}_{\alpha_2}) \cdot (x + \xi^{\alpha_1}_{\alpha_2}) = x + \xi^{\alpha_1}_{\alpha_2}$

$(x + \xi^{\alpha_1}_{\alpha_2}) \cdot (y + \xi^{\alpha_1}_{\alpha_2}) = (y + \xi^{\alpha_1}_{\alpha_2}) \cdot (x + \xi^{\alpha_1}_{\alpha_2}) = xy + \xi^{\alpha_1}_{\alpha_2}$

$(x + \xi^{\alpha_1}_{\alpha_2}) \cdot ((y + \xi^{\alpha_1}_{\alpha_2}) \cdot (z + \xi^{\alpha_1}_{\alpha_2})) = ((x + \xi^{\alpha_1}_{\alpha_2}) \cdot (y + \xi^{\alpha_1}_{\alpha_2})) \cdot (z + \xi^{\alpha_1}_{\alpha_2})$

$(x + \xi^{\alpha_1}_{\alpha_2}) \cdot ((y + \xi^{\alpha_1}_{\alpha_2}) + (z + \xi^{\alpha_1}_{\alpha_2})) = (x + \xi^{\alpha_1}_{\alpha_2}) \cdot (y + \xi^{\alpha_1}_{\alpha_2}) + (x + \xi^{\alpha_1}_{\alpha_2}) \cdot (z + \xi^{\alpha_1}_{\alpha_2})$

$((x + \xi^{\alpha_1}_{\alpha_2}) + (y + \xi^{\alpha_1}_{\alpha_2})) \cdot (z + \xi^{\alpha_1}_{\alpha_2}) = (x + \xi^{\alpha_1}_{\alpha_2}) \cdot (z + \xi^{\alpha_1}_{\alpha_2}) + (y + \xi^{\alpha_1}_{\alpha_2}) \cdot (z + \xi^{\alpha_1}_{\alpha_2})$.

Hence, ϱ is a commutative ring with unity. □

7. Conclusion

In this paper, we have introduced and studied (α_1, α_2)-fuzzy subrings and (α_1, α_2)-fuzzy ideals of a ring. In future this concept can be extended to (α_1, α_2)-intuitionistic fuzzy subrings and (α_1, α_2)-intuitionistic fuzzy ideals of a ring.

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References

1. S. Abou-Zaid, On fuzzy ideals and fuzzy quotient rings of a ring, *Fuzzy Sets and Systems*, **59** (1993), 205–210.
2. P. Khubchandani and J. Khubchandani, Fuzzy α-modularity in fuzzy α lattices, *Journal of Hyperstructures*, **12**(1) (2023), 30–50.
3. W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems*, **8** (1982), 133–139.
4. J. N. Moderson and D. S. Malik, *Fuzzy Commutative Algebra*, World Scientific, River Edge, NJ, USA, 1998.
5. S. Nimbhorkar and J. Khubchandani, L-Fuzzy hollow modules and L-fuzzy multiplication modules, *Kragujevac J. Math.*, **48**(3) (2024), 423–432.
6. S. Nimbhorkar and J. Khubchandani, Fuzzy semi-essential submodules and fuzzy semi-closed submodules, *TWMS J. App. and Eng. Math.*, **13**(2) (2023), 568–575.
7. S. Nimbhorkar and J. Khubchandani, (a, b)-Fuzzy subrings and (a, b)-fuzzy ideals of a ring, *Journal of Hyperstructures*, **9**(2) (2020), 96–114.
8. S. Nimbhorkar and J. Khubchandani, Fuzzy weakly irreducible ideals of a ring, *Journal of Hyperstructures*, **10**(2) (2021), 118–128.
9. A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512–517.
10. P. K. Sharma, α-Fuzzy subgroups, *International Journal of Fuzzy Mathematics and Systems*, **3**(1) (2013), 47–59.
11. L. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353.

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A STUDY OF (α_1, α_2)-FUZZY SUBRINGS AND (α_1, α_2)-FUZZY IDEALS OF A RING

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مطالعه‌ای درباره‌ی زیرحلقه‌های $(\alpha_۱, \alpha_۲)$-فازی و ایده‌آل‌های $(\alpha_۱, \alpha_۲)$-فازی یک حلقه

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به عنوان تعمیمی از تعریف لیو برای زیرحلقه‌ی فازی و ایده‌آل فازی، مفهوم جدیدی از زیرحلقه‌ی $(\alpha_۱, \alpha_۲)$-فازی و ایده‌آل $(\alpha_۱, \alpha_۲)$-فازی برای حلقه معرفی می‌شود. مثال‌هایی ارائه داده‌ایم و خواص آن‌ها را تجزیه و تحلیل کرده‌ایم. علاوه بر این، هم‌مجموعه‌ی $(\alpha_۱, \alpha_۲)$-فازی از یک ایده‌آل $(\alpha_۱, \alpha_۲)$-فازی را برای یک حلقه تعریف کرده و برخی از خواص آن را بررسی نموده‌ایم.

کلمات کلیدی: زیرحلقه‌ی $(\alpha_۱, \alpha_۲)$-فازی، ایده‌آل $(\alpha_۱, \alpha_۲)$-فازی، زیرمجموعه‌ی سطح $(\alpha_۱, \alpha_۲)$-فازی.