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COMMUTATIVITY FOR THE WEAKLY RIGHT CANCELLATIVE SEMIRINGS: AN ENTIRELY NOVEL CATEGORY OF SEMIRINGS AND A WEAK CONDITION FOR COMMUTATIVITY RESEARCH

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Abstract. The goal of this study is to provide an innovation for commutativity research that is less than the strong condition prime ring. This paper will describe weakly right cancellative semirings and examine how commutativity and generalized derivations apply to this class of semirings. A detailed explanation and classification of some of these generalized derivations are also included.

1. Introduction

A semiring is an algebraic structure consisting of a non-empty set S provided with two binary operations, called addition (which is commutative and usually denoted by $+$) and multiplication (usually denoted by $\cdot$) such that the following conditions hold:

(1) $(S,+)$ and $(S,\cdot)$ are semigroups;
(2) multiplication distributes over addition from either side.

Recall that a semiring is said to be commutative if $(S,\cdot)$ is commutative. If there exists a neutral element $0 \in (S,+)$ (resp. $e \in (S,\cdot)$) it is called the zero of $(S,+)$ (resp. the identity of $(S,\cdot)$). Additionally, if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$, then S is called a semiring with absorbing zero. In other words, semirings with absorbing zero are just rings without subtraction. Nontrivial examples of semirings first appeared in the work of Richard Dedekind [5] in 1894, in connection with the algebra of ideals of a commutative ring (one can add and multiply ideals, but one cannot subtract them). Nevertheless, the formal definition of semirings was introduced by H. S. Vandiver in 1934 and has since then been studied by many authors. Semirings constitute a fairly natural generalization of rings and distributive lattices, with broad applications in different areas of mathematics such as combinatorics, functional analysis, topology, graph theory, ring theory including partial ordered rings, optimization theory, automata theory, formal language theory, coding theory

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and the mathematical modeling of quantum physics and parallel computing systems. The basic reference for semirings is [7]. Other valuable results on the structure of semirings are contained in [13] and [16].

It is well known that a semiring's zero need not be absorbing and could even coincide with the identity of S (cf., for example, reference [15]). When for all $a \in S$, $b+a=c+a$ (resp. $a+b=a+c$) gives $b=c$, then the element a is said to be additively right (resp. left) cancellable. A semiring S is said to be additively right (or left) cancellative if all a in S are additively right (or left) cancellable in S. It is stated that S is additively cancellative if and only if it is additively right and left cancellative.

A nonzero element a of S is multiplicatively left cancellable if $ab = ac$ implies $b = c$. A semiring S is considered multiplicatively left cancellative (MLC) when all of its nonzero elements are multiplycatively left cancellable in S.

A left (resp. right) ideal of a semiring S is non-empty subset I of S such that $x+y \in I$ for all $x, y \in I$ and $sx \in I$ (resp. $xs \in I$) for all $x \in I$ and $s \in S$. An ideal of a semiring S is a non-empty subset I of S such that I is both a left and right ideal of S. An additive mapping $d : S \longrightarrow S$ is a derivation on S if $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$. On the other hand, a generalized derivation (F, d) is an additive mapping $F : S \longrightarrow S$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in S$. A semiring S is said to be 2-torsion free if whenever $2x = 0$ for every $x \in S$ implies $x = 0$. The recent literature contains various results which indicate how the global structure of a ring S is often tightly connected to the behaviour of additive mappings defined on S. Recently many authors have studied commutativity in prime and semiprime rings admitting suitably constrained derivations acting on appropriate subsets of the rings. Moreover, several authors have proved comparable results on semirings (for example, see [1, 5, 8, 10, 12]).

Recently, V. DE Filipis, A. Mamouni and L. Oukhtite in [6] defined a new class of semiring which called weakly left cancellative (WLC) semirings and they have studied the connection between commutativity of this class of semirings and derivations. In particular, they proved that, if S is a WLC semiring, I is a nonzero ideal of S and d is a derivation of S such that $d(xy) = d(yx)$ for all $x, y \in I$, or $d(xy) + yx = d(yx) + xy$ for all $x, y \in I$, or $d(x)x = xd(x)$ for all $x \in I$, then S is commutative.

Motivated by the previous results, in the present paper, we introduce the notion of weakly right cancellative (WRC) semirings and we will study the commutativity of a WRC semiring with a generalized derivation F that fulfills certain algebraic properties on an ideal of S. Added to that, we will give a complete description and classification for some of these generalized derivations.

2. Main results

In the current article, the semiring refers to an additively cancellative semiring. We are going to denote the center of S by

$$Z(S) = \{z \in S : zx = xz, \ \forall x \in S\},$$

the commutator $xy - yx$ is denoted as $[x, y]$, while the Jordan product $xy + yx$ is written as $x \circ y$ for every $x, y \in S$.

Based on the concept of right cancellative semirings, we introduce the notion of weakly right cancellative semirings:

Definition 2.1. A semiring S is said to be weakly right cancellative (WRC) if $axb = cxb$ for all $x \in S$, implies either $a = c$ or $b = 0$.

In broad terms, any right cancellative semiring is WRC, but the converse is not guaranteed in general.

Example 2.2. Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{N} \right\}$ where $\mathbb{N}$ is the set of positive integers including 0. It doesn't take much to prove that S is not a right cancellative semiring. Let $M, N, N' \in S$ such that $M \neq 0$, then the relation $NAM = N'AM$ forces $N = N'$ for all $A \in S$ and therefore S is a WRC semiring.

The following facts are going to be referenced periodically during the proofs. Let S be a WRC semiring and I an ideal of S:

Fact 1. If $axb = cxb$ for all $x \in I$, then $a = c$ or $b = 0$.

Fact 2. If I is commutative, then S is commutative. In particular, if $xy = yx$ for all $y \in I$ then $x \in Z(S)$.

Fact 3. If S admits a derivation d such that $d(I) = (0)$, then $d = 0$.

Fact 4. If S admits a generalized derivation F such that $F(I) = (0)$, then $F = 0$.

Proposition 2.3. *Let F be an arbitrary additive mapping of S and d a derivation of S. Then $F(xy) = F(x)y + xd(y)$ for all $x, y \in S$ if and only if $F(xy) = xd(y) + F(x)y$ for all $x, y \in S$. Therefore F is a generalized derivation if and only if $F(xy) = xd(y) + F(x)y$.*

Proof. Given that $F(xy) = F(x)y + xd(y)$ for all $x, y \in S$. Taking into account $x(y + y) = xy + xy$, we have

$$F(x(y + y)) = F(x)(y + y) + xd(y + y) = F(x)y + F(x)y + xd(y) + xd(y)$$

On the other hand

$$F(xy + xy) = F(xy) + F(xy) = F(x)y + xd(y) + F(x)y + xd(y)$$

Subsequently, $F(x)y + xd(y) = xd(y) + F(x)y$, so $F(xy) = xd(y) + F(x)y$. Similar proof confirms the reverse. ☐

Lemma 2.4. *Let S be a 2-torsion free WRC semiring and I be a nonzero ideal of S. If S admits a derivation d such that $d^2(x) = 0$, for all $x \in I$, then $d = 0$.*

Proof. Let's say $d^2(x) = 0$, for all $x \in I$. Replacing x by xy, we get

$$d^2(xy) = 0 = d^2(x)y + 2d(x)d(y) + xd^2(y)$$

for all $x, y \in I$. Yet according to the hypothesis, $d^2(x) = 0 = d^2(y)$. That is why for any $x, y \in I$, we have $2d(x)d(y) = 0$. Knowing that S is 2-torsion free, we conclude $d(x)d(y) = 0$. After changing y by yz, we attain $d(x)yd(z) = 0$ for every $x, y, z \in I$. In particular, for all $x_1, x_2 \in I$, $d(x_1)yd(z) = d(x_2)yd(z)$. Since S is WRC it follows that $d(z) = 0$ or $d(x_1) = d(x_2)$ for all $x_1, x_2, z \in I$. Initially, we go with $d(x_1) = d(x_2)$, which means $d(x_1 + x_2) = d(x_2)$. Thus, $d(x_1) = 0$ for every $x_1 \in I$. Hence in both cases, we arrive at $d(x) = 0$ for all $x \in I$, so through Fact 3 we derive that $d = 0$. ☐

A number of researchers in the field of commutativity research anticipate that the prime ring condition is strong. As a result, we will present some commutativity requirements in the case of weakly right cancellative semirings (WRC) as a weak condition for commutativity.

Theorem 2.5. *Let S be a WRC semiring and I a nonzero ideal of S. If S admits a nonzero generalized derivation F with associated derivation d such that $F(xy) = F(yx)$ for all $x, y \in I$, then S is commutative.*

Proof. Consider that

$$F(xy) = F(yx) \quad \text{for all } x, y \in I. \tag{2.1}$$

Substituting y by yx in (2.1), we acquire

$$F(xy)x + xyd(x) = F(yx)x + yxd(x) \quad \text{for all } x, y \in I. \tag{2.2}$$

By correctly multiplying equation (2.1) by x and comparing the result with equation (2.2), we earn

$$xyd(x) = yxd(x) \quad \text{for all} \ \ x, y \in I. \tag{2.3}$$

When zy is used rather than y in (2.3), we detect that

$$xzyd(x) = zxyd(x) \quad \text{for all} \ \ x, y, z \in I. \tag{2.4}$$

Knowing that S is WRC, equation (2.4) together with Facts 1 and 2, gives rise to $x \in Z(S)$ or $d(x) = 0$ for all $x \in I$. Let $x_0 \in I$ such that $d(x_0) = 0$, for all $x_1 \in Z(S)$ with $d(x_1) \neq 0$, we have $d(x_0+x_1) = d(x_0)+d(x_1) = d(x_1) \neq 0$. It turns out that $x_1+x_0 \in Z(S)$. Thus, we obtain $(x_0+x_1)z = z(x_0+x_1)$ for all $z \in S$. Which implies $x_0z+x_1z = zx_0+zx_1$ for all $z \in S$. Since $x_1 \in Z(S)$, we find $x_0z = zx_0$ for all $z \in S$ and therefore $x_0 \in Z(S)$. Subsquently, $I \subseteq Z(S)$ and Fact 2 implies that S is commutative. □

Every prime ring R is trivially a WRC zero-absorbing semiring. As an outcome of the Theorem 2.5, we get the following Corollary:

Corollary 2.6. ([3], Theorem 3) *Let R be a prime ring and U two-sided ideal of R. If R admits a nonzero derivation d such that $d(xy) = d(yx)$ for all $x, y \in U$, then R is commutative.*

Theorem 2.7. *Let S be a WRC semiring and I a nonzero ideal of S. If S admits a nonzero generalized derivation F with associated nonzero derivation d such that $F(xy)+yx = F(yx)+xy$ for all $x, y \in I$, then S is commutative.*

Proof. Given that

$$F(xy) + yx = F(yx) + xy \quad \text{for all} \ \ x, y \in I. \tag{2.5}$$

Replacing y by yx in (2.5) and using Proposition 2.3, we find that

$$xyd(x) + (F(xy) + yx)x = yxd(x) + (F(yx) + xy)x \quad \text{for all} \ \ x, y \in I. \tag{2.6}$$

In view of equation (2.5), we see that

$$xyd(x) = yxd(x) \quad \text{for all} \ \ x, y \in I. \tag{2.7}$$

By inserting zy for y in (2.7) and applying it, we obtain $xzyd(x) = zxyd(x)$ for all $x, y, z \in I$. With regard to the fact that S is WRC, the last expression used alongside Fact 1 leads to $x \in Z(S)$ or $d(x) = 0$ for all $x \in I$. Now, using the same arguments as used in the end of the proof of Theorem 2.5, we get the required result. □

An identical reasoning as above can be used to proof the following result.

Theorem 2.8. *Let S be a WRC semiring and I a nonzero ideal of S. If S admits a nonzero generalized derivation F with associated nonzero derivation d such that $F(xy)+xy = F(yx)+yx$ for all $x, y \in I$, then S is commutative.*

As a consequence of applying Theorem 2.7 and Theorem 2.8, we acquire the following corollaries:

Corollary 2.9. ([11], Theorem 2.1) *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.*

Corollary 2.10. ([11], Theorem 2.2) *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) + [x, y] = 0$ for all $x, y \in I$, then R is commutative.*

Theorem 2.11. *Let S be a WRC semiring and I a nonzero ideal of S. If S admits a nonzero generalized derivation F with associated nonzero derivation d such that $F(x)x = xF(x)$ for all $x \in I$, then S is commutative.*

Proof. By hypothesis, we have

$$F(x)x = xF(x) \quad \text{for all } \ x \in I. \tag{2.8}$$

Taking $x = x + y$ in equation (2.8) and applying it, we see that

$$F(x)y + F(y)x = xF(y) + yF(x) \quad \text{for all } \ x, y \in I. \tag{2.9}$$

Substituting yx for y in (2.9) and using the fact that $F(x)x = xF(x)$, one can acquire

$$F(x)yx + F(y)x^2 + yd(x)x = xF(y)x + xyd(x) + yF(x)x \quad \text{for all } \ x, y \in I. \tag{2.10}$$

When we right multiply (2.9) by x and compare it to (2.10), we are given

$$xF(y)x + yF(x)x + yd(x)x = xF(y)x + xyd(x) + yF(x)x \quad \text{for all } \ x, y \in I. \tag{2.11}$$

Which means

$$yF(x)x + yd(x)x = xyd(x) + yF(x)x \quad \text{for all } \ x, y \in I. \tag{2.12}$$

By replacing y with ry in equation (2.12), we arrive at

$$ryF(x)x + ryd(x)x = xryd(x) + ryF(x)x \quad \text{for all } \ x, y, r \in I. \tag{2.13}$$

Left multiplying (2.12) by r and merging with (2.13), one can obtain

$$rxyd(x) + ryF(x)x = xryd(x) + ryF(x)x \quad \text{for all } \; x, y, r \in I. \tag{2.14}$$

Hence

$$xryd(x) = rxyd(x) \quad \text{for all } \; x, y, r \in I. \tag{2.15}$$

Based on the fact that S is WRC, then equation (2.15) together with Fact 1 imply that $x \in Z(S)$ or $d(x) = 0$ for all $x \in I$. Following the same reasons as in the end of the proof of the Theorem 2.5, we obtain the required result. □

While every prime ring R is a WRC zero-absorbing semiring, the following Corollary is an application of Theorem 2.11.

Corollary 2.12. ([9], Theorem 3) *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal. If R admits a generalized derivation F such that $[F(u), u] = 0$ for all $u \in J$, then F is a left multiplier or R is commutative.*

Theorem 2.13. *Let S be a 2-torsion free WRC semiring and I a nonzero ideal of S. If S admits a nonzero generalized derivation F associated with a nonzero derivation d such that $F(x)d(y) = d(y)F(x)$ for all $x, y \in I$, then S is commutative.*

Proof. Make the assumption that

$$F(x)d(y) = d(y)F(x) \quad \text{for all } \; x, y \in I.$$

Writing yu instead of y in the preceding expression, we obtain

$$F(x)d(y)u+F(x)yd(u) = d(y)uF(x)+yF(x)d(u) \quad \text{for all } \; x, y, u \in I. \tag{2.16}$$

Putting $u = ud(z)$ in (2.16), we acquire

$$\begin{aligned} &F(x)d(y)ud(z) + F(x)yd(u)d(z) + F(x)yud^2(z) \\ &= d(y)uF(x)d(z) + yF(x)d(u)d(z) + yF(x)ud^2(z). \end{aligned} \tag{2.17}$$

Right multiplying (2.16) by $d(z)$, one can arrive to

$$F(x)d(y)ud(z) + F(x)yd(u)d(z) = d(y)uF(x)d(z) + yF(x)d(u)d(z). \tag{2.18}$$

Invoking (2.18),(2.17), yields

$$\begin{aligned} &F(x)d(y)ud(z) + F(x)yd(u)d(z) + F(x)yud^2(z) \\ &= F(x)d(y)ud(z) + F(x)yd(u)d(z) + yF(x)ud^2(z). \end{aligned} \tag{2.19}$$

Hence

$$F(x)yud^2(z) = yF(x)ud^2(z) \quad \text{for all } \; x, y, u, z \in I. \tag{2.20}$$

While S is WRC, equation (2.20) and Fact 1 ensure that $d^2(z) = 0$ or $F(x)y = yF(x)$ for all x, y, z in I. First we suppose that $d^2(z) = 0$ for all $z \in I$, then by Lemma 2.4, we get $d = 0$, a contradiction. Subsequently, $F(x)y = yF(x)$ for all $x, y \in I$, after taking $y = x$, we find $F(x)x = xF(x)$ for all $x \in I$. Accordingly, by Theorem 2.8 S is commutative. □

Any prime ring R is a WRC zero-absorbing semiring. Applying Theorem 2.13, we get the following Corollary:

Corollary 2.14. ([2], Theorem 2.6) *Let R be a prime ring. If R admits a nonzero generalized derivation F associated with a derivation d such that $[F(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative.*

3. The categorization of some generalized derivations

In the following section, we are going to offer an overview and categorization of several generalized derivations that satisfy specific algebraic characteristics.

Theorem 3.1. *Let S be a 2-torsion free WRC semiring and let I be a nonzero ideal of S. If S admits a generalized derivation F with associated derivation d such that $F(x \circ y) = 0$ for all $x, y \in I$, then $F = 0$.*

Proof. Consider there is a nonzero generalized derivation F such that $F(x \circ y) = 0$ for all $x, y \in I$, thus

$$F(xy) + F(yx) = 0 \quad \text{for all } \ x, y \in I. \tag{3.1}$$

Once, we replace y by yx in equation (3.1), we acquire

$$xyd(x) + F(xy)x + F(yx)x + yxd(x) = 0 \quad \text{for all } \ x, y \in I. \tag{3.2}$$

Applying (3.1), the previously equation gives

$$xyd(x) + yxd(x) = 0 \quad \text{for all } \ x, y \in I. \tag{3.3}$$

Substituting zy for y in (3.3), we obtain

$$xzyd(x) + zyxd(x) = 0 \quad \text{for all } \ x, y, z \in I. \tag{3.4}$$

Left multiplying equation (3.3) by z and comparing with (3.4), to get

$$xzyd(x) = zxyd(x) \quad \text{for all } \ x, y, z \in I \tag{3.5}$$

In light of Fact 1, equation (3.5) implies that $x \in Z(S)$ or $d(x) = 0$ for all $x \in I$. Through the same methods as in Theorem 2.5, we could see that S is commutative. In view of commutativity and equation (3.1), it follows that

$F(xy) + F(yx) = 2F(xy) = 0$ for all $x, y \in I$. By the 2-torsion freeness of S, we arrive at

$$F(xy) = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.6}$$

Changing y to yz allows us to attain

$$F(xy)z + xyd(z) = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{3.7}$$

Which leads to

$$xyd(z) = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.8}$$

Specifically, for any $y \in I$, $xyd(z) = 0yd(z)$. While S is WRC and $I \neq \{0\}$, the above expression gives that $d(z) = 0$ for any $z \in I$. Consequently, $d(I) = 0$ and Fact 3 assures that $d = 0$. On the other hand equation (3.6) becomes $F(x)y = 0$ for all $x, y \in I$. Now, replace y by yz and apply the commutativity of S to attain $yzF(x) = 0 = 0zF(x)$ for all $z \in I$. Given that $I \neq \{0\}$, by utilizing Fact 1, we achieve that $F(x) = 0$ for all $x \in I$. Consequently, $F(I) = (0)$ and Fact 4 forces that $F = 0$. □

Theorem 3.2. *Let S be a 2-torsion free WRC semiring and let I be a nonzero ideal of S. If S admits a generalized derivation F with associated derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then $F = id$.*

Proof. We are given that

$$F(x \circ y) = x \circ y \quad \text{for all} \ \ x, y \in I. \tag{3.9}$$

In equation (3.9), update y with yx and use the fact that $x \circ (yx) = (x \circ y)x$ to achieve

$$F(x \circ y)x + (x \circ y)d(x) = (x \circ y)x \quad \text{for all} \ \ x, y \in I. \tag{3.10}$$

Making use the last equation with equation (3.9), we obtain

$$(x \circ y)d(x) = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.11}$$

Hence

$$xyd(x) + yxd(x) = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.12}$$

While equation (3.12) is identical to equation (3.3), then reasoning as in the proof of Theorem 2.13, we could demonstrate that S is commutative and $d = 0$. In addition, equation (3.9) can be rewritten as

$$F(xy) + F(yx) = xy + yx$$

for all $x, y \in I$. Based on the commutativity of S, it follows that $2F(xy) = 2xy$ for all $x, y \in I$. While S is 2-torsion free and $d = 0$, we attain

$$F(x)y = xy \quad \text{for all } \ x, y \in I. \tag{3.13}$$

Modifying y by zy in (3.13), we acquire

$$F(x)zy = xzy \quad \text{for all } \ x, y, z \in I. \tag{3.14}$$

For the reason that $I \neq \{0\}$, equation (3.14) in combination with Fact 1 lead us to $F(x) = x$ for all $x \in I$, which assures that $F = id$. □

Theorem 3.3. *Let S be a 2-torsion free WRC semiring and let I be a nonzero ideal of S. If S admits a generalized derivation F with associated derivation d such that $F(x \circ y) + x \circ y = 0$ for all $x, y \in I$, then $F + id = 0$.*

Proof. Suppose that

$$F(x \circ y) + x \circ y = 0 \quad \text{for all } \ x, y \in I. \tag{3.15}$$

If we substitute y by yx in (3.15) and applying the fact that $x \circ (yx) = (x \circ y)x$, we are able to derive $(x \circ y)d(x) + (F(x \circ y) + x \circ y)x = 0$ for all $x, y \in I$. In view of equation (3.15) together with the previous expression, one could see that

$$xyd(x) + yxd(x) = 0 \quad \text{for all } \ x, y \in I. \tag{3.16}$$

Knowing that equation (3.16) is identical in form to equation (3.3), then following the same logic as in the proof of the theorem 2.13, we can draw that S is commutative and $d = 0$. Accordingly, equation (3.15) can be rewritten as $F(xy) + F(yx) + xy + yx = 0$ for all $x, y \in I$. Furthermore, commutativity of S shows that $2(F(xy) + xy) = 0$ for all $x, y \in I$. Because of 2-torsion freeness, we get $F(xy) + xy = 0$ and while $d = 0$, we are able to see that

$$F(x)y + xy = 0 \quad \text{for all } \ x, y \in I. \tag{3.17}$$

By changing y to zy in (3.17) as well as applying the commutativity of S, we achieve

$$yz(F(x) + x) = 0 = 0z(F(x) + x) \quad \text{for all } \ x, y, z \in I. \tag{3.18}$$

Given that $I \neq \{0\}$, equation (3.18) combined with Fact 1 gives $F(x) + x = 0$ for all $x \in I$. Consequently, $F + id = 0$. □

The following example demonstrate how in some cases our findings do not hold. In reality, we might conclude from the following example that the condition **WRC-semiring** must be fulfilled.

Example 3.4. Let

$$S = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} |a, b, c \in \mathbb{N} \right\},$$

where $\mathbb{N}$ is the set of positive integers including 0. If we define the maps F and d on the set S given by

$$F\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$d\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

F is a nonzero generalized derivation associated with d of S that satisfies the conditions of Theorems 2.5, 2.7, 2.8 and 2.13. But S is not commutative.

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COMMUTATIVITY FOR THE WEAKLY RIGHT CANCELLATIVE SEMIRINGS:

AN ENTIRELY NOVEL CATEGORY OF SEMIRINGS AND

A WEAK CONDITION FOR COMMUTATIVITY RESEARCH

K. CHARRABI, A. MAMOUNI AND B. NEJJAR

تعویض‌پذیری در نیم‌حلقه‌های ضعیفاً حذف‌پذیر از سمت راست: معرفی دسته‌ای جدید

از نیم‌حلقه‌ها و بررسی شرطی ضعیف برای تعویض‌پذیری

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هدف این مطالعه ارائه نوآوری در تحقیقات تعویض‌پذیری است به گونه‌ای که کمتر از شرط قوی حلقه اول باشد. در این مقاله، نیم‌حلقه‌های ضعیفاً حذف‌پذیر از سمت راست معرفی و بررسی می‌کنیم که تعویض‌پذیری و مشتق‌های تعمیم‌یافته چگونه در این کلاس از نیم‌حلقه‌ها اعمال می‌شوند. به علاوه، توضیحی دقیق و طبقه‌بندی برخی از این مشتق‌های تعمیم‌یافته را ارائه می‌دهیم.

کلمات کلیدی: نیم‌حلقه، نیم‌حلقه حذف‌پذیر، مشتق، مشتق تعمیم‌یافته.