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LAPLACIAN SPECTRUM AND ENERGY OF NON-COMMUTING GRAPHS OF FINITE RINGS

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Abstract. We compute spectrum, energy, Laplacian spectrum/energy and signless Laplacian spectrum/energy of non-commuting graphs of certain finite non-commutative rings. In particular, we consider finite rings R such that $|R| = p^2, p^3, p^4$, p^5, p^2q and p^3q, where p and q are primes. Further, we consider n-centralizer finite rings for $n = 4, 5$, and $p + 2$; more generally, finite rings with central quotients isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Our computations reveal that non-commuting graphs of these rings are L-integral. We also determine whether non-commuting graphs of these rings are integral, Q-integral, hyperenergetic, L-hyperenergetic or Q-hyperenergetic.

1. Introduction

An undirected graph Γ_R on the set of vertices $R \setminus Z(R)$, where R is a finite non-commutative ring and $Z(R)$ is the center of R, is called non-commuting graph of R if two vertices $u \neq v$ are adjacent whenever $uv \neq vu$. Erfanian, Khashyarmanesh and Nafar [6] began researching non-commuting graphs of finite rings. However, in case of finite groups, non-commuting graphs were considered by Neumann [12] in 1976 while answering a question raised by Erdös. Recent results on Γ_R can be found in [3]. In this article, we compute spectrum, energy, Laplacian spectrum, Laplacian energy, signless Laplacian spectrum and signless Laplacian energy of Γ_R for certain classes of finite rings. As a consequence of our results we determine whether Γ_R is integral, L-integral, Q-integral, hyperenergetic, L-hyperenergetic or Q-hyperenergetic for the rings considered in Section 3. Throughout the paper, $\frac{R}{Z(R)}$ denotes an additive quotient group and p, q denote distinct primes. It is worth mentioning that Laplacian spectrum and signless Laplacian spectrum of zero divisor graph of the ring $\mathbb{Z}_n$ for some n were computed in [15] and [13, 14] respectively.

Let $\mathcal{G}$ be a finite graph with $e(\mathcal{G})$ as the set of edges and $v(\mathcal{G})$ as the set of vertices. The spectrum of $\mathcal{G}$ denoted by Spec($\mathcal{G}$) is the set

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$$\{(\alpha_1)^{a_1}, (\alpha_2)^{a_2}, \ldots, (\alpha_n)^{a_n}\},$$

where α_i's represent the eigenvalues of $A(\mathcal{G})$ (adjacency matrix of $\mathcal{G}$) and a_i's are their multiplicities for $1 \leq i \leq n$. If α_i's are integers then $\mathcal{G}$ is said to be integral. We have

$$E(\mathcal{G}) := \sum_{i=1}^{n} a_i |\alpha_i|$$

which is the energy of $\mathcal{G}$. "If $E(\mathcal{G}) > E(K_{|v(\mathcal{G})|}) = 2(|v(\mathcal{G})| - 1)$ then $\mathcal{G}$ is called hyperenergetic". Gutman [8] and Walikar et al. [18] introduced this class of graphs in 1999.

Let $D(\mathcal{G})$ be the degree matrix of $\mathcal{G}$. Then the Laplacian matrix of $\mathcal{G}$ is given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. The Laplacian spectrum of $\mathcal{G}$ denoted by L-spec($\mathcal{G}$) is the set $\{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \ldots, (\beta_m)^{b_m}\}$ where β_j's are the eigenvalues of $L(\mathcal{G})$ and b_j's are their multiplicities for $1 \leq j \leq m$. If β_j's are integers then $\mathcal{G}$ is said to be L-integral. We have

$$LE(\mathcal{G}) := \sum_{j=1}^{m} b_j \left| \beta_j - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right|,$$

the Laplacian energy of $\mathcal{G}$. "If $LE(\mathcal{G}) > LE(K_{|v(\mathcal{G})|}) = 2(|v(\mathcal{G})| - 1)$, then $\mathcal{G}$ is called L-hyperenergetic".

Again, the signless Laplacian matrix of $\mathcal{G}$ is defined by

$$Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G}).$$

The signless Laplacian spectrum of $\mathcal{G}$ denoted by Q-spec($\mathcal{G}$) is the set $\{(\gamma_1)^{c_1}, (\gamma_2)^{c_2}, \ldots, (\gamma_l)^{c_l}\}$, where γ_k's are the eigenvalues of $Q(\mathcal{G})$ and c_k's are their multiplicities for $1 \leq k \leq l$. If γ_k's are integers then $\mathcal{G}$ is said to be Q-integral. We have

$$LE^+(\mathcal{G}) := \sum_{k=1}^{l} c_k \left| \gamma_k - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right|,$$

the signless Laplacian energy of $\mathcal{G}$. "If $LE^+(\mathcal{G}) > LE^+(K_{|v(\mathcal{G})|}) = 2(|v(\mathcal{G})|-1)$ then $\mathcal{G}$ is called Q-hyperenergetic". These classes of graphs were considered in [7].

2. Preliminary results

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two graphs such that $v(\mathcal{H}_1) \cap v(\mathcal{H}_2) = \emptyset$. Then $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ is the graph with

$$v(\mathcal{H}) = v(\mathcal{H}_1) \cup v(\mathcal{H}_2) \text{ and } e(\mathcal{H}) = e(\mathcal{H}_1) \cup e(\mathcal{H}_2).$$

Throughout this paper we write $mK_n = \underbrace{K_n \cup \cdots \cup K_n}_{m\text{-times}}$, where K_n is the complete graph and $|v(K_n)| = n$. Results on the complement of Γ_R were obtained in [4, 7, 11, 16, 17]. Among those, the following results will be used in subsequent sections.

Lemma 2.1. *Let R be a ring.*

(a) *If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $\overline{\Gamma_R} = (p+1)K_{(p-1)\eta}$, where $\eta = |Z(R)|$.* (See [4, Theorem 2.4].)

(b) *Let $|R| = p^2q$ and $Z(R) = \{0\}$.*

(i) *If "$t \in \{p, q, p^2, pq\}$ and $(t-1)$ divides (p^2q-1)", then $\overline{\Gamma_R} = \frac{p^2q-1}{t-1}K_{t-1}$ and* L-spec$(\overline{\Gamma_R}) = \left\{(0)^{\frac{p^2q-1}{t-1}}, (t-1)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$. (See [17, Theorem 2.9(i)] and [7, Theorem 2.5(a)].)

(ii) *If $(p-1)l_1 + (q-1)l_2 + (p^2-1)l_3 + (pq-1)l_4 = p^2q-1$, then*

$$\overline{\Gamma_R} = l_4K_{pq-1} \sqcup l_3K_{p^2-1} \sqcup l_2K_{q-1} \sqcup l_1K_{p-1}$$

and

$$\text{L-spec}(\overline{\Gamma_R}) = \Big\{(0)^{l_1+l_2+l_3+l_4}, (p-1)^{(p-2)l_1}, (q-1)^{(q-2)l_2}, (p^2-1)^{(p^2-2)l_3}, (pq-1)^{(pq-2)l_4}\Big\}.$$

(See [17, Theorem 2.9(ii)] and [7, Theorem 2.5(b)].)

Lemma 2.2. *Suppose that R is a ring with unity.*

(a) *Let $|R| = p^4$.*

(i) *If $Z(R)$ has p elements then $\overline{\Gamma_R} = (1+p+p^2)K_{p(p-1)}$ or $l_1K_{p(p-1)} \sqcup l_2K_{p(p^2-1)}$ and* L-spec$(\overline{\Gamma_R}) = \left\{(0)^{p^2+p+1}, (p^2-p)^{p^4-p^2-2p-1}\right\}$ *or*

$$\left\{(0)^{l_1+l_2}, (p^2-p)^{(p^2-p-1)l_1}, (p^3-p)^{(p^3-p-1)l_2}\right\}$$

respectively, where $p^2+p+1 = l_1 + (p+1)l_2$. (See [16, Theorem 2.5] and [7, Theorem 2.2(a)].)

(ii) *If $Z(R)$ has p^2 elements then $\overline{\Gamma_R} = (p+1)K_{(p^3-p^2)}$.* (See [16, Theorem 2.5] and [7, Theorem 2.2(b)].)

(b) *Let $|R| = p^5$ and $Z(R)$ is not a field.*

(i) *If $Z(R)$ has p^2 elements then $\overline{\Gamma_R} = (1+p+p^2)K_{p^2(p-1)}$ or $l_1K_{p^2(p-1)} \sqcup l_2K_{p^2(p^2-1)}$ and*

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^{p^2+p+1}, (p^3-p^2)^{p^5-2p^2-p-1}\right\}$$

or

$$\left\{(0)^{l_1+l_2}, (p^3-p^2)^{(p^3-p^2-1)l_1}, (p^4-p^2)^{(p^4-p^2-1)l_2}\right\}$$

respectively, where $p^2+p+1 = l_1+(p+1)l_2$*.* (See [16, Theorem 2.7] and [7, Theorem 2.3(a)].)

(ii) *If* $Z(R)$ *has* p^3 *elements then* $\overline{\Gamma_R} = (p+1)K_{(p^4-p^3)}$. (See [16, Theorem 2.7] and [7, Theorem 2.3(b)].)

(c) *If* $|R| = p^3q$ *and* $Z(R)$ *has* pq *elements, then* $\overline{\Gamma_R} = (p+1)K_{p^2q-pq}$ *and*

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^{p+1}, (p^2q-pq)^{(p+1)(p^2q-pq-1)}\right\}.$$

(See [17, Theorem 2.12(iv)] and [7, Theorem 2.6].)

(d) *Let* $|R| = p^3q$ *and* $Z(R)$ *has* p^2 *elements. Then*

(i) $\overline{\Gamma_R} = \frac{pq-1}{p-1}K_{p^3-p^2}$ *and*

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^{\frac{pq-1}{p-1}}, (p^3-p^2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\right\}$$

whenever $(p-1)$ *divides* $(pq-1)$*.* (See [17, Theorem 2.12(i)] and [7, Theorem 2.7(a)].)

(ii) $\overline{\Gamma_R} = \frac{pq-1}{q-1}K_{p^2q-p^2}$ *and*

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^{\frac{pq-1}{q-1}}, (p^2q-p^2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\right\}$$

whenever $(q-1)$ *divides* $(pq-1)$*.* (See [17, Theorem 2.12(ii)] and [7, Theorem 2.7(b)].)

(iii) $\overline{\Gamma_R} = l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}$ *and*

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^{l_1+l_2}, (p^3-p^2)^{l_1(p^3-p^2-1)}, (p^2q-p^2)^{l_2(p^2q-p^2-1)}\right\}$$

whenever $pq-1 = (p-1)l_1+(q-1)l_2$*.* (See [17, Theorem 2.12(iii)] and [7, Theorem 2.7(c)].)

A class of non-commutative rings R, introduced by Erfanian et al. in [6], is referred to as CC-ring if the centralizers $C_R(y)$ are commutative whenever $y \in R \setminus Z(R)$.

Lemma 2.3 ([4, page 3])**.** *If* $S_1, S_2, \dots, S_n$ *are the non-identical centralizers of* $s \in R \setminus Z(R)$*, where* R *is a finite CC-ring and* $|Z(R)| = \eta$*, then* $\overline{\Gamma_R} = \bigsqcup\limits_{i=1}^{n} K_{|S_i|-\eta}$ *and*

$$\text{L-spec}(\overline{\Gamma_R}) = \{(0)^n, (|S_1|-\eta)^{|S_1|-\eta-1}, \dots, (|S_n|-\eta)^{|S_n|-\eta-1}\}.$$

3. Various spectrum and energies

The following results will help us to compute spectrum, Laplacian spectrum and signless Laplacian spectrum of complete r-partite graph in the succeeding sections.

Theorem 3.1 ([19, Corollary 2.3] and [20, Corollary 2.2])**.** *Let $\mathcal{G}$ be a complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1.p_1,a_2.p_2,\ldots,a_s.p_s}$ on n vertices. Then*

(a) *the characteristics polynomial of $A(\mathcal{G})$ is*

$$P_{\mathcal{G}}(x) := x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}\left(\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s}a_jp_j\prod_{i=1, i\neq j}^{s}(x+p_i)\right).$$

(b) *the characteristics polynomial of $Q(\mathcal{G})$ (Q-polynomial) is*

$$Q_{\mathcal{G}}(x) := \prod_{i=1}^{s}(x-n+p_i)^{a_i(p_i-1)}\prod_{i=1}^{s}(x-n+2p_i)^{a_i}\left(1-\sum_{i=1}^{s}\frac{a_ip_i}{x-n+2p_i}\right).$$

The following well-known result gives the spectrum of a strongly regular graph.

Theorem 3.2. *Let $\mathcal{H}$ be a strongly regular graph with parameters (k, λ, μ), then*

$$\operatorname{Spec}(\mathcal{H}) = \left\{(k)^1, \left(\tfrac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}\right)^{n_1}, \left(\tfrac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}\right)^{n_2}\right\},$$

where $n_1 = \frac{1}{2}\left(|v(\mathcal{H})| - 1 + \frac{2k+(|v(\mathcal{H})|-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$ and

$$n_2 = \tfrac{1}{2}\left(|v(\mathcal{H})| - 1 - \tfrac{2k+(|v(\mathcal{H})|-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right).$$

Theorem 3.3. [10, Theorem 3.6] *For any graph $\mathcal{H}$, if*

$$\text{L-spec}(\mathcal{H}) = \left\{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \ldots, (\beta_m)^{b_m}\right\}$$

where $\beta_1 < \beta_2 < \cdots < \beta_m$, then

$$\begin{aligned}\text{L-spec}(\overline{\mathcal{H}}) = \Big\{&(0)^1, (|v(\mathcal{H})| - \beta_m)^{b_m}, (|v(\mathcal{H})| - \beta_{m-1})^{b_{m-1}},\\ &(|v(\mathcal{H})| - \beta_{m-2})^{b_{m-2}}, \ldots, (|v(\mathcal{H})| - \beta_1)^{b_1-1}\Big\}.\end{aligned}$$

We arrive at the following conclusion as a result of Theorems 3.2–3.3.

Theorem 3.4. *If $\mathcal{H} = \overline{mK_n}$, then*

$$\text{Spec}(\mathcal{H}) = \left\{(-n)^{m-1}, (0)^{(n-1)m}, ((m-1)n)^1\right\},$$
$$\text{L-spec}(\mathcal{H}) = \left\{(0)^1, ((m-1)n)^{m(n-1)}, (mn)^{m-1}\right\}$$

and $E(\mathcal{H}) = LE(\mathcal{H}) = 2n(m-1)$.

Proof. If $\mathcal{H} = \overline{mK_n}$ then it is a strongly regular graph where $|v(\mathcal{H})| = mn$, $k = (m-1)n = \mu, \lambda = (m-2)n$. We have $\lambda - \mu = -n$, $k - \mu = 0$ and so $\sqrt{(\lambda-\mu)^2 + 4(k-\mu)} = n$. Therefore,

$$\tfrac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} = -n, \tfrac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} = 0$$

and

$$\tfrac{2k+(|v(\mathcal{H})|-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}} = 2m - mn - 1.$$

Hence, $\text{Spec}(\mathcal{H}) = \left\{(-n)^{m-1}, (0)^{(n-1)m}, ((m-1)n)^1\right\}$ (by Theorem 3.2) and so $E(\mathcal{H}) = (m-1)|-n| + 0 + n|m-1| = 2n(m-1)$.

Since $\text{L-spec}(\overline{\mathcal{H}}) = \left\{(0)^m, (n)^{m(n-1)}\right\}$, by Theorem 3.3 it follows that

$$\text{L-spec}(\mathcal{H}) = \left\{(0)^1, ((m-1)n)^{m(n-1)}, (mn)^{m-1}\right\}.$$

We have $\frac{2|e(\mathcal{H})|}{|v(\mathcal{H})|} = \frac{2}{mn}\left(\frac{mn(mn-1)}{2} - \frac{mn(n-1)}{2}\right) = (m-1)n$. If follows that $|0-n(m-1)| = n(m-1)$, $|n(m-1)-n(m-1)| = 0$ and $|mn-(m-1)n| = n$. Hence, $LE(\mathcal{H}) = n(m-1) + 0 + n(m-1) = 2n(m-1)$. □

Theorem 3.5. *If* $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *and* $|Z(R)| = \eta$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-\eta(p-1))^p, (0)^{\eta(p^2-1)-p-1}, (\eta p(p-1))^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (\eta p(p-1))^{\eta(p^2-1)-p-1}, (\eta(p^2-1))^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta$.

Proof. By Lemma 2.1(a), we have $\overline{\Gamma_R} = (p+1)K_{(p-1)\eta}$. This implies $|v(\Gamma_R)| = (p^2-1)\eta$ and $\Gamma_R = K_{p+1.(p-1)\eta}$. Therefore, Theorem 3.4 leads to the expression of $\text{Spec}(\Gamma_R)$, $\text{L-spec}(\Gamma_R)$, $E(\Gamma_R)$ and $LE(\Gamma_R)$. Using Theorem 3.1(b), we have

$$Q_{\Gamma_R}(x) = (x - (p^2-1)\eta + (p-1)\eta)^{(p+1)((p-1)\eta-1)}$$

$$\times\,(x-(p^2-1)\eta+2(p-1)\eta)^{p+1}\left(1-\frac{(p^2-1)\eta}{x-(p^2-1)\eta+2(p-1)\eta}\right)$$
$$=(x-p(p-1)\eta)^{(p^2-1)\eta-p-1}(x-(p-1)^2\eta)^p(x-2p(p-1)\eta).$$

Thus,

$$\text{Q-spec}(\Gamma_R)=\left\{((p^2-p)\eta)^{(p^2-1)\eta-p-1},((p-1)^2\eta)^p,((2p^2-2p)\eta)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p-1)(p^2-1)\eta^2-(p^2-1)\eta}{2}$. Therefore,

$$|e(\Gamma_R)|=\tfrac{(p^2-1)^2\eta^2-(p^2-1)\eta}{2}-\tfrac{(p-1)(p^2-1)\eta^2-(p^2-1)\eta}{2}=\tfrac{p(p-1)(p^2-1)\eta^2}{2}.$$

Now,

$$\left|p(p-1)\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=\left|p(p-1)\eta-(p^2-p)\eta\right|=0,$$

$$\left|(p-1)^2\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=|(1-p)\eta|=(p-1)\eta$$

and

$$\left|2p(p-1)\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=\left|(p^2-p)\eta\right|=(p^2-p)\eta.$$

Therefore,

$$LE^+(\Gamma_R)=((p^2-1)\eta-p-1)\times 0+p\times(p-1)\eta+(p^2-p)\eta=2p(p-1)\eta$$

□

The following results give various spectra and energies of Γ_R for some n-centralizer rings (see [1, 2, 5]).

Theorem 3.6. *Let R be a finite n-centralizer ring and $|Z(R)|=\eta$.*

(a) *If $n=4$, then*

$$\text{Spec}(\Gamma_R)=\left\{(-\eta)^2,(0)^{3\eta-3},(2\eta)^1\right\},$$
$$\text{Q-spec}(\Gamma_R)=\left\{(2\eta)^{3\eta-3},(\eta)^2,(4\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R)=\left\{(0)^1,(2\eta)^{3\eta-3},(3\eta)^2\right\}$$

and $E(\Gamma_R)=LE^+(\Gamma_R)=LE(\Gamma_R)=4\eta$.

(b) *If* $n = 5$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-2\eta)^3, (0)^{8\eta-4}, (6\eta)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(6\eta)^{8\eta-4}, (4\eta)^3, (12\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (6\eta)^{8\eta-4}, (8\eta)^3\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 12\eta$.

(c) *If* $n = p+2$ *and* $|R| = p^k$ *for* $k \in \mathbb{N}$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-(p-1)\eta)^p, (0)^{(p^2-1)\eta-p-1}, ((p^2-p)\eta)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (\eta\,(p^2-p))^{\eta\,(p^2-1)\,-p\,-1}, (\eta\,(p^2-1))^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta$.

Proof. If $n = 4$, then $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (conf. [2, Theorem 3.2]) and if $n = 5$, then $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ (conf. [2, Theorem 4.3]). Further, if $|R| = p^k$ and $n = p+2$, then $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (conf. [2, Theorem 2.12]). Therefore, Theorem 3.5 leads to the conclusion. □

Theorem 3.7. *Let* $\Pr(R)$ *be the commuting probability of* R *and* $|Z(R)| = \eta$.

(a) *If* $\Pr(R) = \frac{5}{8}$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-\eta)^2, (0)^{3\eta-3}, (2\eta)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(2\eta)^{3\eta-3}, (\eta)^2, (4\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (2\eta)^{3\eta-3}, (3\eta)^2\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 4\eta$.

(b) *If* $\Pr(R) = \frac{p^2+p-1}{p^3}$, *where* p *is the smallest prime divisor of* $|R|$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-\eta(p-1))^p, (0)^{\eta(p^2-1)-p-1}, (\eta p(p-1))^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (\eta p\,(p-1))^{\eta\,(p^2-1)\,-p\,-1}, (\eta\,(p^2-1))^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta$.

Proof. We have $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ if $\Pr(R) = \frac{5}{8}$ and $\frac{p^2+p-1}{p^3}$ (see Theorems 1 and 3 of [9], in the second case p is assumed to be the smallest prime divisor of $|R|$). Therefore, Theorem 3.5 leads to the conclusion. □

Theorem 3.8. *Let R be a non-commutative ring.*

(a) *If $|R| = p^2$, then*

$$\operatorname{Spec}(\Gamma_R) = \left\{(-(p-1))^p, \quad (0)^{p^2-p-2}, (p(p-1))^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(p^2-p)^{p^2-p-2}, \ ((p-1)^2)^p, (2p^2-2p)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p(p-1))^{p^2-p-2}, (p^2-1)^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)$.

(b) *If $|R| = p^3$ with unity, then*

$$\operatorname{Spec}(\Gamma_R) = \left\{(p-p^2)^p, (0)^{p^3-2p-1}, (p^3-p^2)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(p^3-p^2)^{p^3-2p-1}, (p(p-1)^2)^p, (2p^3-2p^2)^1\right\}$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^3-p^2)^{p^3-2p-1}, (p^3-p)^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(p-1)$.

Proof. Note that $|Z(R)| = 1$ if $|R| = p^2$ and $|Z(R)| = p$ if $|R| = p^3$ with unity respectively. In both the cases $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, Theorem 3.5 leads to the conclusion. □

Now we compute various spectra and energies of Γ_R for higher order finite non-commutative ring.

Theorem 3.9. *Let $|R| = p^4$ with unity.*

(a) *Suppose that $Z(R)$ has p elements. Then*

$$\operatorname{Spec}(\Gamma_R) = \left\{(-p(p-1))^{p^2+p}, (0)^{p^4-p^2-2p-1}, (p^4-p^2)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(p^4-p^2)^{p^4-p^2-2p-1}, (p^4-2p^2+p)^{p^2+p}, (2p^4-2p^2)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^4-p^2)^{p^4-p^2-2p-1}, (p^4-p)^{p^2+p}\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(p^2-1)$ or

$$\operatorname{Spec}(\Gamma_R) = \left\{(0)^{p^4-p-l_1-l_2}, (p-p^2)^{l_1-1}, (p-p^3)^{l_2-1}, (x_1)^1, (x_2)^1\right\},$$

where x_1, x_2 *are the roots of the polynomial*

$$x^2 - x(p^4 - p^3 - p^2 + p) - (p^5 - p^4 - p^3 + p^2)(l_1 + l_2 - 1),$$

$$\text{Q-spec}(\Gamma_R) = \Big\{(p^4 - p^2)^{l_1(p^2-p-1)}, (p^4 - p^3)^{l_2(p^3-p-1)}, \\ (p^4 - 2p^2 + p)^{l_1-1}, (p^4 - 2p^3 + p)^{l_2-1}, (x_1)^1, (x_2)^1\Big\},$$

where x_1, x_2 *are the roots of the polynomial*

$$x^2 - x(3p^4 - 2p^3 - 2p^2 + p) + p^8 - 2p^7 - 2p^6 + 6p^5 - 2p^4 - 2p^3 \\ + p^2 + l_1(p^2 - p)(p^4 - 2p^3 + p) + l_2(p^3 - p)(p^4 - 2p^2 + p),$$

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1, (p^4 - p^3)^{l_2(p^3-p-1)}, (p^4 - p^2)^{l_1(p^2-p-1)}, (p^4 - p)^{l_1+l_2-1}\Big\}$$

and $LE(\Gamma_R) = \frac{2}{p^2+p+1}\Big((p^4 - p^2)(l_1 + pl_2) + (p^6 - p^5 - p^4 + p^2)l_1 l_2\Big)$, *where* $l_1 + l_2(p+1) = p^2 + p + 1$.

(b) *Suppose that* $Z(R)$ *has* p^2 *elements. Then*

$$\text{Spec}(\Gamma_R) = \Big\{(-p^3 + p^2)^p, (0)^{p^4-p^2-p-1}, (p^4 - p^3)^1\Big\},$$

$$\text{Q-spec}(\Gamma_R) = \Big\{(p^4 - p^3)^{p^4-p^2-p-1}, (p^4 - 2p^3 + p^2)^p, (2p^4 - 2p^3)^1\Big\},$$

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1, (p^4 - p^3)^{p^4-p^2-p-1}, (p^4 - p^2)^p\Big\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p^3 - p^2)$.

Proof. (a) Lemma 2.2(a)(i) gives $\overline{\Gamma_R} = (1 + p + p^2)K_{(p^2-p)}$ or $l_1 K_{(p^2-p)} \cup l_2 K_{(p^3-p)}$, when $l_1 + l_2(p+1) = p^2 + p + 1$.

If $\overline{\Gamma_R} = (p^2 + p + 1)K_{(p^2-p)}$ then by Theorem 3.4 we have

$$\text{Spec}(\Gamma_R) = \Big\{(-p^2 + p)^{p^2+p}, (0)^{p^4-p^2-2p-1}, (p^4 - p^2)^1\Big\},$$

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1, (p^4 - p^2)^{p^4-p^2-2p-1}, (p^4 - p)^{p^2+p}\Big\}$$

and $E(\Gamma_R) = LE(\Gamma_R) = 2p^2(p^2 - 1)$. Here, $|v(\Gamma_R)| = p^4 - p$ and $\Gamma_R = K_{p^2+p+1.p^2-p}$. Using Theorem 3.1(b), we have

$$Q_{\Gamma_R}(x) = (x - (p^4 - p) + p^2 - p)^{(p^2+p+1)(p^2-p-1)} \\ \times (x - (p^4 - p) + 2(p^2 - p))^{p^2+p+1}\left(1 - \frac{p^4 - p}{x - (p^4 - p) + 2(p^2 - p)}\right) \\ = (x - (p^4 - p^2))^{p^4-p^2-2p-1}(x - (p^4 - 2p^2 + p))^{p^2+p}(x - (2p^4 - 2p^2)).$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^4 - p^2)^{p^4-p^2-2p-1}, (p^4 - 2p^2 + p)^{p^2+p}, \, (2p^4 - 2p^2)^1\right\}.$$

Number of edges of Γ_R^c is $\frac{p^6-p^5-p^4-p^3+p^2+p}{2}$. Therefore,

$$|e(\Gamma_R)| = \tfrac{p^8-2p^5-p^4+p^2+p}{2} - \tfrac{p^6-p^5-p^4-p^3+p^2+p}{2} = \tfrac{p^3(p^3-1)(p^2-1)}{2}.$$

Now,

$$\left|p^4 - p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^4 - p^2 - p^4 + p^2\right| = 0,$$
$$\left|p^4 - 2p^2 + p - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^2 + p\right| = p^2 - p$$

and $\left|2p^4 - 2p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^4 - p^2\right| = p^4 - p^2$. Thus,

$$\begin{aligned} LE^+(\Gamma_R) &= (p^4 - p^2 - 2p - 1) \times 0 + (p^2 + p) \times (p^2 - p) + p^4 - p^2 \\ &= 2p^2(p^2 - 1). \end{aligned}$$

If $\overline{\Gamma_R} = l_1 K_{(p^2-p)} \cup l_2 K_{(p^3-p)}$, then $\Gamma_R = K_{l_1.p^2-p, l_2.p^3-p}$. This implies $|v(\Gamma_R)| = p^4 - p$.

Using Theorem 3.1(a), we have

$$\begin{aligned} &P_{\Gamma_R}(x) \\ &= x^{(p^4-p)-(l_1+l_2)} \prod_{i=1}^{2} (x + p_i)^{a_i - 1} \left(\prod_{i=1}^{2} (x + p_i) - \sum_{j=1}^{2} a_j p_j \prod_{i=1, i\neq j}^{2} (x + p_i) \right) \\ &= x^{p^4-p-l_1-l_2} (x + p^2 - p)^{l_1-1} (x + p^3 - p)^{l_2-1} \\ &\times ((x + p^2 - p)(x + p^3 - p) - l_1(p^2 - p)(x + p^3 - p) \\ &- l_2(p^3 - p)(x + p^2 - p)) \\ &= x^{p^4-p-l_1-l_2} (x - (p - p^2))^{l_1-1} (x - (p - p^3))^{l_2-1} \\ &\times (x^2 - x(p^4 - p^3 - p^2 + p) - (p^5 - p^4 - p^3 + p^2)(l_1 + l_2 - 1)). \end{aligned}$$

Thus, $\text{Spec}(\Gamma_R) = \left\{(0)^{p^4-p-l_1-l_2}, (p - p^2)^{l_1-1}, (p - p^3)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, where x_1, x_2 are the roots of the polynomial

$$x^2 - x(p^4 - p^3 - p^2 + p) - (p^5 - p^4 - p^3 + p^2)(l_1 + l_2 - 1).$$

Using Theorem 3.1(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) &= \prod_{i=1}^{2}(x-(p^4-p)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(p^4-p)+2p_i)^{a_i}\\
&\quad\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(p^4-p)+2p_i}\right)\\
&= (x-(p^4-p)+p^2-p)^{l_1(p^2-p-1)}(x-(p^4-p)+p^3-p)^{l_2(p^3-p-1)}\\
&\quad\times (x-(p^4-p)+2p^2-2p)^{l_1}(x-(p^4-p)+2p^3-2p)^{l_2}\\
&\quad\times\left(1-\frac{l_1(p^2-p)}{x-(p^4-p)+2p^2-2p}-\frac{l_2(p^3-p)}{x-(p^4-p)+2p^3-2p}\right)\\
&= (x-(p^4-p^2))^{l_1(p^2-p-1)}(x-(p^4-p^3))^{l_2(p^3-p-1)}\\
&\quad\times (x-(p^4-2p^2+p))^{l_1}(x-(p^4-2p^3+p))^{l_2}\\
&\quad\times\left(1-\frac{l_1(p^2-p)}{x-(p^4-2p^2+p)}-\frac{l_2(p^3-p)}{x-(p^4-2p^3+p)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q_{\Gamma_R}(x) &= (x-(p^4-p^2))^{l_1(p^2-p-1)}(x-(p^4-p^3))^{l_2(p^3-p-1)}\\
&\quad (x-(p^4-2p^2+p))^{l_1-1}(x-(p^4-2p^3+p))^{l_2-1}f(x),
\end{aligned}$$

where

$$\begin{aligned}
f(x) &= (x^2-x(3p^4-2p^3-2p^2+p)+p^8-2p^7-2p^6+6p^5-2p^4-2p^3\\
&\quad +p^2+l_1(p^2-p)(p^4-2p^3+p)+l_2(p^3-p)(p^4-2p^2+p))
\end{aligned}$$

Thus

$$\begin{aligned}
\text{Q-spec}(\Gamma_R) &= \Big\{(p^4-p^2)^{l_1(p^2-p-1)},(p^4-p^3)^{l_2(p^3-p-1)},\\
&\qquad (p^4-2p^2+p)^{l_1-1},(p^4-2p^3+p)^{l_2-1},(x_1)^1,(x_2)^1\Big\},
\end{aligned}$$

where x_1,x_2 are the roots of the polynomial $f(x)$.

Using Lemma 2.2(a)(i), we have

$$\text{L-spec}(\overline{\Gamma_R}) = \Big\{(0)^{l_1+l_2},\ (p^2-p)^{l_1(p^2-p-1)},(p^3-p)^{l_2(p^3-p-1)}\Big\}.$$

Thus, Theorem 3.3 yields

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1,(p^4-p^3)^{l_2(p^3-p-1)},\ (p^4-p^2)^{l_1(p^2-p-1)},(p^4-p)^{l_1+l_2-1}\Big\}.$$

Here $|v(\Gamma_R)| = p^4 - p$, $|e(\Gamma_R)| = \frac{p^3(p-1)(p^2-1)(l_1+pl_2)}{2}$ and so

$$\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} = \frac{(p^4-p^2)(l_1+pl_2)}{p^2+p+1}.$$

Therefore

$$\left|p^4 - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{(p^3-p^2)l_1}{p^2+p+1},$$

$$\left|p^4 - p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{(p^4-p^2)l_2}{p^2+p+1},$$

$$\left|p^4 - p - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{(p^2-p)(l_1+(p+1)^2l_2)}{p^2+p+1}$$

and so

$$\begin{aligned} LE(\Gamma_R) &= \frac{(p^4-p^2)(l_1+pl_2)}{p^2+p+1} + \frac{(p^3-p^2)(p^3-p-1)l_1l_2}{p^2+p+1} \\ &\quad + \frac{(p^4-p^2)(p^2-p-1)l_1l_2}{p^2+p+1} + \frac{(p^2-p)(l_1+(p+1)^2l_2)(l_1+l_2-1)}{p^2+p+1}. \end{aligned}$$

Consequently, we obtain the necessary expression.

(b) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.2(a)(ii) and Theorem 3.4. By Lemma 2.2(a)(ii), we have $\overline{\Gamma_R} = (p+1)K_{(p^3-p^2)}$. This implies $|v(\Gamma_R)| = p^4 - p^2$ and $\Gamma_R = K_{p+1.p^3-p^2}$. Using Theorem 3.1(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) &= (x-(p^4-p^2)+p^3-p^2)^{(p+1)(p^3-p^2-1)} \\ &\quad \times (x-(p^4-p^2)+2(p^3-p^2))^{p+1}\left(1 - \frac{p^4-p^2}{x-(p^4-p^2)+2(p^3-p^2)}\right) \\ &= (x-(p^4-p^3))^{p^4-p^2-p-1}(x-(p^4-2p^3+p^2))^p(x-(2p^4-2p^3)). \end{aligned}$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^4-p^3)^{p^4-p^2-p-1}, (p^4-2p^3+p^2)^p, (2p^4-2p^3)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p+1)(p^3-p^2)(p^3-p^2-1)}{2}$. Thus

$$|e(\Gamma_R)| = \frac{(p^4-p^2)(p^4-p^2-1)}{2} - \frac{(p+1)(p^3-p^2)(p^3-p^2-1)}{2} = \frac{p^5(p^2-1)(p-1)}{2}.$$

Now

$$\left|p^4 - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^4 - p^3 - p^4 + p^3\right| = 0,$$

$$\left|p^4 - 2p^3 + p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^3 + p^2\right| = p^3 - p^2$$

and

$$\left|2p^4 - 2p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^4 - p^3\right| = p^4 - p^3.$$

Thus, $LE^+(\Gamma_R) = (p^4-p^2-p-1)\times 0 + p\times(p^3-p^2)+p^4-p^3 = 2(p^4-p^3)$. □

Theorem 3.10. *Let $|R| = p^5$ with unity and $Z(R)$ is not a field.*

(a) *Suppose that $Z(R)$ has p^2 elements. Then*

$$\text{Spec}(\Gamma_R) = \left\{(-p^3+p^2)^{p^2+p}, (0)^{p^5-2p^2-p-1}, (p^5-p^3)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(p^5-p^3)^{p^5-2p^2-p-1}, (p^5-2p^3+p^2)^{p^2+p}, (2p^5-2p^3)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^5-p^3)^{p^5-2p^2-p-1}, (p^5-p^2)^{p^2+p}\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^5-p^3)$ or

$$\text{Spec}(\Gamma_R) = \left\{(0)^{p^5-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^4)^{l_2-1}, (x_1)^1, (x_2)^1\right\},$$

where x_1, x_2 are the roots of the polynomial

$$x^2 - x(p^5-p^4-p^3+p^2) - (p^7-p^6-p^5+p^4)(l_1+l_2-1),$$

$$\text{Q-spec}(\Gamma_R) = \Big\{(p^5-p^3)^{l_1(p^3-p^2-1)}, (p^5-p^4)^{l_2(p^4-p^2-1)}, (p^5-2p^3+p^2)^{l_1-1},$$
$$(p^5-2p^4+p^2)^{l_2-1}, (x_1)^1, (x_2)^1\Big\},$$

where x_1, x_2 are the roots of the polynomial

$$x^2 - x(3p^5-2p^4-2p^3+p^2) + p^{10} - 2p^9 - 2p^8 + 6p^7 - 2p^6 - 2p^5 + p^4$$
$$+ l_1(p^3-p^2)(p^5-2p^4+p^2) + l_2(p^4-p^2)(p^5-2p^3+p^2),$$

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1, (p^5-p^4)^{l_2(p^4-p^2-1)}, (p^5-p^3)^{l_1(p^3-p^2-1)},$$
$$(p^5-p^2)^{l_1+l_2-1}\Big\}$$

and

$$LE(\Gamma_R) = \tfrac{2[(p^5-p^3)(l_1+pl_2)+(p^8-p^7-p^6+p^5-p^4+p^3)l_1l_2]}{p^2+p+1},$$

where $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^3$, *then*

$$\text{Spec}(\Gamma_R) = \left\{(-p^4+p^3)^p, (0)^{p^5-p^3-p-1}, (p^5-p^4)^1\right\},$$
$$\text{Q-spec}(\Gamma_R) = \left\{(p^5-p^4)^{p^5-p^3-p-1}, (p^5-2p^4+p^3)^p, (2p^5-2p^4)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^5-p^4)^{p^5-p^3-p-1}, (p^5-p^3)^p\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^5-p^4)$.

Proof. (a) By Lemma 2.2(b)(i), $\overline{\Gamma_R} = (p^2+p+1)K_{p^2(p-1)}$ or $l_1K_{p^2(p-1)} \cup l_2K_{p^2(p^2-1)}$, when $l_1 + l_2(p+1) = p^2+p+1$, if $|Z(R)| = p^2$.

If $\overline{\Gamma_R} = (p^2+p+1)K_{(p^3-p^2)}$ then Theorem 3.4 yields

$$\text{Spec}(\Gamma_R) = \left\{(-p^3+p^2)^{p^2+p}, (0)^{p^5-2p^2-p-1}, (p^5-p^3)^1\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^5-p^3)^{p^5-2p^2-p-1}, (p^5-p^2)^{p^2+p}\right\}$$

and $E(\Gamma_R) = LE(\Gamma_R) = 2(p^5-p^3)$. Here, $|v(\Gamma_R)| = p^5-p^2$ and $\Gamma_R = K_{p^2+p+1.p^3-p^2}$. Using Theorem 3.1(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) &= (x-(p^5-p^2)+p^3-p^2)^{(p^2+p+1)(p^2-p-1)} \\
&\quad \times (x-(p^5-p^2)+2(p^3-p^2))^{p^2+p+1} \\
&\quad \times \left(1 - \frac{p^5-p^2}{x-(p^5-p^2)+2(p^3-p^2)}\right) \\
&= (x-(p^5-p^3))^{p^5-2p^2-p-1}(x-(p^5-2p^3+p^2))^{p^2+p} \\
&\quad \times (x-(2p^5-2p^3)).
\end{aligned}$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^5-p^3)^{p^5-2p^2-p-1}, (p^5-2p^3+p^2)^{p^2+p}, (2p^5-2p^3)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^8-p^7-2p^5+p^4+p^2}{2}$. Therefore

$$|e(\Gamma_R)| = \tfrac{p^{10}-2p^7-p^5+p^4+p^2}{2} - \tfrac{p^8-p^7-2p^5+p^4+p^2}{2} = \tfrac{p^5(p^3-1)(p^2-1)}{2}.$$

Now,

$$\left|p^5-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5-p^3-p^5+p^3\right| = 0,$$
$$\left|p^5-2p^3+p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^3+p^2\right| = p^3-p^2$$

and

$$\left|2p^5 - 2p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5 - p^3\right| = p^5 - p^3.$$

Thus,

$$\begin{aligned} LE^+(\Gamma_R) &= (p^5 - 2p^2 - p - 1) \times 0 + (p^2 + p) \times (p^3 - p^2) + p^5 - p^3 \\ &= 2(p^5 - p^3) \end{aligned}$$

If $\overline{\Gamma_R} = l_1 K_{(p^3-p^2)} \cup l_2 K_{(p^4-p^2)}$, then $\Gamma_R = K_{l_1.p^3-p^2, l_2.p^4-p^2}$. This implies $|v(\Gamma_R)| = p^5 - p^2$.

Using Theorem 3.1(a), we have

$$\begin{aligned} P_{\Gamma_R}(x) &= x^{(p^5-p^2)-(l_1+l_2)} \prod_{i=1}^{2} (x + p_i)^{a_i - 1} \\ &\times \left(\prod_{i=1}^{2} (x + p_i) - \sum_{j=1}^{2} a_j p_j \prod_{i=1, i \neq j}^{2} (x + p_i) \right) \\ &= x^{p^5-p^2-l_1-l_2} (x + p^3 - p^2)^{l_1 - 1} (x + p^4 - p^2)^{l_2 - 1} \\ &\times ((x + p^3 - p^2)(x + p^4 - p^2) - l_1(p^3 - p^2)(x + p^4 - p^2) \\ &- l_2(p^4 - p^2)(x + p^3 - p^2)) \\ &= x^{p^5-p^2-l_1-l_2} (x - (p^2 - p^3))^{l_1 - 1} (x - (p^2 - p^4))^{l_2 - 1} \\ &\times (x^2 - x(p^5 - p^4 - p^3 + p^2) - (p^7 - p^6 - p^5 + p^4)(l_1 + l_2 - 1)). \end{aligned}$$

Thus $\text{Spec}(\Gamma_R) = \left\{(0)^{p^5-p^2-l_1-l_2}, (p^2 - p^3)^{l_1 - 1}, (p^2 - p^4)^{l_2 - 1}, (x_1)^1, (x_2)^1\right\}$, where x_1, x_2 are the roots of the polynomial

$$x^2 - x(p^5 - p^4 - p^3 + p^2) - (p^7 - p^6 - p^5 + p^4)(l_1 + l_2 - 1).$$

Using Theorem 3.1(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) &= \prod_{i=1}^{2} (x - (p^5 - p^2) + p_i)^{a_i(p_i - 1)} \prod_{i=1}^{2} (x - (p^5 - p^2) + 2p_i)^{a_i} \\ &\times \left(1 - \sum_{i=1}^{2} \frac{a_i p_i}{x - (p^5 - p^2) + 2p_i}\right) \\ &= (x - (p^5 - p^2) + p^3 - p^2)^{l_1(p^3-p^2-1)} (x - (p^5 - p^2) + p^4 - p^2)^{l_2(p^4-p^2-1)} \\ &\times (x - (p^5 - p^2) + 2p^3 - 2p^2)^{l_1} (x - (p^5 - p^2) + 2p^4 - 2p^2)^{l_2} \end{aligned}$$

$$\times\left(1-\frac{l_1(p^3-p^2)}{x-(p^5-p^2)+2p^3-2p^2}-\frac{l_2(p^4-p^2)}{x-(p^5-p^2)+2p^4-2p^2}\right)$$
$$=(x-(p^5-p^3))^{l_1(p^3-p^2-1)}(x-(p^5-p^4))^{l_2(p^4-p^2-1)}$$
$$\times(x-(p^5-2p^3+p^2))^{l_1}(x-(p^5-2p^4+p^2))^{l_2}$$
$$\times\left(1-\frac{l_1(p^3-p^2)}{x-(p^5-2p^3+p^2)}-\frac{l_2(p^4-p^2)}{x-(p^5-2p^4+p^2)}\right)$$
$$=(x-(p^5-p^3))^{l_1(p^3-p^2-1)}(x-(p^5-p^4))^{l_2(p^4-p^2-1)}$$
$$\times(x-(p^5-2p^3+p^2))^{l_1-1}(x-(p^5-2p^4+p^2))^{l_2-1}f(x),$$

where

$$\begin{aligned}f(x)\ &=x^2-x(3p^5-2p^4-2p^3+p^2)+p^{10}-2p^9-2p^8+6p^7\\&-2p^6-2p^5+p^4+l_1(p^3-p^2)(p^5-2p^4+p^2)\\&+l_2(p^4-p^2)(p^5-2p^3+p^2).\end{aligned}$$

Thus,

$$\begin{aligned}\text{Q-spec}(\Gamma_R)\ &=\Big\{(p^5-p^3)^{l_1(p^3-p^2-1)},(p^5-p^4)^{l_2(p^4-p^2-1)},\\&\qquad(p^5-2p^3+p^2)^{l_1-1},(p^5-2p^4+p^2)^{l_2-1},(x_1)^1,(x_2)^1\Big\}\end{aligned}$$

where x_1, x_2 are the roots of the polynomial $f(x)$. Using Lemma 2.2(b)(i), we have

$$\text{L-spec}(\overline{\Gamma_R})=\Big\{(0)^{l_1+l_2},(p^3-p^2)^{l_1(p^3-p^2-1)},(p^4-p^2)^{l_2(p^4-p^2-1)}\Big\}.$$

Thus, Theorem 3.3 yields

$$\text{L-spec}(\Gamma_R)=\Big\{(0)^1,\ (p^5-p^4)^{l_2(p^4-p^2-1)},(p^5-p^3)^{l_1(p^3-p^2-1)},(p^5-p^2)^{l_1+l_2-1}\Big\}.$$

Here $|v(\Gamma_R)|=p^5-p^2$, $|e(\Gamma_R)|=\frac{(p^6-p^5)(p^2-1)(l_1+pl_2)}{2}$ and so

$$\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}=\frac{(p^5-p^3)(l_1+pl_2)}{p^2+p+1}.$$

Therefore

$$\left|p^5-p^4-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=\frac{(p^4-p^3)l_1}{p^2+p+1},$$
$$\left|p^5-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=\frac{(p^5-p^3)l_2}{p^2+p+1},$$
$$\left|p^5-p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|=\frac{(p^3-p^2)(l_1+(p+1)^2l_2)}{p^2+p+1}$$

and so

$$LE(\Gamma_R) = \frac{(p^5-p^3)(l_1+pl_2)}{p^2+p+1} + \frac{(p^4-p^3)(p^4-p^2-1)l_1l_2}{p^2+p+1} + \frac{(p^5-p^3)(p^3-p^2-1)l_1l_2}{p^2+p+1} + \frac{(p^3-p^2)(l_1+(p+1)^2l_2)(l_1+l_2-1)}{p^2+p+1}.$$

Consequently, we obtain the necessary expression.

(b) The expressions for $\text{Spec}(\Gamma_R)$, $\text{L-spec}(\Gamma_R)$, $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.2(b)(ii) and Theorem 3.4.

By Lemma 2.2(b)(ii) we have $\overline{\Gamma_R} = (p+1)K_{(p^4-p^3)}$. This implies $|v(\Gamma_R)| = p^5-p^3$ and $\Gamma_R = K_{p+1.p^4-p^3}$. Using Theorem 3.1(b), we have

$$\begin{aligned}Q_{\Gamma_R}(x) &= (x-(p^5-p^3)+p^4-p^3)^{(p+1)(p^4-p^3-1)} \\ &\quad\times (x-(p^5-p^3)+2(p^4-p^3))^{p+1}\left(1-\frac{p^5-p^3}{x-(p^5-p^3)+2(p^4-p^3)}\right) \\ &= (x-(p^5-p^4))^{p^5-p^3-p-1}(x-(p^5-2p^4+p^3))^p(x-(2p^5-2p^4)).\end{aligned}$$

Therefore, $\text{Q-spec}(\Gamma_R) = \left\{(p^5-p^4)^{p^5-p^3-p-1}, (p^5-2p^4+p^3)^p, (2p^5-2p^4)^1\right\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^9-p^8-p^7+p^6-p^5+p^3}{2}$. Therefore,

$$|e(\Gamma_R)| = \tfrac{p^{10}-2p^8+p^6-p^5+p^3}{2} - \tfrac{p^9-p^8-p^7+p^6-p^5+p^3}{2} = \tfrac{p^7(p^2-1)(p-1)}{2}.$$

Now,

$$\left|p^5-p^4-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5-p^4-p^5+p^4\right| = 0,$$

$$\left|p^5-2p^4+p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^4+p^3\right| = p^4-p^3$$

and

$$\left|2p^5-2p^4-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5-p^4\right| = p^5-p^4.$$

Thus,

$$LE^+(\Gamma_R) = (p^5-p^3-p-1)\times 0 + p\times(p^4-p^3) + p^5-p^4 = 2(p^5-p^4).$$

$\square$

Theorem 3.11. *Let $|R| = p^2q$ and $Z(R) = \{0\}$.*

(a) *Suppose that "$t \in \{p, q, p^2, pq\}$ and $(t-1)$ divides (p^2q-1)". Then*

$$\operatorname{Spec}(\Gamma_R) = \left\{(-t+1)^{\frac{p^2q-1}{t-1}-1}, (0)^{\frac{(t-2)(p^2q-1)}{t-1}}, (p^2q-t)^1\right\},$$

$$\text{Q-spec}(\Gamma_R) = \left\{(p^2q-t)^{\frac{p^2q-1}{t-1}(t-2)}, (p^2q-2t+1)^{\frac{p^2q-t}{t-1}}, (2p^2q-2t)^1\right\},$$

$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^2q-t)^{\frac{(t-2)(p^2q-1)}{t-1}}, (p^2q-1)^{\frac{p^2q-1}{t-1}-1}\right\}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^2q-t)$.

(b) *If $(p-1)l_1 + (q-1)l_2 + (p^2-1)l_3 + (pq-1)l_4 = p^2q-1$, then*

$$\operatorname{Spec}(\Gamma_R) = \Big\{(0)^{p^2q-1-l_1-l_2-l_3-l_4}, (1-p)^{l_1-1}, (1-q)^{l_2-1}, (1-p^2)^{l_3-1}, (1-pq)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\},$$

where x_1, x_2, x_3, x_4 are the roots of the polynomial

$$\begin{aligned}&(x+e)(x+f)(x+g)(x+h) - l_1e(x+f)(x+g)(x+h)\\ &-l_2f(x+e)(x+g)(x+h) - l_3g(x+e)(x+f)(x+h)\\ &-l_4h(x+e)(x+f)(x+g),\end{aligned}$$

where $e = p-1$, $f = q-1$, $g = p^2-1$ and $h = pq-1$,

$$\begin{aligned}\text{Q-spec}(\Gamma_R) = \Big\{&(p^2q-p)^{l_1(p-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p^2)^{l_3(p^2-2)},\\ &(p^2q-pq)^{l_4(pq-2)}, (p^2q-2p+1)^{l_1-1}, (p^2q-2q+1)^{l_2-1},\\ &(p^2q-2p^2+1)^{l_3-1}, (p^2q-2pq+1)^{l_4-1}, (x_1)^1,\\ &(x_2)^1, (x_3)^1, (x_4)^1\Big\},\end{aligned}$$

where x_1, x_2, x_3, x_4 are the roots of the polynomial

$$\begin{aligned}&(x-a)(x-b)(x-c)(x-d) - l_1(p-1)(x-b)(x-c)(x-d)\\ &-l_2(q-1)(x-a)(x-c)(x-d) - l_3(p^2-1)(x-a)(x-b)(x-d)\\ &-l_4(pq-1)(x-a)(x-b)(x-c),\end{aligned}$$

where $a = p^2q-2p+1$, $b = p^2q-2q+1$, $c = p^2q-2p^2+1$ and $d = p^2q-2pq+1$ and

$$\begin{aligned}\text{L-spec}(\Gamma_R) = \Big\{&(0)^1, (p^2q-pq)^{l_4(pq-2)}, (p^2q-p^2)^{l_3(p^2-2)},\\ &(p^2q-q)^{l_2(q-2)}, (p^2q-p)^{l_1(p-2)}, (p^2q-1)^{l_1+l_2+l_3+l_4-1}\Big\}.\end{aligned}$$

Proof. (a) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.1(b)(i) and Theorem 3.4. By Lemma 2.1(b)(i), we have $\overline{\Gamma_R} = \frac{p^2q-1}{t-1}K_{t-1}$. This implies $\Gamma_R = K_{\frac{p^2q-1}{t-1}.t-1}$. Using Theorem 3.1(b), we have

$$\begin{aligned}
&Q_{\Gamma_R}(x)\\
&= (x-(p^2q-1)+t-1)^{\frac{p^2q-1}{t-1}(t-1-1)}(x-(p^2q-1)+2(t-1))^{\frac{p^2q-1}{t-1}}\\
&\times\left(1-\frac{p^2q-1}{x-(p^2q-1)+2(t-1)}\right)\\
&= (x-(p^2q-t))^{\frac{p^2q-1}{t-1}(t-2)}(x-(p^2q-2t+1))^{\frac{p^2q-t}{t-1}}(x-(2p^2q-2t)).
\end{aligned}$$

Therefore, Q-spec$(\Gamma_R) = \left\{(p^2q-t)^{\frac{p^2q-1}{t-1}(t-2)}, (p^2q-2t+1)^{\frac{p^2q-t}{t-1}}, (2p^2q-2t)^1\right\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p^2q-1)(t-2)}{2}$. Thus,

$$|e(\Gamma_R)| = \tfrac{(p^2q-1)(p^2q-2)}{2} - \tfrac{(p^2q-1)(t-2)}{2} = \tfrac{(p^2q-1)(p^2q-t)}{2}.$$

Now,

$$\left|p^2q-t-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^2q-t-p^2q+t\right| = 0,$$

$$\left|p^2q-2t+1-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = |-t+1| = t-1$$

and

$$\left|2p^2q-2t-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^2q-t\right| = p^2q-t.$$

Thus, $LE^+(\Gamma_R) = \left(\frac{p^2q-1}{t-1}(t-2)\right)\times 0 + \left(\frac{p^2q-t}{t-1}\right)\times(t-1)+p^2q-t$
$= 2(p^2q-t).$

(b) Lemma 2.1(b)(ii) gives $\overline{\Gamma_R} = l_1K_{p-1}\cup l_2K_{q-1}\cup l_3K_{p^2-1}\cup l_4K_{pq-1}$. This implies $\Gamma_R = K_{l_1.p-1,l_2.q-1,l_3.p^2-1,l_4.pq-1}$.

Using Theorem 3.1(a), we have

$$\begin{aligned}
P_{\Gamma_R}(x) &= x^{(p^2q-1)-(l_1+l_2+l_3+l_4)}\prod_{i=1}^{4}(x+p_i)^{a_i-1}\\
&\times\left(\prod_{i=1}^{4}(x+p_i)-\sum_{j=1}^{4}a_jp_j\prod_{i=1,i\neq j}^{4}(x+p_i)\right)\\
&= x^{p^2q-1-l_1-l_2-l_3-l_4}(x+p-1)^{l_1-1}(x+q-1)^{l_2-1}
\end{aligned}$$

$$\begin{aligned}
&\times (x+p^2-1)^{l_3-1}(x+pq-1)^{l_4-1}\\
&\times ((x+p-1)(x+q-1)(x+p^2-1)(x+pq-1)\\
&- l_1(p-1)(x+q-1)(x+p^2-1)(x+pq-1)\\
&- l_2(q-1)(x+p-1)(x+p^2-1)(x+pq-1)\\
&- l_3(p^2-1)(x+p-1)(x+q-1)(x+pq-1)\\
&- l_4(pq-1)(x+p-1)(x+q-1)(x+p^2-1)).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Spec}(\Gamma_R) = \Big\{&(0)^{p^2q-1-l_1-l_2-l_3-l_4}, (1-p)^{l_1-1}, (1-q)^{l_2-1},\\
&(1-p^2)^{l_3-1}, (1-pq)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\},
\end{aligned}$$

where x_1, x_2, x_3, x_4 are the roots of the polynomial

$$\begin{aligned}
&(x+e)(x+f)(x+g)(x+h) - l_1e(x+f)(x+g)(x+h)\\
&-l_2f(x+e)(x+g)(x+h) - l_3g(x+e)(x+f)(x+h)\\
&-l_4h(x+e)(x+f)(x+g),
\end{aligned}$$

where $e = p-1$, $f = q-1$, $g = p^2-1$ and $h = pq-1$.

Using Theorem 3.1(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) = &\prod_{i=1}^{4}(x-(p^2q-1)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{4}(x-(p^2q-1)+2p_i)^{a_i}\\
&\times\left(1-\sum_{i=1}^{4}\frac{a_ip_i}{x-(p^2q-1)+2p_i}\right)\\
&= (x-(p^2q-1)+p-1)^{l_1(p-2)}(x-(p^2q-1)+q-1)^{l_2(q-2)}\\
&\times (x-(p^2q-1)+p^2-1)^{l_3(p^2-2)}(x-(p^2q-1)+pq-1)^{l_4(pq-2)}\\
&\times (x-(p^2q-1)+2p-2)^{l_1}(x-(p^2q-1)+2q-2)^{l_2}\\
&\times (x-(p^2q-1)+2p^2-2)^{l_3}(x-(p^2q-1)+2pq-2)^{l_4}\\
&\times \left(1-\frac{l_1(p-1)}{x-(p^2q-1)+2p-2}-\frac{l_2(q-1)}{x-(p^2q-1)+2q-2}\right.\\
&\left.-\frac{l_3(p^2-1)}{x-(p^2q-1)+2p^2-2}-\frac{l_4(pq-1)}{x-(p^2q-1)+2pq-2}\right)
\end{aligned}$$

After simplification we get

$$\begin{aligned}
&Q_{\Gamma_R}(x)\\
&= (x-(p^2q-p))^{l_1(p-2)}(x-(p^2q-q))^{l_2(q-2)}(x-(p^2q-p^2))^{l_3(p^2-2)}\\
&\times (x-(p^2q-pq))^{l_4(pq-2)}(x-p^2q+2p-1)^{l_1-1}(x-p^2q+2q-1)^{l_2-1}\\
&\times (x-p^2q+2p^2-1)^{l_3-1}(x-p^2q+2pq-1)^{l_4-1}\\
&\times \Big((x-(p^2q-2p+1))(x-(p^2q-2q+1))\\
&\quad (x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))\\
&- l_1(p-1)(x-(p^2q-2q+1))(x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))\\
&- l_2(q-1)(x-(p^2q-2p+1))(x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))\\
&- l_3(p^2-1)(x-(p^2q-2p+1))(x-(p^2q-2q+1))(x-(p^2q-2pq+1))\\
&- l_4(pq-1)(x-(p^2q-2p+1))(x-(p^2q-2q+1))(x-(p^2q-2p^2+1))\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{Q-spec}(\Gamma_R) = \Big\{&(p^2q-p)^{l_1(p-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p^2)^{l_3(p^2-2)},\\
&(p^2q-pq)^{l_4(pq-2)}, (p^2q-2p+1)^{l_1-1}, (p^2q-2q+1)^{l_2-1},\\
&(p^2q-2p^2+1)^{l_3-1}, (p^2q-2pq+1)^{l_4-1},\\
&(x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\},
\end{aligned}$$

where x_1, x_2, x_3, x_4 are the roots of the polynomial

$$\begin{aligned}
&(x-a)(x-b)(x-c)(x-d) - l_1(p-1)(x-b)(x-c)(x-d)\\
&- l_2(q-1)(x-a)(x-c)(x-d) - l_3(p^2-1)(x-a)(x-b)(x-d)\\
&- l_4(pq-1)(x-a)(x-b)(x-c),
\end{aligned}$$

where $a = p^2q-2p+1$, $b = p^2q-2q+1$, $c = p^2q-2p^2+1$ and $d = p^2q-2pq+1$.

Using Lemma 2.1(b)(ii), we have

$$\begin{aligned}
\text{L-spec}(\overline{\Gamma_R}) = \Big\{&(0)^{l_1+l_2+l_3+l_4}, (p-1)^{(p-2)l_1}, (q-1)^{(q-2)l_2},\\
&(p^2-1)^{(p^2-2)l_3}, (pq-1)^{(pq-2)l_4}\Big\}.
\end{aligned}$$

Thus, Theorem 3.3 yields

$$\text{L-spec}(\Gamma_R) = \Big\{(0)^1, (p^2q-pq)^{l_4(pq-2)}, (p^2q-p^2)^{l_3(p^2-2)},$$

$$(p^2q-q)^{l_2(q-2)}, (p^2q-p)^{l_1(p-2)}, (p^2q-1)^{l_1+l_2+l_3+l_4-1}\Big\}.$$

$\square$

Theorem 3.12. *Let $|R| = p^3q$ with unity.*

(a) *If $Z(R)$ has pq elements, then*

$$\begin{aligned}\text{Spec}(\Gamma_R) &= \Big\{(-p^2q+pq)^p, (0)^{(p+1)(p^2q-pq-1)}, (p^3q-p^2q)^1\Big\},\\ \text{Q-spec}(\Gamma_R) &= \Big\{(p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, (p^3q-2p^2q+pq)^p,\\ &\qquad (2p^3q-2p^2q)^1\Big\},\\ \text{L-spec}(\Gamma_R) &= \Big\{(0)^1, (p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, ((p+1)(p^2q-pq))^p\Big\}\end{aligned}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^3q-p^2q)$.

(b) *Suppose that $Z(R)$ has p^2 elements.*

(i) *If "$(p-1)$ divides $(pq-1)$", then*

$$\begin{aligned}\text{Spec}(\Gamma_R) &= \left\{(-p^3+p^2)^{\frac{pq-1}{p-1}-1}, (0)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3q-p^3)^1\right\},\\ \text{Q-spec}(\Gamma_R) &= \Big\{(p^3q-p^3)^{\frac{pq-1}{p-1}(p^3-p^2-1)}, (p^3q+p^2-2p^3)^{\frac{pq-p}{p-1}},\\ &\qquad (2p^3q-2p^3)^1\Big\},\\ \text{L-spec}(\Gamma_R) &= \left\{(0)^1, (p^3q-p^3)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3q-p^2)^{\frac{pq-1}{p-1}-1}\right\}\end{aligned}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^3(q-1)$.

(ii) *If $(q-1) \mid (pq-1)$, then*

$$\begin{aligned}\text{Spec}(\Gamma_R) &= \left\{(-p^2q+p^2)^{\frac{pq-1}{q-1}-1}, (0)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^3q-p^2)^1\right\},\\ \text{Q-spec}(\Gamma_R) &= \Big\{(p^3q-p^2q)^{\frac{pq-1}{q-1}(p^2q-p^2-1)}, (p^3q+p^2-2p^2q)^{\frac{pq-q}{q-1}},\\ &\qquad (2p^3q-2p^2q)^1\Big\},\\ \text{L-spec}(\Gamma_R) &= \left\{(0)^1, (p^3q-p^2q)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^3q-p^2)^{\frac{pq-1}{q-1}-1}\right\}\end{aligned}$$

and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(pq-q)$.

(iii) *If $pq-1 = (p-1)l_1 + (q-1)l_2$, then $\overline{\Gamma_R} = l_1K_{p^3-p^2} \cup l_2K_{p^2q-p^2}$ and*

$$\text{Spec}(\Gamma_R) = \left\{(0)^{p^3q-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^2q)^{l_2-1}, (x_1)^1, (x_2)^1\right\},$$

where x_1, x_2 are the roots of the polynomial

$$x^2 - x(p^3q - p^3 - p^2q + p^2) - (p^5q - p^5 - p^4q + p^4)(l_1 + l_2 - 1),$$

$$\begin{aligned}\text{Q-spec}(\Gamma_R) = \Big\{&(p^3q-p^3)^{l_1(p^3-p^2-1)}, (p^3q-p^2q)^{l_2(p^2q-p^2-1)},\\ &(p^3q-2p^3+p^2)^{l_1-1}, (p^3q-2p^2q+p^2)^{l_2-1},\\ &(x_1)^1, (x_2)^1\Big\},\end{aligned}$$

where x_1, x_2 are the roots of the polynomial

$$\begin{aligned}&x^2 - x(3p^3q - 2p^2q - 2p^3 + p^2) + p^6q^2 - 2p^6q - 2p^5q^2\\ &+6p^5q - 2p^4q - 2p^5 + p^4 + l_1(p^3 - p^2)(p^3q - 2p^2q + p^2)\\ &+l_2(p^2q - p^2)(p^3q - 2p^3 + p^2),\end{aligned}$$

$$\begin{aligned}\text{L-spec}(\Gamma_R) = \Big\{&(0)^1, (p^3q-p^2q)^{l_2(p^2q-p^2-1)}, (p^3q-p^3)^{l_1(p^3-p^2-1)},\\ &(p^3q-p^2)^{l_1+l_2-1}\Big\}\end{aligned}$$

and

$$LE(\Gamma_R) = \begin{cases}\frac{2(p^3-p^2)[(q-1)(pl_1+ql_2)+(p^2q-p^2-1)(q-p)l_1l_2]}{pq-1}, & \text{if } p < q\\ \frac{2(p^2q-p^2)[(p-1)(pl_1+ql_2)+(p^3-p^2-1)(p-q)l_1l_2]}{pq-1}, & \text{if } p > q.\end{cases}$$

Proof. (a) The expressions for $\text{Spec}(\Gamma_R)$, $\text{L-spec}(\Gamma_R)$, $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.2(c) and Theorem 3.4. By Lemma 2.2(c), we have $\overline{\Gamma_R} = (p+1)K_{p^2q-pq}$. This implies $|v(\Gamma_R)| = p^3q - pq$ and $\Gamma_R = K_{p+1.p^2q-pq}$. Using Theorem 3.1(b), we have

$$\begin{aligned}Q_{\Gamma_R}(x) &= (x - (p^3q - pq) + p^2q - pq)^{(p+1)(p^2q-pq-1)}\\ &\times (x - (p^3q - pq) + 2(p^2q - pq))^{p+1}\\ &\times \left(1 - \frac{p^3q - pq}{x - (p^3q - pq) + 2(p^2q - pq)}\right)\\ &= (x - (p^3q - p^2q))^{(p+1)(p^2q-pq-1)}\\ &\times (x - (p^3q - 2p^2q + pq))^p(x - (2p^3q - 2p^2q)).\end{aligned}$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, (p^3q-2p^2q+pq)^p, (2p^3q-2p^2q)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p+1)(p^2q-pq)(p^2q-pq-1)}{2}$. Therefore,

$$|e(\Gamma_R)| = \tfrac{(p^3q-pq)(p^3q-pq-1)}{2} - \tfrac{(p+1)(p^2q-pq)(p^2q-pq-1)}{2} = \tfrac{q^2(p-1)(p^5-p^3)}{2}.$$

Now

$$\left|p^3q - p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q - p^3q + p^2q\right| = 0,$$

$$\left|p^3q - 2p^2q + pq - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^2q + pq\right| = p^2q - pq$$

and

$$\left|2p^3q - 2p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q\right| = p^3q - p^2q.$$

Thus,

$$\begin{aligned} LE^+(\Gamma_R) &= (p+1)(p^2q - pq - 1) \times 0 + p \times (p^2q - pq) + p^3q - p^2q \\ &= 2(p^3q - p^2q). \end{aligned}$$

(b) (i) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.2(d)(i) and Theorem 3.4. By Lemma 2.2(d)(i), we have $\overline{\Gamma_R} = \frac{pq-1}{p-1}K_{p^3-p^2}$. This implies $\Gamma_R = K_{\frac{pq-1}{p-1}\cdot p^3-p^2}$. Using Theorem 3.1(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) &= (x - (p^3q - p^2) + p^3 - p^2)^{\frac{pq-1}{p-1}(p^3-p^2-1)} \\ &\quad \times (x - (p^3q - p^2) + 2(p^3 - p^2))^{\frac{pq-1}{p-1}} \\ &\quad \times \left(1 - \frac{p^3q - p^2}{x - (p^3q - p^2) + 2(p^3 - p^2)}\right) \\ &= (x - (p^3q - p^3))^{\frac{pq-1}{p-1}(p^3-p^2-1)} \\ &\quad \times (x - (p^3q - 2p^3 + p^2))^{\frac{pq-p}{p-1}}(x - (2p^3q - 2p^3)). \end{aligned}$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^3q - p^3)^{\frac{pq-1}{p-1}(p^3-p^2-1)}, (p^3q - 2p^3 + p^2)^{\frac{pq-p}{p-1}}, (2p^3q - 2p^3)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p^3q-p^2)(p^3-p^2-1)}{2}$. Therefore,

$$|e(\Gamma_R)| = \tfrac{(p^3q-p^2)(p^3q-p^2-1)}{2} - \tfrac{(p^3q-p^2)(p^3-p^2-1)}{2} = \tfrac{(p^3q-p^2)(p^3q-p^3)}{2}.$$

Now,

$$\left|p^3q - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^3 - p^3q + p^3\right| = 0,$$

$$\left|p^3q - 2p^3 + p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^3 + p^2\right| = p^3 - p^2$$

and

$$\left|2p^3q - 2p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^3\right| = p^3q - p^3.$$

Thus,

$$\begin{aligned} LE^+(\Gamma_R) &= \left(\frac{pq-1}{p-1}(p^3 - p^2 - 1)\right) \times 0 + \left(\frac{pq-p}{p-1}\right) \times (p^3 - p^2) + p^3q - p^3 \\ &= 2(p^3q - p^3). \end{aligned}$$

(b) (ii) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Lemma 2.2(d)(ii) and Theorem 3.4. By Lemma 2.2(d)(ii), we have $\overline{\Gamma_R} = \frac{pq-1}{q-1}K_{p^2q-p^2}$. This implies $\Gamma_R = K_{\frac{pq-1}{q-1}.p^2q-p^2}$. Using Theorem 3.1(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) &= (x - (p^3q - p^2) + p^2q - p^2)^{\frac{pq-1}{q-1}(p^2q-p^2-1)} \\ &\quad \times (x - (p^3q - p^2) + 2(p^2q - p^2))^{\frac{pq-1}{q-1}} \\ &\quad \times \left(1 - \frac{p^3q - p^2}{x - (p^3q - p^2) + 2(p^2q - p^2)}\right) \\ &= (x - (p^3q - p^2q))^{\frac{pq-1}{q-1}(p^2q-p^2-1)} \\ &\quad \times (x - (p^3q - 2p^2q + p^2))^{\frac{pq-q}{q-1}}(x - (2p^3q - 2p^2q)). \end{aligned}$$

Therefore,

$$\text{Q-spec}(\Gamma_R) = \left\{(p^3q - p^2q)^{\frac{pq-1}{q-1}(p^2q-p^2-1)}, (p^3q - 2p^2q + p^2)^{\frac{pq-q}{q-1}}, (2p^3q - 2p^2q)^1\right\}.$$

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^2(p^2q-p^2-1)(pq-1)}{2}$. Therefore,

$$|e(\Gamma_R)| = \tfrac{(p^3q-p^2)(p^3q-p^2-1)}{2} - \tfrac{p^2(p^2q-p^2-1)(pq-1)}{2} = \tfrac{(p^3q-p^2)(p^3q-p^2q)}{2}.$$

Now,

$$\left|p^3q - p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q - p^3q + p^2q\right| = 0,$$

$$\left|p^3q - 2p^2q + p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^2q + p^2\right| = p^2q - p^2$$

and

$$\left|2p^3q - 2p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q\right| = p^3q - p^2q.$$

Thus,

$$\begin{aligned} LE^+(\Gamma_R) &= \left(\frac{pq-1}{q-1}(p^2q-p^2-1)\right)\times 0+\left(\frac{pq-q}{q-1}\right)\times(p^2q-p^2)\\ &+p^3q-p^2q\\ &=2(p^3q-p^2q). \end{aligned}$$

(b) (iii) Lemma 2.2(d)(iii) gives $\overline{\Gamma_R} = l_1K_{p^3-p^2}\cup l_2K_{p^2q-p^2}$. This implies $\Gamma_R = K_{l_1.p^3-p^2,l_2.p^2q-p^2}$.

Using Theorem 3.1(a), we have

$$\begin{aligned} P_{\Gamma_R}(x) &= x^{(p^3q-p^2)-(l_1+l_2)}\prod_{i=1}^{2}(x+p_i)^{a_i-1}\\ &\times\left(\prod_{i=1}^{2}(x+p_i)-\sum_{j=1}^{2}a_jp_j\prod_{i=1,i\neq j}^{2}(x+p_i)\right)\\ &=x^{p^3q-p^2-l_1-l_2}(x+p^3-p^2)^{l_1-1}(x+p^2q-p^2)^{l_2-1}\\ &\times\Big((x+p^3-p^2)(x+p^2q-p^2)\\ &-l_1(p^3-p^2)(x+p^2q-p^2)-l_2(p^2q-p^2)(x+p^3-p^2)\Big)\\ &=x^{p^3q-p^2-l_1-l_2}(x-(p^2-p^3))^{l_1-1}(x-(p^2-p^2q))^{l_2-1}\times f(x), \end{aligned}$$

where $f(x)=x^2-x(p^3q-p^3-p^2q+p^2)-(p^5q-p^5-p^4q+p^4)(l_1+l_2-1)$. Thus $\operatorname{Spec}(\Gamma_R) = \left\{(0)^{p^3q-p^2-l_1-l_2},(p^2-p^3)^{l_1-1},(p^2-p^2q)^{l_2-1},(x_1)^1,(x_2)^1\right\}$, where x_1,x_2 are the roots of the polynomial $f(x)$.

Using Theorem 3.1(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) &= \prod_{i=1}^{2}(x-(p^3q-p^2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(p^3q-p^2)+2p_i)^{a_i}\\ &\times\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(p^3q-p^2)+2p_i}\right)\\ &=(x-(p^3q-p^2)+p^3-p^2)^{l_1(p^3-p^2-1)}\\ &\times(x-(p^3q-p^2)+p^2q-p^2)^{l_2(p^2q-p^2-1)}\\ &\times(x-(p^3q-p^2)+2p^3-2p^2)^{l_1}(x-(p^3q-p^2)+2p^2q-2p^2)^{l_2} \end{aligned}$$

$$\begin{aligned}
&\times \left(1 - \frac{l_1(p^3-p^2)}{x-(p^3q-p^2)+2p^3-2p^2} - \frac{l_2(p^2q-p^2)}{x-(p^3q-p^2)+2p^2q-2p^2}\right)\\
&= (x-(p^3q-p^3))^{l_1(p^3-p^2-1)}(x-(p^3q-p^2q))^{l_2(p^2q-p^2-1)}\\
&\times (x-(p^3q-2p^3+p^2))^{l_1-1}(x-(p^3q-2p^2q+p^2))^{l_2-1} \times f(x),
\end{aligned}$$

where

$$\begin{aligned}
f(x) &= x^2 - x(3p^3q-2p^2q-2p^3+p^2)+p^6q^2-2p^6q-2p^5q^2+6p^5q\\
&-2p^4q-2p^5+p^4+l_1(p^3-p^2)(p^3q-2p^2q+p^2)\\
&+l_2(p^2q-p^2)(p^3q-2p^3+p^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{Q-spec}(\Gamma_R) &= \Big\{(p^3q-p^3)^{l_1(p^3-p^2-1)}, (p^3q-p^2q)^{l_2(p^2q-p^2-1)},\\
&\qquad (p^3q-2p^3+p^2)^{l_1-1}, (p^3q-2p^2q+p^2)^{l_2-1}, (x_1)^1, (x_2)^1\Big\},
\end{aligned}$$

where x_1, x_2 are the roots of the polynomial $f(x)$.

Using Lemma 2.2(d)(iii), we have

$$\text{L-spec}(\overline{\Gamma_R}) = \Big\{(0)^{l_1+l_2}, (p^3-p^2)^{l_1(p^3-p^2-1)}, (p^2q-p^2)^{l_2(p^2q-p^2-1)}\Big\}.$$

Thus,
Theorem 3.3 yields

$$\begin{aligned}
\text{L-spec}(\Gamma_R) &= \Big\{(0)^1, (p^3q-p^2q)^{l_2(p^2q-p^2-1)}, (p^3q-p^3)^{l_1(p^3-p^2-1)},\\
&\qquad (p^3q-p^2)^{l_1+l_2-1}\Big\}.
\end{aligned}$$

Here $|v(\Gamma_R)| = p^3q-p^2$, $|e(\Gamma_R)| = \frac{p^4(p-1)(q-1)(pl_1+ql_2)}{2}$ and so

$$\tfrac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} = \tfrac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1}.$$

Therefore

$$\begin{aligned}
\left|p^3q-p^2q-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|\frac{p^2(p-1)(p-q)l_1}{pq-1}\right|\\
&= \begin{cases} \frac{(p^3-p^2)(q-p)l_1}{pq-1}, & \text{if } p<q\\ \frac{(p^3-p^2)(p-q)l_1}{pq-1}, & \text{if } p>q, \end{cases}
\end{aligned}$$

$$\left|p^3q-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|\frac{p^2(q-1)(q-p)l_2}{pq-1}\right|$$

$$= \begin{cases} \frac{(p^2q-p^2)(q-p)l_2}{pq-1}, & \text{if } p < q \\ \frac{(p^2q-p^2)(p-q)l_2}{pq-1}, & \text{if } p > q, \end{cases}$$

and $\left|p^3q - p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{p^2((p-1)^2l_1+(q-1)^2l_2)}{pq-1}$.

If $p < q$, then we have

$$\begin{aligned} LE(\Gamma_R) &= \frac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1} \\ &+ \frac{(p^3-p^2)(q-p)(p^2q-p^2-1)l_1l_2}{pq-1} \\ &+ \frac{(p^2q-p^2)(q-p)(p^3-p^2-1)l_1l_2}{pq-1} \\ &+ \frac{p^2((p-1)^2l_1+(q-1)^2l_2)(l_1+l_2-1)}{pq-1}. \end{aligned}$$

Consequently, we obtain the necessary expression.

If $p > q$, then we have

$$\begin{aligned} LE(\Gamma_R) &= \frac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1} \\ &+ \frac{(p^3-p^2)(p-q)(p^2q-p^2-1)l_1l_2}{pq-1} \\ &+ \frac{(p^2q-p^2)(p-q)(p^3-p^2-1)l_1l_2}{pq-1} \\ &+ \frac{p^2((p-1)^2l_1+(q-1)^2l_2)(l_1+l_2-1)}{pq-1}. \end{aligned}$$

Consequently, we obtain the necessary expression. □

With the following theorem we come to an end of this section.

Theorem 3.13. *If $S_1, S_2, \ldots, S_n$ are the non-identical centralizers of $s \in R \setminus Z(R)$, where R is a finite CC-ring and $|Z(R)| = \eta$, then*

L-spec$(\Gamma_R) = \Big\{(0)^1, (|R|-|S_n|)^{|S_n|-\eta-1}, \ldots, (|R|-|S_1|)^{|S_1|-\eta-1}, (|R|-\eta)^{n-1}\Big\}$

and $LE(\Gamma_R) = \frac{2}{|R|-\eta}\Big(|R|^2 - (\sum_{j=1}^n |S_j|^2) + (n-1)\eta^2\Big)$.

In particular, if $|S| = |S_1| = |S_2| = \cdots = |S_n|$, then

$$\text{Spec}(\Gamma_R) = \Big\{(-|S|+\eta)^{n-1}, (0)^{n(|S|-\eta-1)}, ((n-1)(|S|-\eta))^1\Big\},$$

$$\text{L-spec}(\Gamma_R) = \left\{(0)^1, ((n-1)(|S|-\eta))^{n(|S|-\eta-1)}, (n(|S|-\eta))^{n-1}\right\}$$

and $E(\Gamma_R) = LE(\Gamma_R) = 2(n-1)(|S|-\eta)$.

Proof. Lemma 2.3 gives $\overline{\Gamma_R} = \overset{n}{\underset{i=1}{\cup}} K_{|S_i|-\eta}$ and

$$\text{L-spec}(\overline{\Gamma_R}) = \left\{(0)^n, (|S_1|-\eta)^{|S_1|-\eta-1}, \ldots, (|S_n|-\eta)^{|S_n|-\eta-1}\right\}.$$

Therefore, Theorem 3.3 yields

$$\begin{aligned}\text{L-spec}(\Gamma_R) &= \Big\{(0)^1, (|R|-|S_n|)^{|S_n|-\eta-1}, \ldots, (|R|-|S_1|)^{|S_1|-\eta-1},\\ &\qquad (|R|-\eta)^{n-1}\Big\}.\end{aligned}$$

Again, $|v(\Gamma_R)| = |R|-\eta$, $|R| = \sum_{j=1}^n |S_j| - (n-1)\eta$ and

$$\tfrac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} = \tfrac{|R|^2 - (\sum_{j=1}^n |S_j|^2) + (n-1)\eta^2}{|R|-\eta}.$$

Therefore, for $i = 1, 2, \ldots, n$, we have

$$\left||R| - |S_i| - \tfrac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \tfrac{\sum_{j=1}^n |S_j|^2 + \eta(|S_i| - \eta(n-1)) - (|S_i|+\eta)|R|}{|R|-\eta}$$

and

$$\left||R| - \eta - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{\sum_{j=1}^n |S_j|^2 - (n-2)\eta^2 - 2|R|\eta}{|R|-\eta}.$$

Thus

$$\begin{aligned}&LE(\Gamma_R)\\ &= \frac{|R|^2 - (\sum_{j=1}^n |S_j|^2) + (n-1)\eta^2}{|R|-\eta}\\ &+ \sum_{i=1}^n (|S_i|-\eta-1)\left(\frac{\sum_{j=1}^n |S_j|^2 + \eta(|S_i| - (n-1)\eta) - |R|(\eta + |S_i|)}{|R|-\eta}\right)\\ &+ (n-1)\left(\frac{\sum_{j=1}^n |S_j|^2 - (n-2)\eta^2 - 2|R|\eta}{|R|-\eta}\right).\end{aligned}$$

Consequently, we obtain the necessary expression.

If $|S| = |S_1| = |S_2| = \ldots = |S_n|$, then $\overline{\Gamma_R} = nK_{|S|-\eta}$. Hence, Theorem 3.4 yields the result. $\square$

4. Some consequences

Our findings, in the previous section, imply that Γ_R of the rings we analyze are L-integral. Now, we determine whether Γ_R of the rings considered in this paper are L-hyperenergetic.

Theorem 4.1. *If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $LE(\Gamma_R) \leq LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is not L-hyperenergetic.*

Proof. Since $|v(\Gamma_R)| = (p^2-1)|Z(R)|$, we have

$$LE(K_{|v(\Gamma_R)|}) = 2((p^2-1)|Z(R)|-1) \geq 2p(p-1)|Z(R)|.$$

Hence, Theorem 3.5 yields the result. □

As a consequence of Theorem 4.1, Γ_R of all the rings taken into consideration in Theorem 3.6–3.8 are not L-hyperenergetic.

Theorem 4.2. *Let $|R| = p^4$ with unity.*

(a) *Suppose that $Z(R)$ has p elements. Then $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is not L-hyperenergetic and if $l_1 + l_2(p+1) = p^2+p+1$, then $LE(\Gamma_R) > LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is L-hyperenergetic.*

(b) *If $|Z(R)| = p^2$, then $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is not L-hyperenergetic.*

Proof. (a) Since $|v(\Gamma_R)| = p^4 - p$, $LE(K_{|v(\Gamma_R)|}) = 2(p^4-p-1)$. Therefore, Theorem 3.9(a) yields

$$LE(K_{|v(\Gamma_R)|}) - LE(\Gamma_R) = 2(p^2-p-1) > 0$$

and if $l_1 + l_2(p+1) = p^2+p+1$, we have

$$LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) = \frac{2(p^2+p+1)+2\{(p^6-p^5-p^4+p^2)l_1-(p^4-p^2)\}l_2}{p^2+p+1} > 0$$

as we observe that the term, $\{(p^6-p^5-p^4+p^2)l_1-(p^4-p^2)\}l_2 > 0$, for all p and hence the result follows.

(b) We have $|v(\Gamma_R)| = p^4-p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^4-p^2-1)$. From Theorem 3.9(b), $LE(\Gamma_R) = 2(p^4-p^3) < 2(p^4-p^2-1)$, as $p^3-p^2-1 > 0$. Consequently, we get the required result. □

Theorem 4.3. *Let R has unity, $|R| = p^5$ and $Z(R)$ is not a field.*

(a) *Suppose that $Z(R)$ has p^2 elements. Then $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is not L-hyperenergetic and if $l_1 + l_2(p+1) = p^2+p+1$, then $LE(\Gamma_R) > LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is L-hyperenergetic.*

(b) *If* $|Z(R)| = p^3$, *then* $LE(\Gamma_R) < LE(K_{|R|-|Z(R)|})$; *and hence* Γ_R *is not L-hyperenergetic.*

Proof. Since $Z(R)$ is not a field, $|Z(R)| = p^2$ or $|Z(R)| = p^3$.

Case 1: $|Z(R)| = p^2$

We have $|v(\Gamma_R)| = p^5 - p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^5 - p^2 - 1)$. Theorem 3.10(a), yields

$$LE(K_{|v(\Gamma_R)|}) - LE(\Gamma_R) = 2(p^3 - p^2 - 1) > 0$$

and if $l_1 + l_2(p+1) = p^2 + p + 1$, we have

$$\begin{aligned}
&LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) \\
&= \frac{2(-p^5 + 2p^2 + p + 1)}{p^2 + p + 1} \\
&+ \frac{2\{(p^8 - p^7 - p^6 - p^4)l_1 + (p^5 l_1 - p^5) + (p^3 l_1 + p^3)\}l_2}{p^2 + p + 1} \\
&\geq \frac{p^4(p^4 - p^3 - p^2 - p - 1) + 2p^3 + 2p^2 + p + 1}{p^2 + p + 1} \text{ (since } l_1, l_2 \geq 1) \\
&> 0 \text{ (since } p^4 - p^3 - p^2 - p - 1 > 0 \text{ for all } p).
\end{aligned}$$

Hence the result follows.

Case 2: $|Z(R)| = p^3$

We have $|v(\Gamma_R)| = p^5 - p^3$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^5 - p^3 - 1)$. From Theorem 3.10(b), $LE(\Gamma_R) = 2(p^5 - p^4) < 2(p^5 - p^3 - 1)$, as $p^4 - p^3 - 1 > 0$. Consequently, we get the required result. □

Theorem 4.4. *If "*$|R| = p^2 q$ *and* $Z(R) = \{0\}$ *and* $t \in \{p, q, p^2, pq\}$ *and* $(t-1) \mid (p^2 q - 1)$*" then* Γ_R *is not L-hyperenergetic.*

Proof. Since $|v(\Gamma_R)| = p^2 q - 1$, $LE(K_{|v(\Gamma_R)|}) = 2(p^2 q - 2)$. We know that $p^2 q - t \leq p^2 q - 2$ always, where $t \in \{p, q, p^2, pq\}$. Therefore, by Theorem 3.11(a) the result follows. □

Theorem 4.5. *Let* $|R| = p^3 q$ *with unity.*

(a) *Then* $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$ *whenever* $|Z(R)| = pq$; *and hence* Γ_R *is not L-hyperenergetic.*

(b) *If* $|Z(R)| = p^2$ *and*

  (i) $(p-1)$ *divides* $(pq-1)$ *or* $(q-1)$ *divides* $(pq-1)$, *then* Γ_R *is not L-hyperenergetic.*

  (ii) $pq - 1 = (p-1)l_1 + (q-1)l_2$, *where* $p \neq 2$ *and* $q \neq 3$, *then* Γ_R *is L-hyperenergetic.*

Proof. Since $|Z(R)|$ is not a prime, $|Z(R)| = pq$ or $|Z(R)| = p^2$.

Case 1: $|Z(R)| = pq$

Since $|v(\Gamma_R)| = p^3q - pq$, $LE(K_{|v(\Gamma_R)|}) = 2(p^3q - pq - 1)$. From Theorem 3.12(a), $LE(\Gamma_R) = 2(p^3q - p^2q) < 2(p^3q - pq - 1)$, as $p^2q - pq - 1 > 0$. Hence the result follows.

Case 2: $|Z(R)| = p^2$

We have $|v(\Gamma_R)| = p^3q - p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^3q - p^2 - 1)$.

Subcase 2.1: If $(p - 1) \mid (pq - 1)$, then from Theorem 3.12(b)(i), $LE(\Gamma_R) = 2(p^3q - p^3) < 2(p^3q - p^2 - 1)$, as $p^3 - p^2 - 1 > 0$.

Subcase 2.2: If $(q - 1) \mid (pq - 1)$, then from Theorem 3.12(b)(ii), $LE(\Gamma_R) = 2(p^3q - p^2q) < 2(p^3q - p^2 - 1)$, as $p^2q - p^2 - 1 > 0$.

Subcase 2.3: If $pq-1 = (p-1)l_1+(q-1)l_2$ and $p < q$, then from Theorem 3.12(b)(iii), we have

$$\begin{aligned}
&LE(\Gamma_R) - LE(K_{|v(\Gamma_R)}|) \\
&= \frac{2(pq-1)(1+p^2-p^2q)}{pq-1} \\
&+\frac{2p^2(p-1)(q-p)l_1\{(p^2q-p^2-1)l_2+1\}}{pq-1} \\
&\geq \frac{(pq-1)(1+p^2-p^2q)+p^2(p-1)(q-p)(p^2q-p^2)}{pq-1} \text{ (since } l_1, l_2 \geq 1) \\
&\geq \frac{p^2(q-1)[p^2(p-1)(q-p)-(pq-1)]}{pq-1} \\
&\geq \frac{p^3(q-1)[p(p-1)(q-p)-q]}{pq-1} \\
&\geq \frac{p^3(q-1)[p(q-p)-q]}{pq-1} \\
&> 0,
\end{aligned}$$

since $pq - p^2 - q > 0$ for all p and q such that $p \neq 2$ and $q \neq 3$.

If $pq - 1 = (p - 1)l_1 + (q - 1)l_2$ and $p > q$, then from Theorem 3.12(b)(iii), we have

$$\begin{aligned}
&LE(\Gamma_R) - LE(K_{|v(\Gamma_R)}|) \\
&= \frac{2(pq-1)(1+p^2-p^3)}{pq-1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2p^2(q-1)(p-q)l_2\{(p^3-p^2-1)l_1+1\}}{pq-1}\\
&\geq \frac{(pq-1)(1+p^2-p^3)+p^2(q-1)(p-q)(p^3-p^2)}{pq-1} \text{ (since } l_1, l_2 \geq 1)\\
&\geq \frac{p^2(p-1)[p^2(p-q)(q-1)-(pq-1)]}{pq-1}\\
&\geq \frac{p^3(p-1)[p(p-q)(q-1)-q]}{pq-1}\\
&\geq \frac{p^3(p-1)[p(p-q)-q]}{pq-1}\\
&> 0,
\end{aligned}
$$

since $p^2-pq-q > 0$, for all p and q such that $p \neq 2$ and $q \neq 3$. This concludes the proof. □

The following theorem gives us some finite non-commutative rings R for which Γ_R is integral, Q-integral but not hyperenergetic as well as Q-hyperenergetic.

Theorem 4.6. *Γ_R is integral, Q-integral but not hyperenergetic and Q-hyperenergetic if*

(a) $\frac{R}{Z(R)}$ *is isomorphic to* $\mathbb{Z}_p \times \mathbb{Z}_p$.

(b) $|R| = p^4$ *with unity and*
  (i) $|Z(R)| = p$ *such that* $\Gamma_R = K_{p^2+p+1.p^2-p}$.
  (ii) $|Z(R)| = p^2$.

(c) $|R| = p^5$ *with unity and*
  (i) $|Z(R)| = p^2$ *such that* $\Gamma_R = K_{p^2+p+1.p^3-p^2}$.
  (ii) $|Z(R)| = p^3$.

(d) $|R| = p^2q$ *and* $Z(R) = \{0\}$ *such that* $t \in \{p, q, p^2, pq\}$ *and*

$$(t-1) \mid (p^2q-1).$$

(e) $|R| = p^3q$ *with unity and*
  (i) $|Z(R)| = pq$.
  (ii) $|Z(R)| = p^2$ *such that* $(p-1) \mid (pq-1)$ *or* $(q-1) \mid (pq-1)$.

Proof. For the aforementioned rings R, Spec(Γ_R) and Q-spec(Γ_R) contain only integers so Γ_R is integral as well as Q-integral. Also,

$$LE^+(\Gamma_R) = LE(\Gamma_R) = E(\Gamma_R).$$

Therefore, in view of Theorems 4.1–4.5, Γ_R is not hyperenergetic and Q-hyperenergetic. □

Finally we wrap up this paper noting that as a consequence of Theorem 4.6, Γ_R of all the rings taken into consideration in Theorem 3.6–3.8 are integral, Q-integral but not hyperenergetic as well as Q-hyperenergetic.

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LAPLACIAN SPECTRUM AND ENERGY OF NON-COMMUTING GRAPHS OF FINITE RINGS

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طیف لاپلاسی و انرژی گراف‌های غیرجابجایی حلقه‌های متناهی

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[1,2]گروه علوم ریاضی، دانشگاه تزپور، ناپام، سونیتپور، آسام، هند

در این مقاله، طیف، انرژی، طیف/انرژی لاپلاسی و همچنین طیف/انرژی لاپلاسی بدون علامت گراف‌های غیرجابجایی برخی حلقه‌های غیرجابجایی متناهی بررسی شده است. به‌طور خاص، حلقه‌های متناهی R که $|R| = p^2, p^3, p^4, p^5, p^2q, p^3q$، زمانی که p و q اعداد اول هستند، بررسی می‌شوند. همچنین، حلقه‌های متناهی n-مرکزی برای $n = 4, 5$ و $p+2$ مورد بررسی قرار گرفته‌اند؛ و به‌طور کلی، حلقه‌های متناهی که خارج‌قسمت مرکزی آن‌ها با $\mathbb{Z}_p \times \mathbb{Z}_p$ یکریخت است، مورد مطالعه قرار گرفته‌اند. محاسبات نشان می‌دهد که گراف‌های غیرجابجایی این حلقه‌ها L-صحیح هستند. به علاوه، مشخص کرده‌ایم که آیا گراف‌های غیرجابجایی این حلقه‌ها، صحیح، Q-صحیح، فراانرژتیک، L-فراانرژتیک، یا Q-فراانرژتیک هستند یا خیر.

کلمات کلیدی: طیف گراف، گراف صحیح، گراف غیرجابجایی، حلقه متناهی.