Journal of Algebraic Systems, Vol. 14(No. 2): (2026), pp 281-295.
https://doi.org/10.22044/JAS.2024.13818.1773

CAPABILITY OF LOW-DIMENSIONAL NILPOTENT 3-LIE ALGEBRAS

H. Darabi

Abstract. In this paper, we characterize the capability of nilpotent n-Lie algebras of dimension at most $n+3$ over an arbitrary field when $n>2$ and the capability of 7-dimensional nilpotent 3-Lie algebras over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

1. Introduction

In 1985, Filippov [13] introduced the concept of *n-Lie (Filippov) algebra* as an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$ that satisfies the following generalized Jacobi identity:

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]].$$

Filippov algebras have been intensively studied in recent years since they naturally correspond to certain models of Quantum Mechanics involving particles such as bosons and fermions. This, along with a classical reference [24], shows important relations between Lie Theory and Mathematical Physics in Quantum Mechanics. Definitively, there are additional and more recent interactions between Lie algebras and Quantum Mechanics in [1, 2, 3, 20].

The study of n-Lie algebras is important since it has applications in geometry and physics. Among other problems about n-lie algebras, the classification of n-Lie algebras is a central problem in this field. The algebraic classification of Filippov algebra is also discussed in the literature (for example, see [22, 18, 13, 5, 4, 15, 16, 8]).

Let A be an n-Lie algebra over a field Λ, and let $A \cong F/R$ for a free n-Lie algebra. Then, the Schur multiplier $\mathcal{M}(A)$ of A is isomorphic to $\frac{R \cap F^2}{[R,F,\ldots,F]}$. The multiplier of an n-Lie algebra is always Abelian, and any two multipliers of A are isomorphic.

Several bounds are provided for the dimension of the Schur multiplier of Lie algebras and n-Lie algebras. One of the most important bounds for the

MSC(2020): Primary 17B05; Secondary 17B30, 11Y50.
Keywords: Capable n-Lie algebra; Nilpotent n-Lie algebra; Multiplier.
Received: 10 November 2023, Accepted: 8 September 2024.

dimension of the Schur multiplier of Lie algebras and n-Lie algebras is given in [19] and [9], respectively. Later, these bounds were improved in [21] and [12] for the Schur multiplier of nilpotent Lie algebras and nilpotent n-Lie algebras, respectively.

The n-Lie algebra A is called capable when there exists an n-Lie algebra B such that $A = B/Z(B)$. The capability of nilpotent Lie algebras of small dimensions is given in [23].

In this paper, by using the multiplier, we characterize the capability of n-Lie algebras of dimension at most $n+3$ over an arbitrary field when $n > 2$ and the capability of 7-dimensional nilpotent 3-Lie algebras over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

2. Preliminaries

This section contains some definitions, theorems, and results, which we need in other sections. Assume that $A_1, \ldots, A_n$ are subalgebras of an n-Lie algebra A. Then, the subalgebra of A generated by all vectors $[x_1, \ldots, x_n]$ $(x_i \in A_i)$ will be represented by $[A_1, \ldots, A_n]$. The subalgebra $A^2 = [A, \ldots, A]$ is called the derived subalgebra of A. The center of n-Lie algebra A is defined as follows:

$$Z\left(A\right) = \left\{x \in A : [x, A, \ldots, A] = 0\right\}.$$

The notion of nilpotent Filippov algebra is defined by Kasymov [17]. A Filippov algebra A is nilpotent if $A^s = 0$, where s is a nonnegative integer number. Note that A^i is defined as inducted by $A^1 = A,\ A^{i+1} = [A^i, A, \ldots, A]$.

We call $Z^*\left(A\right)$, the smallest ideal I of A, whenever A/I is capable. Therefore, the n-Lie algebra A is capable if and only if $Z^*\left(A\right) = 0$. The Abelian n-Lie algebra $F(d)$ is capable if and only if $d \geq n$. Also, the n-Lie algebra $H(n,m) \oplus F(k)$ is capable if and only if $m = 1$, when

$$\begin{aligned} H(n,m) = \langle x, x_1, \ldots, x_{mn} : [&x_{n(i-1)+1}, x_{n(i-1)+2}, \ldots, x_{ni}] = x, \\ &i = 1, \ldots, m\rangle, \end{aligned}$$

(see [11, 12] for more information).

The following lemma plays a key role in the study of the capability of n-Lie algebras.

Lemma 2.1. *The n-Lie algebra A is capable if and only if*

$$\dim \mathcal{M}(A) > \dim \mathcal{M}(\tfrac{A}{I}) - \dim(A^2 \cap I)$$

for each central ideal I of A.

Proof. By corollary 4.3 of [14], it is obvious. □

Theorem 2.2 ([9]). *Let A and B be two finite-dimensional n-Lie algebras. Then*

$$\dim \mathcal{M}(A \oplus B) = \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n},$$

where $a = \dim A/A^2$ and $b = \dim B/B^2$.

Theorem 2.3 ([10]). *Let A be a d-dimensional nilpotent n-Lie algebra with $\dim A^2 = 1$. Then $A \cong H(n,m) \oplus F(d-mn-1)$ for some $m \geq 1$. Moreover,*

$$\dim \mathcal{M}(A) = \begin{cases} \binom{d-1}{n} + n - 1, & m = 1, \\ \binom{d-1}{n} - 1, & m \geq 2. \end{cases}$$

The nilpotent n-Lie algebras of dimension at most $n+3$ over an arbitrary field are known. These algebras are collected in the following theorem.

Theorem 2.4. *Let A be a d-dimensional nilpotent n-Lie algebra.*

(i) *If $d \leq n$, then A is isomorphic to $F(d)$.*
(ii) *If $d = n+1$, then A is isomorphic to $F(n+1)$ or $H(n,1)$.*
(iii) *If $d = n+2$, then A is isomorphic to $F(n+2)$, $H(n,1) \oplus F(1)$, or $A_{n,n+2,1}$.*
(iv) *If $d = n+3$, then A is isomorphic to $F(n+3)$, $H(n,1) \oplus F(2)$, $A_{n,n+2,1} \oplus F(1)$, $A_{n,n+3,1}$, $A_{n,n+3,2}$, $A_{n,n+3,3}$, $A_{n,n+3,4}$, or $A_{n,n+3,5}$ when $n > 2$.*

The structure and dimension of the Schur multiplier of d-dimensional nilpotent n-Lie algebras with $d \leq n+3$ are listed in Tables 1 and 2.

Table 1. The structure of d-dimensional indecomposable nilpotent n-Lie algebras with $d \leq n+3$ and $\dim A^2 \geq 2$ over an arbitrary field.

Name	Non-zero multiplication
$A_{n,n+2,1}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}$
$A_{n,n+3,1}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,2}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = [e_1,e_3,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,3}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}, [e_1,e_3,\dots,e_{n+1}] = e_{n+3}$
$A_{n,n+3,4}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,5}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2},$ $[e_2,\dots,e_n,e_{n+2}] = [e_1,e_3,\dots,e_{n+1}] = e_{n+3}$

The classification of $(n+4)$-dimensional nilpotent n-Lie algebras does not done yet for each n, but 7-dimensional nilpotent 3-Lie algebras are classified in [7], as follows.

Table 2. The dimension of the Schur multiplier of d-dimensional nilpotent n-Lie algebras with $d \leq n+3$ over an arbitrary field.

Name	$\dim \mathcal{M}(A)$
$F(d)$	$\binom{d}{n}$
$H(n,1) \oplus F(2)$	$\frac{1}{2}n(n+5)$
$H(n,1)$, $A_{n,n+2,1}$	n
$H(n,1) \oplus F(1)$, $A_{n,n+2,1} \oplus F(1)$, $A_{n,n+3,2}$	$2n$
$A_{n,n+3,1}$	$4n-2$
$A_{n,n+3,3}$, $A_{n,n+3,5}$	$n+1$
$A_{n,n+3,4}$	$2n-1$

Theorem 2.5 ([7]). *The 7-dimensional nilpotent 3-Lie algebras are $F(7)$, $H(3,1) \oplus F(3)$, $H(3,2)$, $A_{3,5,1} \oplus F(2)$, $A_{3,6,i} \oplus F(1)$ and $A_{3,7,j}$, where $1 \leq i \leq 5$ and $1 \leq j \leq 27$.*

The structure of 7-dimensional indecomposable nilpotent 3-Lie algebra with $\dim Z(A) = 1$, $\dim Z(A) = 2$,and $\dim Z(A) = 3$ are presented in Tables 3, 4, and 5, respectively.

The following lemma is useful for recognizing the capability of decomposable n-Lie algebras.

Lemma 2.6. *Let A be a capable n-Lie algebra. Then $A \oplus F(1)$ is capable.*

Proof. Since A is capable,

$$\dim \mathcal{M}(A) > \dim \mathcal{M}(\frac{A}{I}) - \dim(A^2 \cap I), \tag{2.1}$$

for each central ideal I, by Lemma 2.1. Let $A = \langle e_1, e_2, \ldots, e_d \rangle$, and let $B = A \oplus F(1)$. Then $B = \langle e_1, e_2, \ldots, e_d, e_{d+1} \rangle$ such that e_{d+1} is central. The central ideals of B are I, $\langle e_{d+1} \rangle$, and $I \oplus \langle e_{d+1} \rangle$, when I is a central ideal of A. By Theorem 2.2,

$$\dim \mathcal{M}(B) = \dim \mathcal{M}(A) + \binom{d - \dim A^2}{n-1}. \tag{2.2}$$

On the other hand, $\frac{B}{I} = \frac{A}{I} \oplus \langle e_{d+1} \rangle$ and $B^2 \cap I = A^2 \cap I$. By Theorem 2.2,

$$\dim \mathcal{M}(\frac{B}{I}) = \dim \mathcal{M}(\frac{A}{I}) + \binom{d - \dim A^2 + \dim(A^2 \cap I) - \dim I}{n-1}. \tag{2.3}$$

By using relations (2.1), (2.2), and (2.3), we have

$$\dim \mathcal{M}(B) > \dim \mathcal{M}(\tfrac{B}{I}) - \dim(B^2 \cap I).$$

TABLE 3. The structure of 7-dimensional indecomposable nilpotent 3-Lie algebras with $\dim Z(A) = 1$ over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

Name	Non-zero multiplication
$A_{3,7,3}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = [e_1, e_2, e_5] = e_7$
$A_{3,7,4}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_3, e_4, e_5] = e_7$
$A_{3,7,5}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_4, e_5] = e_7$
$A_{3,7,9}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_5] = e_7$
$A_{3,7,10}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6,$ $[e_2, e_3, e_6] = [e_1, e_3, e_5] = [e_1, e_3, e_4] = e_7$
$A_{3,7,11}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6,$ $[e_2, e_3, e_6] = [e_1, e_3, e_5] = [e_1, e_2, e_4] = e_7$
$A_{3,7,13}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = e_7$
$A_{3,7,14}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_6] = [e_3, e_4, e_5] = e_7$
$A_{3,7,15}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = [e_1, e_3, e_4] = e_7$
$A_{3,7,16}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = [e_1, e_2, e_4] = e_7$
$A_{3,7,17}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = [e_1, e_3, e_5] = e_6,$ $[e_2, e_3, e_6] = -[e_3, e_4, e_5] = e_7$
$A_{3,7,21}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6,$ $[e_1, e_3, e_5] = [e_2, e_3, e_6] = e_7$
$A_{3,7,22}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6,$ $[e_1, e_2, e_4] = [e_1, e_3, e_5] = [e_2, e_3, e_6] = e_7$
$A_{3,7,23}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_6] = e_7$
$A_{3,7,24}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6,$ $[e_2, e_3, e_6] = [e_1, e_3, e_4] = e_7$
$A_{3,7,25}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6,$ $[e_3, e_4, e_5] = [e_1, e_3, e_6] = e_7$
$A_{3,7,26}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = [e_1, e_3, e_4] = e_6,$ $[e_3, e_4, e_5] = [e_1, e_3, e_6] = e_7$
$A_{3,7,27}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = [e_1, e_3, e_4] = e_6,$ $[e_1, e_3, e_5] = [e_2, e_3, e_6] = e_7$

For the ideal $\langle e_{d+1} \rangle$ of B, we have $\frac{B}{\langle e_{d+1} \rangle} = A$ and $B^2 \cap \langle e_{d+1} \rangle = A^2 \cap \langle e_{d+1} \rangle = 0$. Since, for the d-dimensional nilpotent n-Lie algebra A, $\dim A^2 \leq d - n$, we have $\binom{d-\dim A^2}{n-1} \geq n$. Thus, $\dim \mathcal{M}(B) > \dim \mathcal{M}(\frac{B}{\langle e_{d+1} \rangle}) - \dim(B^2 \cap \langle e_{d+1} \rangle)$.

Table 4. The structure of 7-dimensional indecomposable nilpotent 3-Lie algebras with $\dim Z(A) = 2$ over a field $\mathcal{K}$ with $\mathrm{char}\mathcal{K} \neq 2$.

Name	Non-zero multiplication
$A_{3,7,1}$	$[e_1, e_2, e_3] = e_6$, $[e_3, e_4, e_5] = e_7$
$A_{3,7,6}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_6$, $[e_1, e_3, e_4] = [e_2, e_3, e_5] = e_7$
$A_{3,7,7}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_6$, $[e_2, e_3, e_5] = e_7$
$A_{3,7,8}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_6$, $[e_1, e_3, e_5] = e_7$
$A_{3,7,12}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_5] = e_6$, $[e_2, e_3, e_4] = [e_1, e_3, e_5] = e_7$
$A_{3,7,19}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_5$, $[e_1, e_3, e_4] = e_6$, $[e_2, e_3, e_5] = e_7$
$A_{3,7,20}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_5$, $[e_1, e_3, e_4] = e_6$, $[e_1, e_2, e_4] = [e_2, e_3, e_5] = e_7$

Table 5. The structure of 7-dimensional indecomposable nilpotent 3-Lie algebras with $\dim Z(A) = 3$ over a field $\mathcal{K}$ with $\mathrm{char}\mathcal{K} \neq 2$.

Name	Non-zero multiplication
$A_{3,7,2}$	$[e_1, e_2, e_3] = e_4$, $[e_3, e_4, e_5] = e_6$, $[e_1, e_3, e_5] = e_7$
$A_{3,7,18}$	$[e_1, e_2, e_3] = e_4$, $[e_2, e_3, e_4] = e_5$, $[e_1, e_3, e_4] = e_6$, $[e_1, e_2, e_4] = e_7$

Finally, for the ideal $I \oplus \langle e_{d+1} \rangle$ of B, we have $\frac{B}{I \oplus \langle e_{d+1} \rangle} = \frac{A}{I}$ and $B^2 \cap (I \oplus \langle e_{d+1} \rangle) = A^2 \cap I$. Thus,

$$\dim \mathcal{M}(B) > \dim \mathcal{M}(\tfrac{B}{I \oplus \langle e_{d+1} \rangle}) - \dim(B^2 \cap (I \oplus \langle e_{d+1} \rangle)).$$

Therefore B is capable by Lemma 2.1. □

Theorem 2.7. *Let A be a 7-dimensional indecomposable nilpotent 3-Lie algebra with $\dim Z(A) = 1$. Then A is isomorphic to one of the algebras $A_{3,7,3}$, $A_{3,7,4}$, $A_{3,7,5}$, $A_{3,7,9}$, $A_{3,7,10}$, $A_{3,7,11}$, $A_{3,7,13}$, $A_{3,7,14}$, $A_{3,7,15}$, $A_{3,7,16}$, $A_{3,7,17}$, $A_{3,7,21}$, $A_{3,7,22}$, $A_{3,7,23}$, $A_{3,7,24}$, $A_{3,7,25}$, $A_{3,7,26}$, or $A_{3,7,27}$.*

Proof. By Theorem 2.5, the 7-dimensional indecomposable nilpotent 3-Lie algebra with $\dim Z(A) = 1$ is estimated.

□

Theorem 2.8. *Let A be a 7-dimensional indecomposable nilpotent 3-Lie algebra with $\dim Z(A) = 2$. Then A is isomorphic to one of the algebras $A_{3,7,1}$, $A_{3,7,6}$, $A_{3,7,7}$, $A_{3,7,8}$, $A_{3,7,12}$, $A_{3,7,19}$, or $A_{3,7,20}$.*

Proof. According to Theorem 2.5, the 7-dimensional indecomposable nilpotent 3-Lie algebra with $\dim Z(A) = 1$ is estimated. □

Lemma 2.9. *Let A be a 7-dimensional indecomposable nilpotent 3-Lie algebra with* $\dim Z(A) = 3$*. Then A is isomorphic to $A_{3,7,2}$ or $A_{3,7,18}$.*

Proof. By Theorem 2.5, it is clear. □

3. Main results

In this section, we characterize the capability of n-Lie algebras of dimension at most $n+3$ over an arbitrary field when $n > 2$ and the capability of 7-dimensional nilpotent 3-Lie algebras over the field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

First, we investigate the capability of nilpotent n-Lie algebras of dimension at most $n+3$ over an arbitrary field. By [11, Theorem 2.1], all d-dimensional nilpotent n-Lie algebras are non-capable when $d < n$.

Theorem 3.1. *All nilpotent n-Lie algebras of dimension n, $n+1$, $n+2$, and $n+3$ are capable when $n \geq 3$.*

Proof. By [11, Theorems 2.1 and 2.2], the algebras $F(n)$, $F(n+1)$, $F(n+2)$, $F(n+3)$, $H(n,1)$, $H(n,1) \oplus F(1)$, and $H(n,1) \oplus F(2)$ are capable. Since $\frac{A_{n,n+3,4}}{Z(A_{n,n+3,4})} \cong A_{n,n+2,1}$, the algebra $A_{n,n+2,1}$ is capable. By [15, Lemma 4.6], the algebra $A_{n,n+3,1}$ is capable.

We know that only central ideal of $A_{n,n+3,2}$, $A_{n,n+3,4}$, and $A_{n,n+3,5}$ is $Z(A) = \langle e_{n+3} \rangle$. By using Theorems 2.3 and 2.4, we get

$$\dim \mathcal{M}\left(\frac{A_{n,n+3,2}}{Z(A_{n,n+3,2})}\right) = 2n,$$

$$\dim \mathcal{M}\left(\frac{A_{n,n+3,4}}{Z(A_{n,n+3,4})}\right) = \dim \mathcal{M}\left(\frac{A_{n,n+3,5}}{Z(A_{n,n+3,5})}\right) = n.$$

So, by lemma 2.1, $A_{n,n+3,2}$, $A_{n,n+3,4}$, and $A_{n,n+3,5}$ are capable.

We know that only central ideals of $A_{n,n+3,3}$ are $\langle e_{n+2} \rangle$, $\langle e_{n+3} \rangle$, and $Z(A) = \langle e_{n+2}, e_{n+3} \rangle$.

By using Theorems 2.4 and 2.3, we have $\dim \mathcal{M}(A_{n,n+3,3}) = n+1$ and

$$\dim \mathcal{M}\left(\frac{A_{n,n+3,3}}{\langle e_{n+2} \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{n,n+3,3}}{\langle e_{n+3} \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{n,n+3,3}}{Z(A_{n,n+3,3})}\right) = n.$$

So, by lemma 2.1, $A_{n,n+3,3}$ is capable. □

We now investigate the capability of 7-dimensional nilpotent 3-Lie algebras over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

Table 6. The dimension of the Schur multiplier of 7-dimensional nilpotent 3-Lie algebras with $\dim Z(A) = 1$ over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$.

algebra A	$\dim \mathcal{M}(A)$	A/I	$\dim \mathcal{M}(A/I)$
$A_{3,7,3}, A_{3,7,4}, A_{3,7,5}$	12	$H(3,1) \oplus F(2)$	12
$A_{3,7,9}$	8	$A_{3,5,1} \oplus F(1)$	6
$A_{3,7,10}, A_{3,7,11}$	7	$A_{3,5,1} \oplus F(1)$	6
$A_{3,7,13}, A_{3,7,15}, A_{3,7,16}$	9	$A_{3,6,1}$	10
$A_{3,7,14}$	10	$A_{3,6,1}$	10
$A_{3,7,17}$	7	$A_{3,6,2}$	6
$A_{3,7,21}$	7	$A_{3,6,3}$	4
$A_{3,7,22}$	6	$A_{3,6,3}$	4
$A_{3,7,23}, A_{3,7,24}$	5	$A_{3,6,4}$	5
$A_{3,7,25}$	4	$A_{3,6,4}$	5
$A_{3,7,26}$	3	$A_{3,6,5}$	4
$A_{3,7,27}$	4	$A_{3,6,5}$	4

The dimension of the Schur multiplier of 7-dimensional indecomposable nilpotent 3-Lie algebras with $\dim Z(A) = 1$ over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$ is presented in Table 6.

The dimension of the Schur multiplier of 7-dimensional indecomposable nilpotent 3-Lie algebras with $\dim Z(A) = 2$ over a field $\mathcal{K}$ with char$\mathcal{K} \neq 2$ is presented in Table 7.

Lemma 3.2. *The 3-Lie algebras $A_{3,7,2}$ and $A_{3,7,18}$ are capable.*

Proof. We know that only central ideals of $A_{3,7,2}$ are $\langle e_4 \rangle$, $\langle e_6 \rangle$, $\langle e_7 \rangle$, $\langle e_4, e_6 \rangle$, $\langle e_4, e_7 \rangle$, $\langle e_6, e_7 \rangle$, and $\langle e_4, e_6, e_7 \rangle$.

By applying a similar method as in [10, Theorem 3.2], we get

$$\dim \mathcal{M}(A_{3,7,2}) = 15$$

and

$$\dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_4 \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_6 \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_7 \rangle}\right) = 10,$$

TABLE 7. The dimension of the Schur multiplier of 7-dimensional nilpotent 3-Lie algebras with $\dim Z(A) = 2$ over a field $\mathcal{K}$ with $\text{char}\mathcal{K} \neq 2$.

algebra A	$\dim \mathcal{M}(A)$	A/I	$\dim \mathcal{M}(A/I)$	$\dim(A^2 \cap I)$
$A_{3,7,1}$	15	$\frac{A}{\langle e_6 \rangle} \cong H(3,1) \oplus F(2)$	12	1
		$\frac{A}{\langle e_7 \rangle} \cong H(3,1) \oplus F(2)$	12	1
		$\frac{A}{\langle e_6, e_7 \rangle} \cong F(5)$	10	2
$A_{3,7,6}$	7	$\frac{A}{\langle e_6 \rangle} \cong A_{3,6,2}$	6	1
		$\frac{A}{\langle e_7 \rangle} \cong A_{3,5,1} \oplus F(1)$	6	1
		$\frac{A}{\langle e_6, e_7 \rangle} \cong H(3,1) \oplus F(1)$	6	2
$A_{3,7,7}, A_{3,7,8}$	10	$\frac{A}{\langle e_6 \rangle} \cong A_{3,6,1}$	10	1
		$\frac{A}{\langle e_7 \rangle} \cong A_{3,5,1} \oplus F(1)$	6	1
		$\frac{A}{\langle e_6, e_7 \rangle} \cong H(3,1) \oplus F(1)$	6	2
$A_{3,7,12}$	10	$\frac{A}{\langle e_6 \rangle} \cong A_{3,6,2}$	6	1
		$\frac{A}{\langle e_7 \rangle} \cong A_{3,6,1}$	10	1
		$\frac{A}{\langle e_6, e_7 \rangle} \cong H(3,1) \oplus F(1)$	6	2
$A_{3,7,19}$, $A_{3,7,20}$	6	$\frac{A}{\langle e_6 \rangle} \cong A_{3,6,5}$	4	1
		$\frac{A}{\langle e_7 \rangle} \cong A_{3,6,3}$	4	1
		$\frac{A}{\langle e_6, e_7 \rangle} \cong A_{3,5,1}$	3	2

$$\dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_4, e_6 \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_4, e_7 \rangle}\right) = \dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_6, e_7 \rangle}\right) = 6,$$

and $\dim \mathcal{M}\left(\frac{A_{3,7,2}}{\langle e_4, e_6, e_7 \rangle}\right) = 4$. Thus, by Lemma 2.1, $A_{3,7,2}$ is capable. Similarly, we conclude that $A_{3,7,18}$ is capable. □

Theorem 3.3. *The only non-capable* 7*-dimensional nilpotent* 3*-Lie algebras are* $H(3,2)$, $A_{3,7,13}$, $A_{3,7,15}$, $A_{3,7,16}$, $A_{3,7,25}$, *and* $A_{3,7,26}$.

Proof. By Theorem 2.5, the 7-dimensional nilpotent 3-Lie algebras are $F(7)$, $H(3,1) \oplus F(3)$, $H(3,2)$, $A_{3,5,1} \oplus F(2)$, $A_{3,6,i} \oplus F(1)$ and $A_{3,7,j}$, where $1 \leq i \leq 5$ and $1 \leq j \leq 27$.

By [11, Theorems 2.1 and 2.2], the algebras $F(7)$ and $H(3,1) \oplus F(3)$ are capable, and $H(3,2)$ is non-capable.

According to Theorem 3.1, the algebras $A_{3,5,1} \oplus F(1)$ and $A_{3,6,i}$, $1 \leq i \leq 5$ are capable. Thus, $A_{3,5,1} \oplus F(2)$ and $A_{3,6,i} \oplus F(1)$, $1 \leq i \leq 5$ are capable by Lemma 2.6. The algebras $A_{3,7,2}$ and $A_{3,7,18}$ are capable, by Lemma 3.2.

According to Table 7 and Lemma 2.1, the algebras $A_{3,7,1}$, $A_{3,7,6}$, $A_{3,7,7}$, $A_{3,7,8}$, $A_{3,7,12}$, $A_{3,7,19}$, and $A_{3,7,20}$ are capable. Also, by Table 6 and Lemma 2.1, the algebras $A_{3,7,3}$, $A_{3,7,4}$, $A_{3,7,5}$, $A_{3,7,9}$, $A_{3,7,10}$, $A_{3,7,11}$, $A_{3,7,14}$, $A_{3,7,17}$, $A_{3,7,21}$, $A_{3,7,22}$, $A_{3,7,23}$, $A_{3,7,24}$, and $A_{3,7,27}$ are capable and the algebras $A_{3,7,13}$, $A_{3,7,15}$, $A_{3,7,16}$, $A_{3,7,25}$, and $A_{3,7,26}$ are non-capable. □

Due to the fact that the dimension of the multiplier of nilpotent algebras plays an essential role in this article, we dedicate the last discussion of this section to the method of calculating the dimension of the multiplier of algebras.

The multiplier for Lie algebras was defined in [19]. It is obvious that the multiplier of a Lie algebra is Abelian, and therefore, only the dimension of the multiplier of a Lie algebra is important. The method of calculating the dimension of the multiplier of a Lie algebra is given in [6]. The method of calculating the dimension of the multiplier of n-Lie algebras is similar to Lie algebras and is given in [10]. In what follows, as an example, we calculate the dimension of the multiplier of $A_{3,7,26}$. The structure of this algebra is

$$\begin{aligned} A_{3,7,26} = \langle x_1, \ldots, x_7 : &[x_1, x_2, x_3] = x_4, [x_2, x_3, x_4] = x_5, \\ &[x_2, x_3, x_5] = [x_1, x_3, x_4] = x_6, \\ &[x_3, x_4, x_5] = [x_1, x_3, x_6] = x_7\rangle. \end{aligned}$$

We use the method of [6]. Start with

$$\begin{aligned}
&[x_1, x_2, x_3] = x_4 + s_1, && [x_1, x_2, x_4] = s_2, && [x_1, x_2, x_5] = s_3, \\
&[x_1, x_2, x_6] = s_4, && [x_1, x_2, x_7] = s_5, && [x_1, x_3, x_4] = x_7 + s_6, \\
&[x_1, x_3, x_5] = s_7, && [x_1, x_3, x_6] = x_7 + s_8, && [x_1, x_3, x_7] = s_9, \\
&[x_1, x_4, x_5] = s_{10}, && [x_1, x_4, x_6] = s_{11}, && [x_1, x_4, x_7] = s_{12}, \\
&[x_1, x_5, x_6] = s_{13}, && [x_1, x_5, x_7] = s_{14}, && [x_1, x_6, x_7] = s_{15}, \\
&[x_2, x_3, x_4] = x_5 + s_{16}, && [x_2, x_3, x_5] = x_6 + s_{17}, && [x_2, x_3, x_6] = s_{18}, \\
&[x_2, x_3, x_7] = s_{19}, && [x_2, x_4, x_5] = s_{20}, && [x_2, x_4, x_6] = s_{21}, \\
&[x_2, x_4, x_7] = s_{22}, && [x_2, x_5, x_6] = s_{23}, && [x_2, x_5, x_7] = s_{24}, \\
&[x_2, x_6, x_7] = s_{25}, && [x_3, x_4, x_5] = x_7 + s_{26}, && [x_3, x_4, x_6] = s_{27}, \\
&[x_3, x_4, x_7] = s_{28}, && [x_3, x_5, x_6] = s_{29}, && [x_3, x_5, x_7] = s_{30},
\end{aligned}$$

$$[x_3, x_6, x_7] = s_{31}, \quad [x_4, x_5, x_6] = s_{32}, \quad [x_4, x_5, x_7] = s_{33},$$
$$[x_4, x_6, x_7] = s_{34}, \quad [x_5, x_6, x_7] = s_{35},$$

where $s_1, \ldots, s_{35}$ generate $\mathcal{M}(A_{3,7,26})$. Use of the Jacobi identity on all possible triples shows that

$$\begin{aligned}
s_7 - s_{18} =& [[x_2, x_3, x_4], x_1, x_3] - [[x_1, x_3, x_4], x_2, x_3] \\
=& [x_2, x_3, [x_4, x_1, x_3]] - [[x_1, x_3, x_4], x_2, x_3] \\
=& 0, \\
s_8 - s_{26} =& [[x_2, x_3, x_5], x_1, x_3] + [[x_1, x_2, x_3], x_3, x_5] \\
=& [[x_2, x_1, x_3], x_3, x_5] + [[x_1, x_2, x_3], x_3, x_5] \\
=& 0, \\
s_{19} - s_{27} =& [[x_3, x_4, x_5], x_2, x_3] - [[x_2, x_3, x_5], x_3, x_4] \\
=& [x_3, x_4, [x_5, x_2, x_3]] - [[x_2, x_3, x_5], x_3, x_4] \\
=& 0, \\
s_{19} + s_{27} =& [[x_1, x_3, x_6], x_2, x_3] - [[x_1, x_2, x_3], x_3, x_6] \\
=& [[x_1, x_2, x_3], x_3, x_6] - [[x_1, x_2, x_3], x_3, x_6] \\
=& 0, \\
s_3 =& [[x_2, x_3, x_4], x_1, x_2] \\
=& [[x_2, x_1, x_2], x_3, x_4] + [x_2, [x_3, x_1, x_2], x_4] \\
& + [x_2, x_3, [x_4, x_1, x_2]] \\
=& 0, \\
s_4 =& [[x_1, x_3, x_4], x_1, x_2] \\
=& [[x_1, x_1, x_2], x_3, x_4] + [x_1, [x_3, x_1, x_2], x_4] \\
& + [x_1, x_3, [x_4, x_1, x_2]] \\
=& 0, \\
s_5 =& [[x_3, x_4, x_5], x_1, x_2] \\
=& [[x_3, x_1, x_2], x_4, x_5] + [x_3, [x_4, x_1, x_2], x_5] \\
& + [x_3, x_4, [x_5, x_1, x_2]] \\
=& 0, \\
s_9 =& [[x_3, x_4, x_5], x_1, x_3] \\
=& [[x_3, x_1, x_3], x_4, x_5] + [x_3, [x_4, x_1, x_3], x_5]
\end{aligned}$$

$$
\begin{aligned}
&+[x_3,x_4,[x_5,x_1,x_3]]\\
=&0,\\
s_{10}=&-[[x_1,x_2,x_3],x_1,x_5]\\
=&-[[x_1,x_1,x_5],x_2,x_3]-[x_1,[x_2,x_1,x_5],x_3]\\
&-[x_1,x_2,[x_3,x_1,x_5]]\\
=&0,\\
s_{11}=&-[[x_1,x_2,x_3],x_1,x_6]\\
=&-[[x_1,x_1,x_6],x_2,x_3]-[x_1,[x_2,x_1,x_6],x_3]\\
&-[x_1,x_2,[x_3,x_1,x_6]]\\
=&0,\\
s_{12}=&-[[x_1,x_2,x_3],x_1,x_7]\\
=&-[[x_1,x_1,x_7],x_2,x_3]-[x_1,[x_2,x_1,x_7],x_3]\\
&-[x_1,x_2,[x_3,x_1,x_7]]\\
=&0,\\
s_{13}=&[[x_2,x_3,x_5],x_1,x_5]\\
=&[[x_2,x_1,x_5],x_3,x_5]+[x_2,[x_3,x_1,x_5],x_5]\\
&+[x_2,x_3,[x_5,x_1,x_5]]\\
=&0,\\
s_{14}=&[[x_1,x_3,x_6],x_1,x_5]\\
=&[[x_1,x_1,x_5],x_3,x_6]+[x_1,[x_3,x_1,x_5],x_6]\\
&+[x_1,x_3,[x_6,x_1,x_5]]\\
=&0,\\
s_{15}=&-[[x_2,x_3,x_5],x_1,x_7]\\
=&-[[x_2,x_1,x_7],x_3,x_5]-[x_2,[x_3,x_1,x_7],x_5]\\
&-[x_2,x_3,[x_5,x_1,x_7]]\\
=&0,\\
s_{20}=&-[[x_1,x_2,x_3],x_2,x_5]\\
=&-[[x_1,x_2,x_5],x_2,x_3]-[x_1,[x_2,x_2,x_5],x_3]\\
&-[x_1,x_2,[x_3,x_2,x_5]]\\
=&0,
\end{aligned}
$$

$$
\begin{aligned}
s_{21} =& -[[x_1,x_2,x_3],x_2,x_6]\\
=& -[[x_1,x_2,x_6],x_2,x_3]-[x_1,[x_2,x_2,x_6],x_3]\\
& -[x_1,x_2,[x_3,x_2,x_6]]\\
=& 0,\\
s_{22} =& -[[x_1,x_2,x_3],x_2,x_7]\\
=& -[[x_1,x_2,x_7],x_4,x_5]-[x_1,[x_2,x_2,x_7],x_3]\\
& -[x_1,x_2,[x_3,x_2,x_7]]\\
=& 0,\\
s_{23} =& -[[x_2,x_3,x_4],x_2,x_6]\\
=& -[[x_2,x_2,x_6],x_3,x_4]-[x_2,[x_3,x_2,x_6],x_4]\\
& -[x_2,x_3,[x_4,x_2,x_6]]\\
=& 0,\\
s_{24} =& -[[x_2,x_3,x_4],x_2,x_7]\\
=& -[[x_2,x_2,x_7],x_3,x_4]-[x_2,[x_3,x_2,x_7],x_4]\\
& -[x_2,x_3,[x_4,x_2,x_7]]\\
=& 0,\\
s_{25} =& -[[x_2,x_3,x_5],x_2,x_7]\\
=& -[[x_2,x_2,x_7],x_3,x_5]-[x_2,[x_3,x_2,x_7],x_5]\\
& -[x_2,x_3,[x_5,x_2,x_7]]\\
=& 0,\\
s_{28} =& -[[x_1,x_2,x_3],x_3,x_7]\\
=& -[[x_1,x_3,x_7],x_2,x_3]-[x_1,[x_2,x_3,x_7],x_3]\\
& -[x_1,x_2,[x_3,x_3,x_7]]\\
=& 0,\\
s_{29} =& -[[x_2,x_3,x_4],x_3,x_6]\\
=& -[[x_2,x_3,x_6],x_3,x_4]-[x_2,[x_3,x_3,x_6],x_4]\\
=& -[x_2,x_3,[x_4,x_3,x_6]]\\
=& 0,\\
s_{30} =& -[[x_2,x_3,x_4],x_3,x_7]\\
=& -[[x_2,x_3,x_7],x_3,x_4]-[x_2,[x_3,x_3,x_7],x_4]
\end{aligned}
$$

$$
\begin{aligned}
&- [x_2, x_3, [x_4, x_3, x_7]] \\
=&0, \\
s_{31} =&- [[x_2, x_3, x_5], x_3, x_7] \\
=&- [[x_2, x_3, x_7], x_3, x_5] - [x_2, [x_3, x_3, x_7], x_5] \\
&- [x_2, x_3, [x_5, x_3, x_7]] \\
=&0, \\
s_{32} =&[[x_1, x_2, x_3], x_5, x_6] \\
=&[[x_1, x_5, x_6], x_2, x_3] + [x_1, [x_2, x_5, x_6], x_3] \\
&+ [x_1, x_2, [x_3, x_5, x_6]] \\
=&0, \\
s_{33} =&[[x_1, x_2, x_3], x_5, x_7] \\
=&[[x_1, x_5, x_7], x_2, x_3] + [x_1, [x_2, x_5, x_7], x_3] \\
&+ [x_1, x_2, [x_3, x_5, x_7]] \\
=&0, \\
s_{34} =&[[x_1, x_2, x_3], x_6, x_7] \\
=&[[x_1, x_6, x_7], x_2, x_3] + [x_1, [x_2, x_6, x_7], x_3] \\
&+ [x_1, x_2, [x_3, x_6, x_7]] \\
=&0, \\
s_{35} =&[[x_2, x_3, x_4], x_6, x_7] \\
=&[[x_2, x_6, x_7], x_3, x_4] + [x_2, [x_3, x_6, x_7], x_4] \\
&+ [x_2, x_3, [x_4, x_6, x_7]] \\
=&0.
\end{aligned}
$$

We set $x'_4 = x_4 + s_1$, $x'_5 = x_5 + s_{16}$, $x'_6 = x_6 + s_6$, and $x'_7 = x_7 + s_8$. A change of variables allows us $s_1 = s_{16} = s_6 = s_8 = 0$. As $s_{19} - s_{27} = 0$, $s_{19} + s_{27} = 0$, and the characteristic of the field is not equal 2, we have $s_{19} = s_{27} = 0$. Thus $\mathcal{M}(A_{3,7,26}) = \langle s_2, s_7, s_{17} \rangle$ and $\dim(\mathcal{M}(A_{3,7,26})) = 3$.

References

1. F. Bagarello and F. G. Russo, A description of pseudo-bosons in terms of nilpotent Lie algebras, *J. Geom. Phys.*, **125** (2018), 1–11.
2. F. Bagarello and F. G. Russo, On the presence of families of pseudo-bosons in nilpotent Lie algebras of arbitrary corank, *J. Geom. Phys.*, **137** (2019), 124–131.

3. F. Bagarello and F. G. Russo, Realization of Lie algebras of high dimension via pseudo-bosonic operators, *J. Lie Theory*, **30** (2020), 925–938.
4. R. Bai, G. Song and Y. Zhang, On classification of n-Lie algebras, *Front. Math. China*, **6**(4) (2011), 581–606.
5. R. Bai, X. Wang, W. Xiao and H. An, Structure of low dimensional n-Lie algebras over a field of characteristic 2, *Linear Algebra Appl.*, **428** (2008), 1912–1920.
6. P. Batten, K. Moneyhun and E. Stitzinger, On characterizing nilpotent Lie algebras by their multipliers, *Comm. Algebra*, **24** (1996), 4319–4330.
7. H. Darabi, The algebraic classification of 7-dimensional nilpotent 3-Lie algebras, *J. Algebra Relat. Topics*, (2026), Article in Press.
8. H. Darabi and M. Imanparast, On classification of 9-dimensional nilpotent 3-ary algebras of class two, *Bull. Iranian Math. Soc.*, **47** (2021), 929–937.
9. H. Darabi and F. Saeedi, On the Schur multiplier of n-Lie algebras, *J. Lie Theory*, **27** (2017), 271–281.
10. H. Darabi, F. Saeedi and M. Eshrati, A characterization of finite dimensional nilpotent Filippov algebras, *J. Geom. Phys.*, **101** (2016), 100–107.
11. H. Darabi, F. Saeedi and M. Eshrati, Capable n-Lie algebras and the classification of nilpotent n-Lie algebras with $s(A) = 3$, *J. Geom. Phys.*, **110** (2016), 25–29.
12. M. Eshrati, F. Saeedi and H. Darabi, On the multiplier of nilpotent n-Lie algebras, *J. Algebra*, **450** (2016), 162–172.
13. V. T. Filippov, n-Lie algebras, *Sib. Math. Zh.*, **26**(6) (1985), 126–140.
14. Z. Hoseini, F. Saeedi and H. Darabi, Characterization of capable nilpotent n-Lie algebras of class two by their Schur multipliers, *Rend. Circ. Mat. Palermo. (2)*, **68** (2019), 541–555.
15. Z. Hoseini, F. Saeedi and H. Darabi, On Classification of $(n+5)$-dimensional nilpotent n-Lie algebras of class two, *Bull. Iranian Math. Soc.*, **45**(4) (2019), 939–949.
16. M. Jamshidi, F. Saeedi and H. Darabi, Classification of $(n+6)$-dimensional nilpotent n-Lie algebras of class two with $n \geq 4$, *Arab J. Math. Sci.*, **27**(2) (2021), 139–150.
17. S. M. Kasymov, Theory of n-Lie algebras, *Algebra Logika*, **26**(3) (1987), 277–297.
18. I. Kaygorodov and Y. Volkov, Degenerations of Filippov algebras, *J. Math. Phys.*, **61**(2), (2020), Article ID: 021701.
19. K. Moneyhun, Isoclinisms in Lie algebras, *Algebras Groups Geom.*, **11** (1994), 9–22.
20. J. C. Ndogmo and P. Winternitz, Solvable Lie algebras with abelian nilradical, *J. Phys. A: Math. Gen.*, **27** (1994), 405–423.
21. P. Niroomand and F. G. Russo, A note on the Schur multiplier of nilpotent Lie algebra, *Comm. Algebra*, **39** (2011), 1293–1297.
22. A. P. Pozhidaev, On simple n-Lie algebras, *Algebra Logic*, **38**(3) (1999), 181–192.
23. F. P. Shanbehbazari, P. Niroomand, F. G. Russo and A. Shamsaki, Capability of nilpotent Lie algebras of small dimension, *Bull. Iranian Math. Soc.*, **48** (2022), 1153–1167.
24. L. Snobl and P. Winternitz, *Classification and Identification of Lie Algebras*, CRM Monograph Series, 33, American Mathematical Society, Providence, RI, 2014.

Hamid Darabi
Department of Mathematics, Esfarayen University of Technology, Esfarayen, Iran.
Email: darabi@esfarayen.ac.ir

Journal of Algebraic Systems

CAPABILITY OF LOW DIMENSIONAL NILPOTENT 3-LIE ALGEBRAS

H. DARABI

توانایی ۳-جبرهای لی پوچتوان از بعد پایین

حمید دارابی

گروه ریاضی، مجتمع آموزش عالی فنی و مهندسی اسفراین، اسفراین، ایران

در این مقاله، توانایی n-جبرهای لی پوچتوان از بعد حداکثر ۳ + n روی میدان دلخواه وقتی ۲ < n است و توانایی ۳-جبرهای لی پوچتوان ۷ بعدی روی میدان $\mathcal{K}$ با ۲ ≠ $\mathrm{char}\mathcal{K}$ را مشخص می‌کنیم.

کلمات کلیدی: n-جبر لی توانا، n-جبر لی پوچتوان، ضربگر.