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$K-bi-g-$FRAMES IN HILBERT SPACES

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Abstract. This paper will introduce the new concept of K-bi-$g-$frames for Hilbert spaces. Then we examine some characterizations with the help of a biframe operator. Finally, we investigate several results about the stability of K-bi-g-frames produced using frame theory methods.

1. Introduction

Duffin and Schaffer introduced the notion of frames in Hilbert spaces [8] in 1952 to research certain difficult nonharmonic Fourier series problems. Following the fundamental paper [6] by Daubechies, Grossman, and Meyer, frame theory started to become popular, especially in the more specific context of Gabor frames and wavelet frames [11]. A sequence $\{\Phi_i\}_{i\in I}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, \Phi_i\rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

For more detailed information on frame theory, readers are recommended to consult: [2, 4, 5, 13, 14, 16, 21, 20, 19].

The concept of K-frames was introduced by Laura Găvruţa [12] and serves as a tool for investigating atomic systems for a bounded linear operator K in a separable Hilbert space. A sequence $\{\Phi_i\}_{i\in I}$ in $\mathcal{H}$ is called a K-frame for $\mathcal{H}$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|K^*x\|^2 \leq \sum_{i\in I} |\langle x, \Phi_i\rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

The notion of K-frames generalizes ordinary frames in that the lower frame bound applies only to elements within the range of K. After that, Xiao et al. [23] introduced the concept of a K-g-frame, which is a more general framework than both g-frames and K-frames in Hilbert spaces.

The idea of pair frames, which refers to a pair of sequences in a Hilbert space, was first presented in [9] by Fereydooni and Safapour. Parizi, Alijani,

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and Dehghan [10] studied biframe, which is a generalization of a controlled frame in Hilbert space. The concept of a frame is defined by a single sequence, but to define a biframe we will need two sequences. The concept of a biframe is a generalization of the controlled frames and a special case of pair frames.

This paper will introduce the concept of K-bi-g-frames in Hilbert space and present some examples of this type of frame. Moreover, we investigate a characterization of K-bi-g-frames by using the biframe operator. Finally, in our exploration of biframes, we investigate some results about the stability of K-bi-g-frames produced via the use of frame theory.

2. Notation and preliminaries

Throughout this paper, $\mathcal{H}$ represents a separable Hilbert space. The notation $\mathcal{B}(\mathcal{H},\mathcal{K})$ denotes the collection of all bounded linear operators from $\mathcal{H}$ to the Hilbert space $\mathcal{K}$. When $\mathcal{H}=\mathcal{K}$, this set is denoted simply as $\mathcal{B}(\mathcal{H})$. We will use $\mathcal{N}(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$ to denote the null and range space of an operator $\mathcal{T}\in\mathcal{B}(\mathcal{H})$. Also, $\mathrm{GL}(\mathcal{H})$ is the collection of all invertible, bounded linear operators acting on $\mathcal{H}$. Let $\{\mathcal{K}_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{H}$, where I is a finite or countable index set. $\ell^2\left(\{\mathcal{K}_i\}_{i\in I}\right)$ is defined by

$$\ell^2\left(\{\mathcal{K}_i\}_{i\in I}\right)=\left\{\{x_i\}_{i\in I}: x_i\in\mathcal{K}_i,\quad i\in I,\quad \sum_{i\in I}\|x_i\|^2<+\infty\right\},$$

with the inner product

$$\left\langle\{x_i\}_{i\in I},\{y_i\}_{i\in I}\right\rangle=\sum_{i\in I}\langle x_i,y_i\rangle .$$

Certainly, let's begin with some preliminaries. Before diving into the details, let's briefly recall the definition of a biframe:

Definition 2.1. [10] A pair $(\Phi,\Psi)=\left(\{\Phi_i\}_{i\in I},\{\Psi_i\}_{i\in I}\right)$ in $\mathcal{H}$ is called a biframe for $\mathcal{H}$ if there exist two constants $0<A\leq B<\infty$ such that

$$A\|x\|^2\leq\sum_{i\in I}\langle x,\Phi_i\rangle\langle\Psi_i,x\rangle\leq B\|x\|^2,\ \text{for all } x\in\mathcal{H}.$$

Theorem 2.2. [1] *$\mathcal{T}\in\mathcal{B}(\mathcal{H})$ is an injective and closed range operator if and only if there exists a constant $c>0$ such that $c\|x\|^2\leq\|\mathcal{T}x\|^2$, for all $x\in\mathcal{H}$.*

Definition 2.3. [15] Let $\mathcal{H}$ be a Hilbert space, and suppose that $\mathcal{T}\in\mathcal{B}(\mathcal{H})$ has a closed range. Then there exists an operator $\mathcal{T}^\dagger\in\mathcal{B}(\mathcal{H})$ for which

$$N\left(\mathcal{T}^\dagger\right)=\mathcal{R}(\mathcal{T})^\perp,\quad R\left(\mathcal{T}^\dagger\right)=N(\mathcal{T})^\perp,\quad \mathcal{T}\mathcal{T}^\dagger x=x,\quad x\in\mathcal{R}(\mathcal{T}).$$

We call the operator $\mathcal{T}^\dagger$ the pseudo-inverse of $\mathcal{T}$. This operator is uniquely determined by these properties. In fact, if $\mathcal{T}$ is invertible, then we have $\mathcal{T}^{-1} = \mathcal{T}^\dagger$.

Theorem 2.4. [7] *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent:*

(1) $\mathcal{R}(\mathcal{T}_1) \subset \mathcal{R}(\mathcal{T}_2)$;
(2) $\mathcal{T}_1\mathcal{T}_1^* \leq \lambda^2 \mathcal{T}_2\mathcal{T}_2^*$ *for some* $\lambda \geq 0$;
(3) $\mathcal{T}_1 = \mathcal{T}_2 U$ *for some* $U \in \mathcal{B}(\mathcal{H})$.

Lemma 2.5. [3] *Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a linear operator, and assume that there exist constants $\alpha, \beta \in [0; 1)$ such that $\|\mathcal{T}x - x\| \leq \alpha\|x\| + \beta\|\mathcal{T}x\|$, for all $x \in \mathcal{H}$. Then $\mathcal{T} \in \mathcal{B}(\mathcal{H})$, and*

$$\frac{1-\alpha}{1+\beta}\|x\| \leq \|\mathcal{T}x\| \leq \frac{1+\alpha}{1-\beta}\|x\|, \ \frac{1-\beta}{1+\alpha}\|x\| \leq \left\|\mathcal{T}^{-1}x\right\| \leq \frac{1+\beta}{1-\alpha}\|x\|, \forall x \in \mathcal{H}.$$

3. K-bi-g-frames in Hilbert spaces

In this section, we introduce the concept of a K-bi-g-frame and subsequently establish some of its properties. However, before proceeding, we first define the notions of a g-frame and bi-g-frame in Hilbert spaces. Throughout the remainder of this part (sections 3,4 and 5), we denote:

$$(\Phi, \Psi)_K = \left(\{\Phi_i \ : \ \Phi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}, \{\Psi_i \ : \ \Psi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}\right)$$

Definition 3.1. [22] A sequence $\{\Phi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}$ is called a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if there are two positive constants A and B such that

$$A\|x\|^2 \leqslant \sum_{j\in I} \|\Phi_i x\|^2 \leqslant B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

The constants A and B are called the lower and upper g-frame bounds, respectively.

Definition 3.2. [18] A pair $(\Phi, \Psi)_K$ of sequences is called a bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{\in I}$, if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} \langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Definition 3.3. Let $K \in \mathcal{B}(\mathcal{H})$. A pair $(\Phi, \Psi)_K$ of sequences is called a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{\in I}$, if there exist two constants $0 < A \leq B < \infty$ such that

$$A \left\|K^* x\right\|^2 \leq \sum_{i \in I} \langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

The numbers A and B are called respectively the lower and upper bounds for the K-bi-g-frames $(\Phi, \Psi)_K$ respectively. If K is equal to $\mathcal{I}_{\mathcal{H}}$, the identity operator on $\mathcal{H}$, then K-bi-g-frames is bi-g-frames.

Let $(\Phi, \Psi) = \left(\{\Phi_i\}_{i \in I}, \{\Psi_i\}_{i \in I}\right)$ be a bi-g-frame for $\mathcal{H}$. We define the bi-g-frame operator $S_{\Phi,\Psi}$ as follows:

$$S_{\Phi,\Psi} : \mathcal{H} \longrightarrow \mathcal{H}, \quad S_{\Phi,\Psi}(x) := \sum_{i \in I} \Psi_i^* \Phi_i x.$$

Definition 3.4. [17] Let $C, C' \in GL(\mathcal{H})$. The family

$$\Phi = \{\Phi_i \in \mathcal{B}\left(\mathcal{H}, \mathcal{K}_i\right) : i \in I\}$$

will be called a (C, C')-controlled g-frame for $\mathcal{H}$, if Φ is a g-Bessel sequence and there exists constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i \in I} \left\langle \Phi_i C f, \Phi_i C' f\right\rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

A and B will be called controlled frame bounds. If $C' = I$, we call $\Phi = \{\Phi_i\}$ a C-controlled g-frame for $\mathcal{H}$ with bounds A and B.

Remark 3.5. According to Definition 3.3, the following statements are true for a sequence $\Phi = \{\Phi_i \ : \ \Phi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{\in I}$:

(1) If (Φ, Φ) is a K-bi-g-frame for $\mathcal{H}$, then Φ is a K-g-frame for $\mathcal{H}$.
(2) If $(\Phi, C\Phi)$ is a K-bi-g-frame for some $C \in \mathrm{GL}(\mathcal{H})$, then Φ is a C-controlled K-g-frame for $\mathcal{H}$.
(3) If $(C_1\Phi, C_2\Phi)$ is a K-bi-g-frame for some C_1 and C_2 in $\mathrm{GL}(\mathcal{H})$, then Φ is a (C_1, C_2)-controlled K-g-frame for $\mathcal{H}$.

Example 3.6. Let $\mathcal{H} = \mathbb{C}^4$ and $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $\mathcal{H}$ and $\mathcal{K}_1 = \mathcal{K}_2 = \overline{\text{span}}\,\{e_1\}, \mathcal{K}_3 = \overline{\text{span}}\,\{e_3\}, \mathcal{K}_4 = \overline{\text{span}}\,\{e_4\}$. Define

$$K : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad Kx = \langle x, e_1\rangle\, e_2.$$

We consider two sequences, $\Phi = \{\Phi_i\}_{i=1}^4$ and $\Psi = \{\Psi_i\}_{i=1}^4$ defined as follows:

$$\begin{aligned}
&\Phi_1 : \mathcal{H} \to \mathcal{K}_1, \Phi_1 x = \langle x, e_1 \rangle e_1, && x \in \mathcal{H},\\
&\Phi_2 : \mathcal{H} \to \mathcal{K}_2, \Phi_2 x = \langle x, e_1 \rangle e_1, && x \in \mathcal{H},\\
&\Phi_3 : \mathcal{H} \to \mathcal{K}_3, \Phi_3 x = 3 \langle x, e_2 \rangle e_3, && x \in \mathcal{H},\\
&\Phi_4 : \mathcal{H} \to \mathcal{K}_4, \Phi_4 x = 4 \langle x, e_3 \rangle e_4, && x \in \mathcal{H}.
\end{aligned}$$

And

$$\begin{aligned}
&\Psi_1 : \mathcal{H} \to \mathcal{K}_1, \Psi_1 x = \langle x, e_1 \rangle e_1, && x \in \mathcal{H},\\
&\Psi_2 : \mathcal{H} \to \mathcal{K}_2, \Psi_2 x = \langle x, e_1 \rangle e_1, && x \in \mathcal{H},\\
&\Psi_3 : \mathcal{H} \to \mathcal{K}_3, \Psi_3 x = \frac{1}{3} \langle x, e_2 \rangle e_3, && x \in \mathcal{H},\\
&\Psi_4 : \mathcal{H} \to \mathcal{K}_4, \Psi_4 x = \frac{1}{4} \langle x, e_3 \rangle e_4, && x \in \mathcal{H}.
\end{aligned}$$

Next, we establish that $K^* x = \langle x, e_2 \rangle e_1, x \in \mathcal{H}$. Indeed, for any $x, y \in \mathcal{H}$, we obtain:

$$\begin{aligned}
\langle K^* x, y \rangle &= \langle x, Ky \rangle \\
&= \langle x, \langle y, e_1 \rangle e_2 \rangle \\
&= \langle x, e_2 \rangle \overline{\langle y, e_1 \rangle} \\
&= \langle x, e_2 \rangle \langle e_1, y \rangle \\
&= \langle \langle x, e_2 \rangle e_1, y \rangle
\end{aligned}$$

For $x \in \mathcal{H}$, we have $\sum_{i=1}^4 \langle \Phi_i x, \Psi_i x \rangle = 2 |\langle x, e_1 \rangle|^2 + |\langle x, e_2 \rangle|^2 + |\langle x, e_3 \rangle|^2$. Hence, for every $x \in \mathcal{H}$, we have

$$\begin{aligned}
\|K^* x\|^2 &= \|\langle x, e_2 \rangle e_1\|^2 \\
&= |\langle x, e_2 \rangle|^2 \\
&\leq 2 |\langle x, e_1 \rangle|^2 + |\langle x, e_2 \rangle|^2 + |\langle x, e_3 \rangle|^2 \\
&= \sum_{i=1}^4 \langle \Phi_i x, \Psi_i x \rangle \\
&\leq 2\|x\|^2.
\end{aligned}$$

Therefore, $(\Phi, \Psi)_K$ is a K-bi-g-frame with bounds 1 and 2.

Definition 3.7. Let $K \in \mathcal{B}(\mathcal{H})$. A pair $(\Phi, \Psi)_K$ of sequences in $\mathcal{H}$ is said to be a δ - tight K-bi-g-frame with bound A if

$$\delta \|K^*x\|^2 = \sum_{i\in I} \langle \Phi_i x, \Psi_i x\rangle, \text{ for all } x \in \mathcal{H}.$$

When $\delta = 1$, it is called a Parseval K-bi-g-frame.

Example 3.8. Let $\mathcal{H} = \mathbb{C}^4$ and $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $\mathcal{H}$, and $\mathcal{K}_1 = \overline{\text{span}}\{e_1\}, \mathcal{K}_2 = \overline{\text{span}}\{e_2\}, \mathcal{K}_3 = \overline{\text{span}}\{e_3\}, \mathcal{K}_4 = \overline{\text{span}}\{e_4\}$. Define

$$K : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad Kx = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3 + \langle x, e_4\rangle e_4.$$

We consider two sequences, $\Phi = \{\Phi_i\}_{i=1}^4$ and $\Psi = \{\Psi_i\}_{i=1}^4$ defined as follows:

$$\begin{aligned}
&\Phi_1 : \mathcal{H} \to \mathcal{K}_1, \Phi_1 x = \langle x, e_1\rangle e_1, \quad x \in \mathcal{H},\\
&\Phi_2 : \mathcal{H} \to \mathcal{K}_2, \Phi_2 x = 2\langle x, e_2\rangle e_2, \quad x \in \mathcal{H},\\
&\Phi_3 : \mathcal{H} \to \mathcal{K}_3, \Phi_3 x = 3\langle x, e_3\rangle e_3, \quad x \in \mathcal{H},\\
&\Phi_4 : \mathcal{H} \to \mathcal{K}_4, \Phi_4 x = 4\langle x, e_4\rangle e_4, \quad x \in \mathcal{H}.
\end{aligned}$$

And

$$\begin{aligned}
&\Psi_1 : \mathcal{H} \to \mathcal{K}_1, \Psi_1 x = \langle x, e_1\rangle e_1, \quad x \in \mathcal{H},\\
&\Psi_2 : \mathcal{H} \to \mathcal{K}_2, \Psi_2 x = \frac{1}{2}\langle x, e_2\rangle e_2, \quad x \in \mathcal{H},\\
&\Psi_3 : \mathcal{H} \to \mathcal{K}_3, \Psi_3 x = \frac{1}{3}\langle x, e_3\rangle e_3, \quad x \in \mathcal{H},\\
&\Psi_4 : \mathcal{H} \to \mathcal{K}_4, \Psi_4 x = \frac{1}{4}\langle x, e_4\rangle e_4, \quad x \in \mathcal{H}.
\end{aligned}$$

For $x \in \mathcal{H}$ we have,

$$K^*x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3 + \langle x, e_4\rangle e_4, \quad x \in \mathcal{H}.$$

Indeed, for any $x, y \in \mathcal{H}$, we obtain:

$$\begin{aligned}
\langle K^*x, y\rangle &= \langle x, Km\rangle\\
&= \langle x, \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 + \langle y, e_3\rangle e_3 + \langle y, e_4\rangle e_4)\rangle\\
&= \langle x, \langle y, e_1\rangle e_1\rangle + \langle x, \langle y, e_2\rangle e_2\rangle + \langle x, \langle y, e_3\rangle e_3\rangle + \langle x, \langle y, e_4\rangle e_4\rangle\\
&= \langle x, e_1\rangle \overline{\langle y, e_1\rangle} + \langle x, e_2\rangle \overline{\langle y, e_2\rangle} + \langle x, e_3\rangle \overline{\langle y, e_3\rangle} + \langle x, e_4\rangle \overline{\langle y, e_4\rangle}\\
&= \langle x, e_1\rangle \langle e_1, y\rangle + \langle x, e_2\rangle \langle e_2, y\rangle + \langle x, e_3\rangle \langle e_3, y\rangle + \langle x, e_4\rangle \langle e_4, y\rangle\\
&= \langle \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3 + \langle x, e_4\rangle e_4, y\rangle .
\end{aligned}$$

Also for $x \in \mathcal{H}$ we have,

$$\begin{aligned}\|K^*x\|^2 &= \|\langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3 + \langle x, e_4\rangle e_4\|^2 \\ &= |\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 + |\langle x, e_3\rangle|^2 + |\langle x, e_4\rangle|^2 \\ &= \sum_{i=1}^{4}\langle \Phi_i x, \Psi_i x\rangle.\end{aligned}$$

Therefore, $(\Phi, \Psi)_K$ is a Parseval K-bi-g-frame for $\mathcal{H}$.

Theorem 3.9. *$(\Phi, \Psi)_K$ is a K-bi-g-frame if and only if*

$$(\Psi, \Phi)_K = \left(\{\Psi_i\}_{i\in I}, \{\Phi_i\}_{i\in I}\right)$$

is a K-bi-g-frame.

Proof. Let $(\Phi, \Psi)_K$ be a K-bi-g-frame with bounds A and B. Then, for every $x \in \mathcal{H}$,

$$A\|K^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2.$$

Now, we can write

$$\begin{aligned}\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle &= \overline{\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle} \\ &= \sum_{i\in I}\overline{\langle \Phi_i x, \Psi_i x\rangle} \\ &= \sum_{i\in I}\langle \Psi_i x, \Phi_i x\rangle.\end{aligned}$$

Therefore, $A\|K^*x\|^2 \leq \sum_{i\in I}\langle \Psi_i x, \Phi_i x\rangle \leq B\|x\|^2$. This implies that, $(\Psi, \Phi)_K$ is a K-bi-g-frame with bounds A and B. The reverse of this statement can be proved similarly.. □

Theorem 3.10. *Let $K_1, K_2 \in \mathcal{B}(\mathcal{H})$. If $(\Phi, \Psi)_{K_j}$ is a K_j-bi-g-frame for $j \in \{1,\ 2\}$ and α_1, α_2 are scalars, then the following holds:*

(1) *$(\Phi, \Psi)_K$ is a $(\alpha_1 K_1 + \alpha_2 K_2)$-bi-$g$-frame.*
(2) *$(\Phi, \Psi)_K$ is a K_1K_2-bi-g-frame.*

Proof. (1) Let $(\Phi, \Psi)_{K_j}$ be a K_j-bi-g-frame for $j = 1, 2$. For $j = 1$, there exist two constants $0 < A \leq B < \infty$ such that

$$A\|K_1^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Similarly, for $j = 2$, there exist two constants $0 < C \leq D < \infty$ such that

$$C\left\|K_2^* x\right\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq D\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Now, we can write

$$\begin{aligned}\left\|(\alpha_1 K_1 + \alpha_2 K_2)^* x\right\|^2 &\leq |\alpha_1|^2\left\|K_1^* x\right\|^2 + |\alpha_2|^2\left\|K_2^* x\right\|^2 \\ &\leq |\alpha_1|^2\left(\frac{1}{A}\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle\right) + |\alpha_2|^2\left(\frac{1}{C}\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle\right) \\ &= \left(\frac{|\alpha_1|^2}{A} + \frac{|\alpha_2|^2}{C}\right)\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle.\end{aligned}$$

It follows that

$$\left(\frac{AC}{C|\alpha_1|^2 + A|\alpha_2|^2}\right)\left\|(\alpha_1 K_1 + \alpha_2 K_2)^* x\right\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle.$$

Hence, $(\Phi, \Psi)_K$ satisfies the lower K-bi-g-frame condition. We have

$$\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq \min\{B, D\}\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

It follows that

$$\begin{aligned}&\left(\frac{AC}{C|\alpha|^2 + A|\alpha_2|^2}\right)\left\|(\alpha_1 K_1 + \alpha_2 K_2)^* x\right\|^2 \\ &\leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq \min\{B, D\}\|x\|^2, \quad \text{for all } x \in \mathcal{H}.\end{aligned}$$

Therefore, $(\Phi, \Psi)_K$ is a $(\alpha_1 K_1 + \alpha_2 K_2)$-bi-$g$-frame.

(2) Now, for each $x \in \mathcal{H}$, we have

$$\left\|(K_1 K_2)^* x\right\|^2 = \left\|K_2^* K_1^* x\right\|^2 \leq \left\|K_2^*\right\|^2\left\|K_1^* x\right\|^2.$$

Since $(\Phi, \Psi)_K$ is a K_1-bi-g-frame, there exist two constants $0 < A \leq B < \infty$ such that

$$A\left\|K_1^* x\right\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Therefore,

$$\frac{1}{\left\|K_2^*\right\|^2}\left\|(K_1 K_2)^* x\right\|^2 \leq \left\|K_1^* x\right\|^2 \leq \frac{1}{A}\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq \frac{B}{A}\|x\|^2.$$

This implies that

$$\frac{A}{\|K_2^*\|^2}\left\|(K_1K_2)^* x\right\|^2 \leq \sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Therefore, $(\Phi,\Psi)_K$ is a K_1K_2-bi-g-frame for $\mathcal{H}$. □

Corollary 3.11. *Let $n\in\mathbb{N}\setminus\{0,1\}$ and $K_i\in\mathcal{B}(\mathcal{H})$ for $j\in[\![1;n]\!]$. If $(\Phi,\Psi)_K$ is a K_j-bi-g-frame for $j\in[\![1;n]\!]$ and $\alpha_1,\alpha_2\cdots,\alpha_n$ are non-zero scalars, then the following holds:*

(1) $(\Phi,\Psi)_K$ *is a* $(\sum_{j=1}^{n}\alpha_j K_i)$*-bi-g-frame.*
(2) $(\Phi,\Psi)_K$ *is a* $(K_1K_2\cdots K_n)$*-bi-g-frame.*

Proof. (1)Suppose that $n\in\mathbb{N}\setminus\{0,1\}$ and for every $j\in[\![1;n]\!]$, $(\Phi,\Psi)_K$ is a K_j-bi-g-frame. Then for each $j\in[\![1;n]\!]$, there exist positive constants $0<A_j\leq B_j<\infty$ such that

$$A_j\left\|K_i^* x\right\|^2 \leq \sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle \leq B_j\|x\|^2, \text{ for all } x\in\mathcal{H}.$$

Now, we can write

$$\begin{aligned}
&\left\|\left(\sum_{j=1}^{n}\alpha_j K_i\right)^* x\right\|^2\\
&= \left\|\alpha_1 K_1^* x + (\alpha_2K_2+\cdots+\alpha_nK_n)^* x\right\|^2\\
&\leq |\alpha_1|^2\left\|K_1^* x\right\|^2 + \left\|(\alpha_2K_2+\cdots+\alpha_nK_n)^* x\right\|^2\\
&\leq |\alpha_1|^2\left\|K_1^* x\right\|^2 + \cdots + |\alpha_n|^2\left\|K_n^* x\right\|^2\\
&\leq |\alpha_1|^2\left(\frac{1}{A_1}\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\right) + \cdots + |\alpha_n|^2\left(\frac{1}{A_n}\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\right)\\
&= \left(\frac{|\alpha_1|^2}{A_1}+\cdots+\frac{|\alpha_n|^2}{A_n}\right)\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\\
&= \left(\sum_{j=1}^{n}\frac{|\alpha_j|^2}{A_j}\right)\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle.
\end{aligned}$$

Hence, $(\Phi,\Psi)_K$ satisfies the lower frame condition. And we have

$$\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle \leq \min_{j\in[\![1;n]\!]}\{B_j\}\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

It follows that

$$\left(\sum_{j=1}^{n}\frac{|\alpha_j|^2}{A_j}\right)^{-1}\left\|\left(\sum_{j=1}^{n}\alpha_j K_i\right)^{*}x\right\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle$$
$$\leq \min_{j\in[\![1;n]\!]}\{B_j\}\|x\|^2, \text{ for all } x\in\mathcal{H}.$$

Hence, $(\Phi,\Psi)_K$ is a $(\sum_{j=1}^{n}\alpha_j K_i)$-bi-$g$-frame.

(2) Now, for each $x\in\mathcal{H}$, we have

$$\|(K_1K_2\cdots K_n)^*x\|^2 = \|K_n^*\cdots K_1^*x\|^2 \leq \|K_n^*\cdots K_2^*\|^2\,\|K_1^*x\|^2\,.$$

Since $(\Phi,\Psi)_K$ is a K_1-bi-g-frame, there exist two constants $0 < A_1 \leq B_1 < \infty$ such that

$$A_1\|K_1^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B_1\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Therefore,

$$\frac{1}{\|K_n^*\cdots K_2^*\|^2}\|(K_1K_2\cdots K_n)^*x\|^2 \leq \frac{1}{A_1}\sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq \frac{B_1}{A_1}\|x\|^2.$$

This implies that

$$\frac{A_1}{\|K_n^*\cdots K_2^*\|^2}\|(K_1K_2\cdots K_n)^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B_1\|x\|^2, \text{ for all } x\in\mathcal{H}.$$

Therefore, $(\Phi,\Psi)_K$ is a $(K_1K_2\cdots K_n)$-bi-g-frame for $\mathcal{H}$.

□

Theorem 3.12. *Let $K\in\mathcal{B}(\mathcal{H})$ with $\|K\|\geq 1$. Then every ordinary bi-g-frame is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{\in I}$.*

Proof. Suppose that $(\Phi,\Psi)_K$ is a bi-g-frame for $\mathcal{H}$. Then there exist two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

For $K\in\mathcal{B}(\mathcal{H})$, we have $\|K^*x\|^2 \leq \|K\|^2\|x\|^2$, for all $x\in\mathcal{H}$. Since $\|K\|\geq 1$, we obtain $\frac{1}{\|K\|^2}\|K^*x\|^2 \leq \|x\|^2$, for all $x\in\mathcal{H}$. Therefore,

$$\frac{A}{\|K\|^2}\|K^*x\|^2 \leq A\|x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Therefore, $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$. □

Theorem 3.13. *Let $(\Phi,\Psi)_K$ be a bi-g-frame for $\mathcal{H}$. Then $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if and only if there exists $A>0$ such that $S_{\Phi,\Psi}\geq AKK^*$, where $S_{\Phi,\Psi}$ is the bi-g-frame operator for $(\Phi,\Psi)_K$.*

Proof. $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with frame bounds A,B and bi-g-frame operator $S_{\Phi,\Psi}$ if and only if

$$A\left\|K^*x\right\|^2\leq\langle S_{\Phi,\Psi}x,x\rangle=\langle\sum_{i\in I}\Psi_i^*\Phi_i x,x\rangle=\sum_{i\in I}\langle\Phi_i x,\Psi_i x\rangle\leq B\|x\|^2,\ \forall x\in\mathcal{H},$$

that is, $\langle AKK^*x,x\rangle\leq\langle S_{\Phi,\Psi}x,x\rangle\leq\langle Bx,x\rangle$, for all $x\in\mathcal{H}$. So the conclusion holds. □

Corollary 3.14. *Let $(\Phi,\Psi)_K$ be a bi-g-frame for $\mathcal{H}$. Then $(\Phi,\Psi)_K$ is a tight K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if and only if there exists $A>0$ such that $S_{\Phi,\Psi}=AKK^*$, where $S_{\Phi,\Psi}$ is the bi-g-frame operator for $(\Phi,\Psi)_K$.*

Proof. The proof is evident; one can simply utilize the definition of a tight K-bi-g-frame (see Definition 3.7).K □

Theorem 3.15. *Let $(\Phi,\Psi)_K$ be a bi-g-frame for $\mathcal{H}$, with bi-g-frame operator $S_{\Phi,\Psi}$ such that $S_{\Phi,\Psi}^{\frac{1}{2}^*}=S_{\Phi,\Psi}^{\frac{1}{2}}$. Then $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if and only if $K=S_{\Phi,\Psi}^{\frac{1}{2}}U$, for some $U\in\mathcal{B}(\mathcal{H})$.*

Proof. Assume that $(\Phi,\Psi)_K$ is a K-bi-g-frame. By Theorem 3.13, there exists $A>0$ such that $AKK^*\leq S_{\Phi,\Psi}^{\frac{1}{2}}S_{\Phi,\Psi}^{\frac{1}{2}^*}$. Then, $KK^*\leq\frac{1}{A}S_{\Phi,\Psi}^{\frac{1}{2}}S_{\Phi,\Psi}^{\frac{1}{2}^*}$. So, $KK^*\leq\lambda^2 S_{\Phi,\Psi}^{\frac{1}{2}}S_{\Phi,\Psi}^{\frac{1}{2}^*}$, where $\lambda=\frac{1}{\sqrt{A}}>0$. Therefore, by Theorem 2.4, $K=S_{\Phi,\Psi}^{\frac{1}{2}}U$, for some $U\in\mathcal{B}(\mathcal{H})$.

Conversely, assume $K=S_{\Phi,\Psi}^{\frac{1}{2}}W$, for some $W\in\mathcal{B}(\mathcal{H})$. Then by Theorem 2.4, there exists a positive number λ such that $KK^*\leq\lambda^2 S_{\Phi,\Psi}^{\frac{1}{2}}S_{\Phi,\Psi}^{\frac{1}{2}^*}$. Then, $\mu KK^*\leq S_{\Phi,\Psi}^{\frac{1}{2}}S_{\Phi,\Psi}^{\frac{1}{2}^*}$, where $\mu=\frac{1}{\sqrt{\lambda}}>0$. Since $S_{\Phi,\Psi}^{\frac{1}{2}^*}=S_{\Phi,\Psi}^{\frac{1}{2}}$, by Theorem 3.13, $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$. □

4. Operators on K-bi-g-frame in Hilbert Spaces

In the following proposition, we will require a necessary condition for the operator $\mathcal{T}$ such that $(\Phi,\Psi)_K$ will be a $\mathcal{T}$-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$.

Proposition 4.1. *Let $(\Phi,\Psi)_K$ be a K-bi-g-frame for $\mathcal{H}$ with $K\in\mathcal{B}(\mathcal{H})$. Let $\mathcal{T}\in\mathcal{B}(\mathcal{H})$ with $R(\mathcal{T})\subseteq\mathcal{R}(K)$. Then $(\Phi,\Psi)_K$ is a $\mathcal{T}$-bi-g-frame for $\mathcal{H}$.*

Proof. Suppose that $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$. Then there are positive constants $0<A\leq B<\infty$ such that

$$A\left\|K^*x\right\|^2\leq\sum_{i\in I}\langle\Phi_i x,\Psi_i x\rangle\leq B\|x\|^2,\quad\text{for all } x\in\mathcal{H}.$$

Since $R(\mathcal{T})\subseteq\mathcal{R}(K)$, by Theorem 2.4, there exists $\alpha>0$ such that $\mathcal{T}\mathcal{T}^*\leq\alpha^2KK^*$. Hence,

$$\frac{A}{\alpha^2}\left\|\mathcal{T}^*x\right\|^2\leq A\left\|K^*x\right\|^2\leq\sum_{i\in I}\langle\Phi_i x,\Psi_i x\rangle\leq B\|x\|^2,\quad\text{for all } x\in\mathcal{H}.$$

Hence, $(\Phi,\Psi)_K$ is a $\mathcal{T}$-bi-g-frame for $\mathcal{H}$. □

Theorem 4.2. *Let $(\Phi,\Psi)_K$ be a K-bi-g-frame for $\mathcal{H}$ with bi-g-frame operator $S_{\Phi,\Psi}$ and let $\mathcal{T}$ be a positive operator. Then*

$$(\Phi+\mathcal{T}\Phi,\Psi+\mathcal{T}\Psi)_K=\left(\{\Phi_i+\mathcal{T}\Phi_i\}_{i\in I},\{\Psi_i+\mathcal{T}\Psi_i\}_{i\in I}\right)$$

is a K-bi-g-frame.

Moreover, for any positive integer n, $\left(\{\Phi_i+\mathcal{T}^n\Phi_i\}_{i\in I},\{\Psi_i+\mathcal{T}^n\Psi_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$.

Proof. Suppose that $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$. Then by Theorem 3.13, there exists $m>0$ such that $S_{\Phi,\Psi}\geq mKK^*$. For every $x\in\mathcal{H}$, we have

$$\begin{aligned}S_{(\Phi+\mathcal{T}\Phi),(\Psi+\mathcal{T}\Psi)}&=\sum_{i\in I}\left(\Psi_i+\mathcal{T}\Psi_i\right)^*\left(\Phi_i+\mathcal{T}\Phi_i\right)\\&=(I+\mathcal{T})^*\sum_{i\in I}\Psi_i^*\Phi_i(I+\mathcal{T})\\&=(I+\mathcal{T})^*S_{\Phi,\Psi}(I+\mathcal{T}).\end{aligned}$$

Hence, the frame operator for $(\Phi+\mathcal{T}\Phi,\Psi+\mathcal{T}\Psi)_K$ is $(I+\mathcal{T})^*S_{\Phi,\Psi}(I+\mathcal{T})$. Since $\mathcal{T}$ is positive operator we get,

$$(I+\mathcal{T})^*S_{\Phi,\Psi}(I+\mathcal{T})=S_{\Phi,\Psi}+S_{\Phi,\Psi}\mathcal{T}+\mathcal{T}^*S_{\Phi,\Psi}+\mathcal{T}^*S_{\Phi,\Psi}\mathcal{T}\geq S_{\Phi,\Psi}\geq mKK^*,$$

Once again, applying Theorem 3.13, we can conclude that $(\Phi+\mathcal{T}\Phi,\Psi+\mathcal{T}\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$.

Now, for any positive integer n, the frame operator for

$$S_{(\Phi+\mathcal{T}^n\Phi),(\Psi+\mathcal{T}^n\Psi)}=\left(I+\mathcal{T}^n\right)^*S_{\Phi,\Psi}\left(I+\mathcal{T}^n\right)\geq S_{\Phi,\Psi}.$$

Hence, $\left(\{\Phi_i+\mathcal{T}^n\Phi_i\}_{i\in I},\{\Psi_i+\mathcal{T}^n\Psi_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$. □

Theorem 4.3. *Let $K \in \mathcal{B}(\mathcal{H})$ and $(\Phi,\Psi)_K$ be a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$, and let $M \in \mathcal{B}(\mathcal{H})$ be an operator with a closed range such that $MK = KM$. If $\mathcal{R}(K^*) \subset \mathcal{R}(M)$, then*

$$(\Phi M^*, \Psi M^*)_K = \left(\{\Phi_i M^*\}_{i\in I}, \{\Psi_i M^*\}_{i\in I}\right)$$

is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$.

Proof. For every $x \in \mathcal{H}$, we have $A\|K^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle \leq B\|x\|^2$. Then for $M \in \mathcal{B}(\mathcal{H})$, we get

$$\sum_{i\in I}\langle \Phi_i M^* x, \Psi_i M^* x\rangle \leq B\|M^*x\|^2 \leq B\|M\|^2\|x\|^2.$$

Since M has a closed range and $\mathcal{R}(K^*) \subset \mathcal{R}(M)$,

$$\begin{aligned}\|K^*x\|^2 &= \left\|MM^\dagger K^*x\right\|^2\\ &= \left\|\left(M^\dagger\right)^* M^*K^*x\right\|^2\\ &= \left\|\left(M^\dagger\right)^* K^*M^*x\right\|^2\\ &\leq \left\|M^\dagger\right\|^2 \|K^*M^*x\|^2.\end{aligned}$$

On the other hand, we have

$$\sum_{i\in I}\langle \Phi_i M^* x, \Psi_i M^* x\rangle \geq A\|K^*M^*x\|^2 \geq A\left\|M^\dagger\right\|^{-2}\|K^*x\|^2.$$

Hence, $(\Phi M^*, \Psi M^*)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. □

Theorem 4.4. *Let $K, M \in B(\mathcal{H})$ and $(\Phi,\Psi)_K$ be a δ-tight K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. If $\mathcal{R}(K^*) = \mathcal{H}$ and $MK = KM$, then $(\Phi M^*, \Psi M^*)_K = \left(\{\Phi_i M^*\}_{i\in I}, \{\Psi_i M^*\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if and only if M is surjective.*

Proof. Suppose that $\left(\{\Phi_i M^*\}_{i\in I}, \{\Psi_i M^*\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with frame bounds A and B. That is, for every $x \in \mathcal{H}$,

$$A\|K^*x\|^2 \leq \sum_{i\in I}\langle \Phi_i M^* x, \Psi_i M^* x\rangle \leq B\|x\|^2.$$

and we have

$$\delta\|K^*x\|^2 = \sum_{i\in I}\langle \Phi_i x, \Psi_i x\rangle,\ \text{for all } x \in \mathcal{H}.$$

Since $K^*M^* = M^*K^*$, we obtain

$$\delta\|M^*K^*x\|^2 = \delta\|K^*M^*x\|^2 = \textstyle\sum_{i\in I}\langle \Phi_i M^* x, \Psi_i M^* x\rangle.$$

Hence, $\|M^*K^*x\|^2 = \dfrac{1}{\delta}\sum_{i\in I}\langle \Phi_i M^* x, \Psi_i M^* x\rangle \geq \dfrac{A}{\delta}\|K^*x\|^2$. From which we conclude that M^* is injective. Since $\mathcal{R}(K^*) = \mathcal{H}$, M is surjective as a consequence. $\square$

5. Stability of K-bi-g-frames for Hilbert spaces

Theorem 5.1. *Suppose that $K \in \mathcal{B}(\mathcal{H})$ and K has closed range. Let $\Phi = \{\Phi_i : \Phi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}$ and $\Psi = \{\Psi_i : \Psi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}$ be two g-Bessel sequences with bounds B_Φ and B_Ψ respectively. Assume that $(\Phi, \Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds A and B, and let $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ be a pair of sequences in $\mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ for $i \in I$. If there exist constants $\alpha, \beta, \gamma \in [0, 1)$ such that $\max\{\alpha + \gamma, \beta\} < 1$ and*

$$\left\|\sum_{i\in J}(\Psi_i^*\Phi_i - \Gamma_i^*\Lambda_i)x\right\| \leq \alpha\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| + \gamma\|x\|,$$

where J is any finite subset of I and $\|S_{\Phi,\Psi}^{-1}x\| \leq \|x\|$, where $S_{\Phi,\Psi}$ is the bi-g-frame operator of $(\Phi, \Psi)_K$. Then $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds

$$A\tfrac{[1-(\alpha+\gamma)]}{(1+\beta)}, \tfrac{(1+\alpha)\sqrt{B_\Phi B_\Psi}+\gamma}{1-\beta}.$$

Proof. Suppose that $J \subset I, |J| < +\infty$. For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| &\leq \left\|\sum_{i\in J}(\Gamma_i^*\Lambda_i - \Psi_i^*\Phi_i)x\right\| + \left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| \\ &\leq (1+\alpha)\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| + \gamma\|x\|.\end{aligned}$$

Then $\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| \leq \frac{1+\alpha}{1-\beta}\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \frac{\gamma}{1-\beta}\|x\|$. Since

$$\begin{aligned}\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| &= \sup_{\|y\|=1}\left|\left\langle\sum_{i\in J}\Psi_i^*\Phi_i x, y\right\rangle\right| \\ &= \sup_{\|y\|=1}\left|\left\langle\sum_{i\in J}\Phi_i x, \Psi_i y\right\rangle\right| \\ &\leq \left(\sum_{i\in J}\|\Phi_i x\|^2\right)^{\frac{1}{2}}\sup_{\|y\|=1}\left(\sum_{i\in J}\|\Psi_i y\|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{B_\Phi B_\Psi}\,\|x\|.\end{aligned}$$

Hence, for all $x \in H$, we have

$$\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| \leq \frac{(1+\alpha)\sqrt{B_\Phi B_\Psi}}{1-\beta}\|x\| + \frac{\gamma}{1-\beta}\|x\| = \frac{(1+\alpha)\sqrt{B_\Phi B_\Psi}+\gamma}{1-\beta}\|x\|.$$

We considere

$$\mathcal{M}:\mathcal{H}\to\mathcal{H}, \quad \mathcal{M}x = \sum_{i\in J}\Gamma_i^*\Lambda_i x, x\in\mathcal{H}.$$

Then $\mathcal{M}$ is well-defined, bounded, and $\|\mathcal{M}\| \leq \frac{(1+\alpha)\sqrt{B_\Phi B_\Psi}+\gamma}{1-\beta}$. For every $x \in \mathcal{H}$, we have

$$\langle\mathcal{M}x, x\rangle = \langle\sum_{i\in J}\Gamma_i^*\Lambda_i x, x\rangle = \sum_{i\in J}\langle\Lambda_i x, \Gamma_i x\rangle \leq \|\mathcal{M}\|\|x\|^2. \tag{5.1}$$

It implies that $\left(\{\Lambda_i\}_{i\in J}, \{\Gamma_i\}_{i\in J}\right)$ is a bi-g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in J}$. Let $S_{\Phi,\Psi}$ be the bi-g-frame operator of $(\Phi,\Psi)_K$. According to the theorem hypothesis, we obtain

$$\|(S_{\Phi,\Psi}-\mathcal{M})x\| \leq \alpha\|S_{\Phi,\Psi}x\| + \beta\|\mathcal{M}x\| + \gamma\|x\|, \forall x\in\mathcal{H}.$$

Then,

$$\begin{aligned}\left\|x-\mathcal{M}S_{\Phi,\Psi}^{-1}x\right\| &\leq \alpha\|x\| + \beta\left\|\mathcal{M}S_{\Phi,\Psi}^{-1}x\right\| + \gamma\|S_{\Phi,\Psi}^{-1}x\| \\ &\leq (\alpha+\gamma)\|x\| + \beta\left\|\mathcal{M}S_{\Phi,\Psi}^{-1}x\right\|.\end{aligned}$$

Since $0 \leq \max\{\alpha+\gamma, \beta\} < 1$, According to Lemma 2.5 , we get

$$\frac{1-\beta}{1+(\alpha+\gamma)} \leq \left\|S_{\Phi,\Psi}\mathcal{M}^{-1}\right\| \leq \frac{1+\beta}{1-(\alpha+\gamma)}.$$

Since $\|S_{\Phi,\Psi}\| = \|S_{\Phi,\Psi}\mathcal{M}^{-1}\mathcal{M}\| \leq \|S_{\Phi,\Psi}\mathcal{M}^{-1}\|\|\mathcal{M}\|$. Therefore,

$$\|\mathcal{M}\| \geq \frac{A}{\|S_{\Phi,\Psi}\mathcal{M}^{-1}\|}\|KK^*\| \geq A\frac{[1-(\alpha+\gamma)]}{(1+\beta)}\|KK^*\|. \tag{5.2}$$

Hence, by Theorem 3.13, we can conclude that $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. □

Corollary 5.2. *Suppose that $K \in \mathcal{B}(\mathcal{H})$ and K has closed range. Let $\Phi = \{\Phi_i : \Phi_i \in \mathcal{B}(\mathcal{H},\mathcal{K}_i)\}_{i\in I}$ and $\Psi = \{\Psi_i : \Psi_i \in \mathcal{B}(\mathcal{H},\mathcal{K}_i)\}_{i\in I}$ be two g-Bessel sequences with bounds B_Φ, B_Ψ respectively. Assume that $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds A and B, and*

$\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ *is a pair of sequences for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i\}_{i\in I}$. *If there exists a constant* $0 < D < A$ *such that*

$$\left\|\sum_{i\in I}\left(\Psi_i^*\Phi_i - \Gamma_i^*\Lambda_i\right)x\right\| \leq D\left\|K^*x\right\|, \quad \forall x \in \mathcal{H},$$

and $\|S_{\Phi,\Psi}^{-1}x\| \leq \|x\|$, *where* $S_{\Phi,\Psi}$ *is the bi-g-frame operator of* $(\Phi,\Psi)_K$, *then* $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ *is a* K*-bi-g-frame for* $\mathcal{H}$ *with respect to* $\{K_i\}_{i\in I}$ *with bounds* $A\left(1 - D\sqrt{\frac{B}{A}}\right)$ *and* $\left(\sqrt{B_\Phi B_\Psi} + D\sqrt{\frac{B}{A}}\right)$.

Proof. For any $x \in \mathcal{H}$, we have $\|K^*x\| \leq \dfrac{1}{\sqrt{A}}\left(\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\right)^{\frac{1}{2}}$. Then

$$\begin{aligned}\left\|\sum_{i\in I}\left(\Psi_i^*\Phi_i - \Gamma_i^*\Lambda_i\right)x\right\| &\leq D\left\|K^*x\right\| \\ &\leq \frac{1}{\sqrt{A}}\left(\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\right)^{\frac{1}{2}} \\ &\leq D\sqrt{\frac{B}{A}}\|x\|.\end{aligned}$$

By letting $\alpha, \beta = 0, \gamma = D\sqrt{\dfrac{B}{A}}$ in Theorem 5.1, $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds $A\left(1 - D\sqrt{\dfrac{B}{A}}\right)$ and $\left(\sqrt{B_\Phi B_\Psi} + D\sqrt{\dfrac{B}{A}}\right)$. □

Theorem 5.3. *Suppose that* $K \in \mathcal{B}(\mathcal{H})$ *and* K *has closed range. Let* $\Phi = \{\Phi_i : \Phi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}$ *and* $\Psi = \{\Psi_i : \Psi_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in I}$ *be two g-Bessel sequences with bounds* B_Φ *and* B_Ψ, *respectively. Assume that* $(\Phi,\Psi)_K$ *is a* K*-bi-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i\}_{i\in I}$ *with bounds* A *and* B, *and let* $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ *be a pair of sequences for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i\}_{i\in I}$. *If there exist constants* $\alpha, \beta, \gamma \in [0, 1)$ *such that* $\max\left\{\alpha + \gamma\sqrt{\frac{B}{A}}, \beta\right\} < 1$ *and*

$$\left\|\sum_{i\in J}\left(\Psi_i^*\Phi_i - \Gamma_i^*\Lambda_i\right)x\right\| \leq \alpha\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| + \gamma\left\|K^*x\right\|, \quad (5.3)$$

where J is any finite subset of I and $\|S_{\Phi,\Psi}^{-1}x\| \leq \|x\|$, where $S_{\Phi,\Psi}$ is the bi-g-frame operator of $(\Phi,\Psi)_K$, then $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds

$$A\frac{1-\left(\alpha+\gamma\sqrt{\frac{B}{A}}\right)}{1+\beta}, \quad \frac{\left[(1+\alpha)\sqrt{B_\Phi B_\Psi}+\gamma\sqrt{\frac{B}{A}}\right]}{1-\beta}.$$

Proof. For any $x \in \mathcal{H}$, we have

$$\|K^*x\| \leq \frac{1}{\sqrt{A}}\left(\sum_{i\in I}\langle\Phi_i x, \Psi_i x\rangle\right)^{\frac{1}{2}} \leq \sqrt{\frac{B}{A}}\|x\|. \tag{5.4}$$

Then, the hypothesis 5.3 is equivalent to:

$$\left\|\sum_{i\in J}\left(\Psi_i^*\Phi_i-\Gamma_i^*\Lambda_i\right)x\right\| \leq \alpha\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| + \gamma\sqrt{\frac{B}{A}}\|x\|.$$

Therefore, the remaining part of the proof is similar to the proof of Theorem 5.1. □

Theorem 5.4. *Suppose that $K \in \mathcal{B}(\mathcal{H})$ and K has closed range. Let $\Phi = \{\Phi_i : \Phi_i \in \mathcal{B}(\mathcal{H},\mathcal{K}_i)\}_{i\in I}$ and $\Psi = \{\Psi_i : \Psi_i \in \mathcal{B}(\mathcal{H},\mathcal{K}_i)\}_{i\in I}$ be two g-Bessel sequences with bounds B_Φ, B_Ψ respectively. Assume that $(\Phi,\Psi)_K$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds A and B and $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ be a pair of sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. If there exist constants $\alpha,\beta,\sigma,\gamma \in [0,1)$ such that* $\max\left\{\alpha+\sigma+\gamma\sqrt{\dfrac{B}{A}}, \beta\right\} < 1$ *and*

$$\left\|\sum_{i\in J}\left(\Psi_i^*\Phi_i-\Gamma_i^*\Lambda_i\right)x\right\| \leq \alpha\left\|\sum_{i\in J}\Psi_i^*\Phi_i x\right\| + \beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_i x\right\| + \sigma\|x\| + \gamma\|K^*x\|, \tag{5.5}$$

where J is any finite subset of I and $\|S_{\Phi,\Psi}^{-1}x\| \leq \|x\|$, where $S_{\Phi,\Psi}$ is the bi-g-frame operator of $(\Phi,\Psi)_K$. Then $\left(\{\Lambda_i\}_{i\in I}, \{\Gamma_i\}_{i\in I}\right)$ is a K-bi-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ with bounds

$$A\frac{\left[1-\left(\alpha+\sigma+\gamma\sqrt{\dfrac{B}{A}}\right)\right]}{(1+\beta)}, \quad \frac{\left[(1+\alpha)\sqrt{B_\Phi B_\Psi}+\sigma+\gamma\sqrt{\dfrac{B}{A}}\right]}{1-\beta}.$$

Proof. With the inequality 5.4, our hypothesis 5.5 will be equivalent to

$$\left\|\sum_{i\in J}\left(\Psi_i^*\Phi_i-\Gamma_i^*\Lambda_i\right)x\right\|\leq\alpha\left\|\sum_{i\in J}\Psi_i^*\Phi_ix\right\|+\beta\left\|\sum_{i\in J}\Gamma_i^*\Lambda_ix\right\|+\left(\sigma+\gamma\sqrt{\frac{B}{A}}\right)\|x\|. \tag{5.6}$$

Therefore, the remaining part of the proof is similar to the proof of Theorem 5.1. □

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$K-bi-g-$FRAMES IN HILBERT SPACES

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$K-bi-g$-قاب‌ها در فضاهای هیلبرت

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این مقاله مفهوم جدیدی از $K-bi-g$-قاب‌ها را برای فضاهای هیلبرت معرفی می‌کند. سپس با کمک یک عملگر بای‌قاب، برخی ویژگی‌ها را بررسی می‌کنیم. در نهایت، چندین نتیجه درباره پایداری $K-bi-g$-قاب‌ها که با استفاده از روش‌های نظریه قاب به‌دست آمده‌اند، را بررسی می‌کنیم.

کلمات کلیدی: قاب، K-قاب، بای‌قاب، $K-bi-g$-قاب‌ها، فضاهای هیلبرت.