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THE PARTITION DIMENSION AND k-DOMINATION NUMBER OF TWO SPECIFIC GRAPHS

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Abstract. For an ordered k-partition $\Omega = \{S_1, S_2, ..., S_k\}$ of vertex set of a connected graph G and a vertex v of G, the representation of v with respect to Ω is defined as the k-tuple $r(v|\Omega) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. The partition Ω is called a resolving partition of G, if $r(u|\Omega) \neq r(v|\Omega)$ for all distinct $u, v \in V(G)$. The partition dimension of a graph G, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G. A subset $D \subseteq V(G)$ is k-dominating in G, if every vertex of $V(G) \setminus D$ has at least k neighbors in D. The minimum cardinality among all k-dominating sets is called the k-domination number of G, denoted by $\gamma_k(G)$. In this paper, we determine the partition dimension of cocktail party graph $CP(m+1)$ and corona product $G \circ \overline{K_m}$. Moreover, we obtain k-domination numbers for $CP(m+1)$ and corona product $C_n \circ \overline{K_m}$.

1. Introduction

Given a set of vertices $S = \{v_1, ..., v_k\}$ of a connected simple graph G, the metric representation of a vertex v of G with respect to S is the vector $r(v|S) = (d(v, v_1), ..., d(v, v_k))$, where $d(v, v_i)$, $i \in \{1, ..., k\}$ denotes the distance between v and v_i. The set S is a resolving set of G, if for every pair of vertices u,v of $V(G)$, $r(u|S) \neq r(v|S)$. The metric dimension $dim(G)$ of G is the minimum cardinality of any resolving set of G. This parameter has studied well in literature.

One of the useful concepts in graph theory is determining the partition dimension of a graph, which first proposed in [5].

It is well known that the problem of determining the partition dimension is NP-complete, see [10]. Before giving the formal definition, we present detailed explanations showing how usual concept of partition dimension is a natural generalization of metric dimension. The problem of determining the partition dimension of a graph has a long history, as it has many applications in chemistry to represent the chemical compounds [15, 16], network discovery and verification [2], digital world to recognize the pattern, robotics for image processing [19], and others [4, 6].

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This concept came from the study of metric dimension which was firstly studied by Harary and Melter [12], and independently by Slater [27]. Indeed, partition dimension is as a generalization of resolving set when the vertices are classified in different types.

For an ordered k-partition $\Omega = \{S_1, S_2, ..., S_k\}$ of vertex set of a connected graph G and a vertex v of G, the representation of v with respect to Ω is defined as the k-tuple

$$r(v|\Omega) = (d(v, S_1), d(v, S_2), ..., d(v, S_k)),$$

representing a unique code of v relative to Ω, where the distance $d(v, S_i)$ between v and S_i is defined as

$$d(v, S_i) = min\{d(v, x)|x \in S_i\}.$$

The partition Ω is called a resolving partition of G, if $r(u|\Omega) \neq r(v|\Omega)$ for all distinct $u, v \in V(G)$. The partition dimension of a graph G, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G. It is well known that, for any connected graph G of order $n \geq 2$, $2 \leq pd(G) \leq n$. Authors in [5] proved that for connected graph G of order $n \geq 2$, $pd(G) = 2$ if and only if $G = P_n$, where P_n is the path of order n and $pd(G) = n$ if and only if $G = K_n$, where K_n is the complete graph of order n.

The partition dimension of some classes of graphs such as some wheel-related graphs [14], hexagonal and honeycomb networks [22], unicyclic graphs [23], trees [25], a homogeneous firecrackers [1], Nanotubes [26], fullerene graphs [18], kayak paddle graph, cycle graph with chords [28], Cayley digraphs [9] and corona product [24] have been studied.

The concept of domination set was first introduced by Oystein Ore in 1962, see [21], and the study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In fact, a domination problem and its related parameters, the problem of placing fire stations in an optimum way is significant. Cockayne and Hedetniemi [7] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. A subset $D \subseteq V(G)$ is said to be a dominating set of G if every vertex of $V(G) \setminus D$ is adjacent to an element in D. In [3], Borowiecki and Kuzak have generalized the concept of a dominating set in a graph. A subset $D \subseteq V(G)$ is k-dominating in G if every vertex of $V(G) \setminus D$ has at least k neighbors in D. The minimum cardinality among all k-dominating sets is called the k-domination number of G, denoted by $\gamma_k(G)$. For $k = 1$ a k-dominating set D is an ordinary dominating set and the classical domination number $\gamma_1(G)$, and it is well known that every $(k+1)$-dominating set is also a k-dominating set, and so $\gamma_k(G) \leq \gamma_{k+1}(G)$. Some variants of dominating set in graphs and its related concepts can be found in [8], and [13].

Suppose G is a finite group and Ω a subset of G that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, \Omega)$ is a graph whose vertex set and edge set are defined as follows ([11]):

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in \Omega\}.$$

Let G and H be two graphs of order n and m, respectively. The corona product $G \circ H$ of two graphs G and H of order n and m, respectively, is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G.

In this paper, we consider determining the partition dimension of cocktail party graph $CP(m+1)$ and corona product $G \circ \overline{K_m}$. Moreover, we obtain k-domination numbers for $CP(m+1)$ and corona product $C_n \circ \overline{K_m}$.

2. Results for cocktail party graph $CP(m+1)$

Based on [17, 20], we can see that if $n \geq 4$ is an even integer and $m = \frac{n}{2} - 1$, then the Cayley graph $\Gamma = Cay(\mathbb{Z}_n, S_m)$, where $\mathbb{Z}_n$ is the cyclic additive group and $S_1 = \{1, n-1\}$, ..., $S_m = S_{m-1} \cup \{m, n-m\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $m \in \mathbb{N}$, $1 \leq m \leq [\frac{n}{2}] - 1$ is isomorphic to the cocktail party graph $CP(m+1)$. Some resolving parameters for cocktail party graph $CP(m+1)$ has been computed, see [17]. In this section, we determine the partition dimension and k-domination number of cocktail party graph $CP(m+1)$.

2.1. The partition dimension of $CP(m+1)$.

We need the following result:

Theorem 2.1. [5] *If G is a nontrivial connected graph, then*

$$pd(G) \leq dim(G) + 1.$$

The following theorem gives the partition dimension of $CP(m+1)$ for $m = \frac{n}{2} - 1$.

Theorem 2.2. *If $n \geq 6$ is an even integer and $m = \frac{n}{2} - 1$, then*

$$pd(CP(m+1)) = m + 2.$$

Proof. Let G be a graph with vertex set $V(G) = \{1, 2, ..., n\}$ and let G be isomorphic to the cocktail party graph $CP(m+1)$. Now, let $\Sigma = \{R_1, R_2, ..., R_p\}$

be a partition set of the vertex set of G so that Σ is a resolving partition of G. In the following, we show that the partition dimension of G is $m+2$.

Claim 1. Each pair of vertices u and v with distance two in G must lies in distinct parts of the partition set Σ. For this purpose, suppose on contrary that both vertices u and v lie in the same part, say $R_i \in \Sigma$. Since G is a vertex transitive graph of order n with valency $2m$, and all neighbors of the vertices u and v in other parts of Σ are identical, it follows that $r(u|\Sigma) = r(v|\Sigma)$, which is a contradiction.

Claim 2. We show that the size of all parts R_i of Σ cannot be greater than or equal to 2. Suppose on contrary that the size of each $R_i \in \Sigma$ is greater than or equal to 2, and let u and v be distinct pair of vertices in G so that these vertices lie in the same part of Σ, say R_i. Based on Claim 1, $d(u,v) = 1$, that is all the elements of each $R_i \in \Sigma$ are neighbors. Now, we consider the following subcases.

Subcase 2.1. Let $R_j \in \Sigma$ be an arbitrary part of Σ, $j \neq i$, and let w and z be distinct pair of vertices in G so that these vertices lie in $R_j \in \Sigma$, $d(u,w) = 2$ and $d(v,z) = 1$, hence $d(u,z) = 1$ because for any vertex of graph G, there is exactly one vertex of G is at the distance 2, and hence $r(u|R_j) = r(v|R_j)$, it follows that $r(u|\Sigma) = r(v|\Sigma)$, which is a contradiction.

Subcase 2.2. Let $R_j \in \Sigma$ be an arbitrary part of Σ, $j \neq i$, and let w and z be distinct pair of vertices in G so that these vertices lie in $R_j \in \Sigma$, $d(u,w) = 2$ and $d(v,z) = 2$, hence $d(u,z) = 1$ and $d(v,w) = 1$, and hence $r(u|R_j) = r(v|R_j)$, it follows that $r(u|\Sigma) = r(v|\Sigma)$, which is a contradiction.

Based on Claim 1, if there is a part $R_i \in \Sigma$ so that the size of R_i is $m+2$, then the partition set Σ of G cannot be a resolving partition of G, because in this case there are at least two elements of R_i is at the distance 2, also based on Claim 2, the size of some parts of Σ must be equal to 1. Therefore, the partition set Σ of G may be a minimal resolving partition of G if there is a part $R_i \in \Sigma$, so that the size of R_i is at most $m+1$, and hence if there is a part $R_i \in \Sigma$, so that the size of R_i is $m+1$, say $R_i = \{1, 2, ..., m+1\}$, then other $m+1$ vertices of G, say $m+2, ..., n$, must lie in distinct parts, it follows that $pd(G) > m+1$. On the other hand, based on Theorem 5 of [17], we know that $dim(G) = m+1$. In particular, based on Theorem 2.1, we have $pd(G) \leq dim(G) + 1$. Thus the partition dimension of G is $m+2$. □

Example 2.3. Let G be a graph with vertex set $V(G) = \{1, 2, ..., 12\}$ and let G be isomorphic to the cocktail party graph $CP(6)$. Then we can see that the partition set

$$\Delta = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\},$$

is a resolving partition of G, and the partition sets

$$\Pi = \{\{1,2,3,4,5,6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\}\},$$

and

$$\Sigma = \{\{1,2,3,4,5\},\{6,7\},\{8\},\{9\},\{10\},\{11\},\{12\}\},$$

are minimal resolving partition of G.

2.2. **k-domination number of $CP(m+1)$.** In this subsection, we obtain $\gamma_k(CP(m+1))$. First we consider $k=1$.

Proposition 2.4. *For any integer $m \geq 1$, if $n = 2m+2$, then*

$$\gamma_1(CP(m+1)) = 2.$$

Proof. Let G be a graph with vertex set $V(G) = \{1,2,...,n\}$ and let G be isomorphic to the cocktail party graph $CP(m+1)$. If u and v are distinct pair of vertices in G, then the length of a shortest path from u to v is $d(u,v) = 1$ or 2, since $diam(G) = 2$. On the other hand, the size of any clique in G is $m+1$, it follows that the size of any independent set of vertices in G is 2, and this implies that any 1-dominating set in G has cardinality at least 2, that is $\gamma_1(G) \geq 2$. Let $D = \{u,v\}$. Since every vertex of $V(G) \setminus D$ has at least one neighbor in D it follows that $\gamma_1(G) = 2$. □

Proposition 2.5. *For any integer $m \geq 1$, if $n = 2m+2$, then*

$$\gamma_2(CP(m+1)) = 2.$$

Proof. Let G be a graph with vertex set $V(G) = \{1,2,...,n\}$ and let G be isomorphic to the cocktail party graph $CP(m+1)$. Since G is a vertex transitive graph of order n with valency $2m$, diameter 2, and the size of any independent set of vertices in G is 2, it follows that for every vertex u in G, there is exactly one vertex of G say v, so that $d(u,v) = 2$. Specially, every 2-dominating set of G is a 1-dominating set of G and it follows that $\gamma_1(G) \leq \gamma_2(G)$, and so any 2-dominating set in G has cardinality greater than or equal to 2. If we now consider the subset $D = \{u,v\}$ of vertices of G with $d(u,v) = 2$, then we see that D is a 2-dominating set of cocktail party graph $CP(m+1)$. Because every vertex of $V(G) \setminus D$ has exactly 2 neighbors in D, and so $\gamma_2(G) = 2$. □

Theorem 2.6. *For any integer $m \geq 2$, if $n = 2m+2$, and $3 \leq k \leq 2m$, then*

$$\gamma_k(CP(m+1)) = \begin{cases} k & \text{if } k \text{ is an even integer,} \\ k+1 & \text{if } k \text{ is an odd integer.} \end{cases}$$

Proof. Let G be a graph with vertex set $V(G) = \{1, 2, ..., n\}$ and let G be isomorphic to the cocktail party graph $CP(m+1)$. We can verify that if D is a subset of vertices of G with $|D| \leq k-1$, then D cannot be a k-dominating set of G, and so any k-dominating set in G has cardinality greater than or equal to k. In the following, we consider two parts:

(i) If k is an even integer and D is a subset of vertices of G with $|D| = k$ and for every $u \in D$ there is $v \in D$ so that $d_G(u, v) = 2$, then the set D is a k-dominating set of G. Because every vertex of $V(G) \setminus D$ has exactly k neighbors in D, and so $\gamma_k(G) = k$.

(ii) Now, suppose that k is an odd integer and D is a k-dominating set of G. We claim that $|D| > k$. For this purpose, suppose on contrary that $|D| = k$. Since k is an odd integer then there are $u \in D$ and $v \in V(G) \setminus D$ such that $d_G(u, v) = 2$ and so D cannot be a k-dominating set of G. Because the vertex $v \in V(G) \setminus D$ has exactly $(k-1)$ neighbors in D which is a contradiction. Now, if k is an odd integer and D is a subset of vertices of G such that $|D| = k+1$, and for every $u \in D$ there is $v \in D$ such that $d_G(u, v) = 2$, then this set is a k-dominating set of G. Therefore $\gamma_k(G) = k+1$.

□

3. Results for $G \circ \overline{K_m}$

Let n and m be fixed positive integers, with $n \geq 3$ and $m \geq 2$, and let C_n, K_m and $\overline{K_m}$ denote the cycle graph of order n on vertices $V(C_n) = \{v_1, ..., v_n\}$, the complete graph on m vertices, and the complement of K_m, respectively. The partition dimension of corona product has been widely studied in literature (see e.g. [24]). Rodríguez-Velázquez, Yero and Kuziak in [24] proved that for $n \geq 2m + 1 \geq 5$, $pd(P_n \circ K_m) = m + 2$ and for $n = |V(G)| > \beta(H) \geq 2$, $pd(G \circ H) \geq \beta(H) + 1$, where $\beta(H)$ is the number of isolated vertices of H. In this section, we determine the partition dimension of $G \circ \overline{K_m}$. Moreover, we obtain k-domination numbers of $C_n \circ \overline{K_m}$.

Observation 3.1. [24] For any connected graph G of order $n > m \geq 2$,

(i) $pd(G \circ \overline{K_m}) \geq m + 1$.

(ii) For $n \geq m \geq 2$, $pd(P_n \circ \overline{K_m}) = m + 1$.

Theorem 3.2. *For any graph G of order $n \geq 3$, with $n \geq m \geq 2$, we have*

(i) *If $m = n - 1$, then any partition set of the vertex set of $G \circ \overline{K_m}$ with the cardinality m cannot be a resolving partition of $G \circ \overline{K_m}$.*

(ii) *If $m = n$, then $pd(G \circ \overline{K_n}) = n$.*

(iii) *If* $s = n-1 \geq 2$, *then* $pd(G \circ \overline{K_s}) = n$.
(iv) *If* G *is connected and* $t = n-2 \geq 2$, *then the cardinality of a minimum resolving partition of* $G \circ \overline{K_t}$ *is at most* n.

Proof. (i) Let $V(G \circ \overline{K_m}) = V_1 \cup V_2$, where $V_1 = V(G) = \{v_1, ..., v_n\}$ and $V_2 = \{v_{11}, ..., v_{1m}, ..., v_{n1}, ..., v_{nm}\}$, so that every vertex $v_i \in V_1$ is adjacent to the vertices $v_{i1}, v_{i2}, ..., v_{im}$, and $deg(v_{ij}) = 1$ for every vertex $v_{ij} \in V_2$. Now, let $\Delta = \{S_1, S_2, ..., S_m\}$ be a partition set of the vertex set of $G \circ \overline{K_m}$. Since, the set V_1 contains n vertices and $m < n$, then there is a part $S_r \in \Delta$, with at least two vertices of V_1, say v_i, v_j; belong to S_r. Hence $r(v_i|\Delta) = r(v_j|\Delta)$, because none of the vertices $v_{i1}, v_{i2}, ..., v_{im}$ cannot belong to the same part S_i of the partition set Δ. Thus if $m = n-1$, then any partition set of the vertex set of $G \circ \overline{K_m}$ as the cardinality m cannot be a resolving partition of $G \circ \overline{K_m}$.

(ii) Let $V(G \circ \overline{K_n}) = V_1 \cup V_2$, where V_1 and V_2 is defined in the proof of Part (i). Now, let $\Sigma = \{S_1, S_2, ..., S_k\}$ be a partition set of the vertex set of $G \circ \overline{K_n}$ so that Σ is a resolving partition of $G \circ \overline{K_n}$. Since every vertex $v_i \in V_1$ is adjacent to all the vertices $v_{i1}, v_{i2}, ..., v_{in} \in V_2$ and all the vertices $v_{i1}, v_{i2}, ..., v_{in}$ are pendant, then none of the vertices $v_{i1}, v_{i2}, ..., v_{in}$ cannot belong to the same part S_i of the resolving partition Σ, and hence k must be greater than or equal to n. Therefore, the cardinality of any minimum resolving partition of $G \circ \overline{K_n}$ must be greater than or equal to n. For $i \in \{1, 2, ..., n\}$, if we consider $S_i = \{v_i, v_{1i}, v_{2i}, ..., v_{ni}\}$, then $\Omega = \{S_1, S_2, ..., S_n\}$ is a resolving partition of $G \circ \overline{K_n}$. Because according to the choice of each $S_i \in \Omega$, it is sufficient to show that the representations of all the elements of S_i with respect to Ω are not identical. For this aim, since $v_i \in S_i$ and v_i is adjacent to an element of each $S_j \in \Omega$, then we have

$$r(v_i|\Omega) = (1, ..., 1, \overbrace{\ 0\ }^{i^{\text{th}}}, 1, ..., 1),$$

also for $v_{ii} \in S_i$ we have

$$r(v_{ii}|\Omega) = (2, ..., 2, \overbrace{\ 0\ }^{i^{\text{th}}}, 2, ..., 2),$$

in particular, for every $v_{ki} \in S_i$; $k, i \in \{1, ..., n\}$, $k \neq i$; may be $k > i$ or $k < i$. So if $k > i$, then we have

$$r(v_{ki}|\Omega) = (2, ..., 2, \overbrace{\ 0\ }^{i^{\text{th}}}, 2, ..., 2, \overbrace{\ 1\ }^{k^{\text{th}}}, 2, ..., 2).$$

Hence, the partition set Ω is a resolving partition of $G \circ \overline{K_n}$. Thus the size of any minimum resolving partition of $G \circ \overline{K_n}$ is n.

(iii) Let $[n] = \{1, 2, ..., n\}$ and $V(G \circ \overline{K_s}) = V_1 \cup V_2$, where

$$V_1 = V(G) = \{v_1, ..., v_n\}, \;\; V_2 = \{A_{1j}, A_{2j}, ..., A_{nj}\},$$

and let $A_{ij} = \cup_{j=1}^{s} \{v_{ij}\}$, $1 \leq i \leq n$ so that every vertex $v_i \in V_1$ is adjacent to the vertices $v_{i1}, v_{i2}, ..., v_{is}$, and $deg(v_{ij}) = 1$ for every vertex $v_{ij} \in V_2$. Based on Theorem 3.2 (i), the size of any minimum resolving partition of $G \circ \overline{K_s}$ must be greater than or equal to n. Now, for $i \in [n]$, if we consider $v_i \in S_i$ and let S_i contains exactly one element of each A_{kj} for $k \in [n] - \{i\}$, then the partition set $\Upsilon = \{S_1, S_2, ..., S_n\}$ is a minimal resolving partition of $G \circ \overline{K_s}$. Because according to choice of each $S_i \in \Upsilon$, it is sufficient to show that the representations of all the elements of S_i with respect to Υ are not identical. For this aim, since, for $i \in [n]$, we have $v_i \in S_i$ and v_i is adjacent to an element of each $S_k \in \Upsilon$, $k \neq i$, then we have

$$r(v_i|\Upsilon) = (1, ..., 1, \overbrace{0}^{i^{\text{th}}}, 1, ..., 1),$$

also for $v_{kj} \in S_i$ we have

$$r(v_{kj}|\Upsilon) = (2, ..., 2, \overbrace{0}^{i^{\text{th}}}, 2, ..., 2, \overbrace{1}^{k^{\text{th}}}, 2, ..., 2).$$

Hence, the partition set Υ is a resolving partition of $G \circ \overline{K_s}$. Thus the size of any minimum resolving partition of $G \circ \overline{K_s}$ is n.

(iv) Let $V(G \circ \overline{K_t}) = V_1 \cup V_2$, where V_1 and V_2 can be defined similarly to in the proof of Part (i). Based on the Observation 3.1, $pd(G \circ \overline{K_t}) \geq n - 1$. For $i \in \{1, 2, ..., n\} - \{n-1, n\}$, if we consider $S_i = \{v_i, v_{1i}, v_{2i}, ..., v_{ni}\}$, then by similar methods in previous parts we can show that the partition set $\Pi = \{S_1, S_2, ..., S_{n-2}, \{v_{n-1}\}, \{v_n\}\}$ is a resolving partition of $G \circ \overline{K_t}$. Due to the uncertainty of the graph structure, we can not show that the partition set Π is a minimal resolving partition. Therefore the size of any minimum resolving partition of $G \circ \overline{K_t}$ is at most n. □

Example 3.3. Let G be any graph of order 3, and let $G \circ \overline{K_2}$ be a graph with vertex set $V_1 \cup V_2$, where $V_1 = V(G) = \{v_1, v_2, v_3\}$ and

$$V_2 = \{v_{11}, v_{12}; v_{21}, v_{22}; v_{31}, v_{32}\},$$

so that every vertex $v_i \in V_1$ is adjacent to the vertices v_{i1}, v_{i2}, and $deg(v_{ij}) = 1$, for every vertex $v_{ij} \in V_2$. Then we can see that the partition set

$$\Delta = \{\{v_1, v_{21}, v_{31}\}, \{v_2, v_{11}, v_{32}\}, \{v_3, v_{12}, v_{22}\}\},$$

is a resolving partition of $G \circ \overline{K_2}$, and hence based on Theorem 3.2 (iii), $pd(G \circ \overline{K_2}) = 3$.

Corollary 3.4. *Let n be fixed positive integer, so that $n \geq 5$. If $m = n-3 \geq 2$, then $pd(C_n \circ \overline{K_m}) < n$.*

Proof. Let $V(C_n \circ \overline{K_m}) = V_1 \cup V_2$, where $V_1 = V(C_n) = \{v_1, ..., v_n\}$, $V_2 = \{A_{1j}, A_{2j}, ..., A_{nj}\}$, and let $A_{ij} = \cup_{j=1}^{m}\{v_{ij}\}$, $1 \leq i \leq n$ so that every vertex $v_i \in V_1$ is adjacent to the vertices $v_{i1}, v_{i2}, ..., v_{im}$, and $deg(v_{ij}) = 1$ for every vertex $v_{ij} \in V_2$. Based on the Observation 3.1, $pd(C_n \circ \overline{K_m}) \geq n-2$. If we consider $S_1 = \{v_1, v_{11}, v_{21}, ..., v_{n1}\}$, and for $i \in \{2, 3, ..., n-3\}$, we consider $S_i = \{v_{i+1}, v_{1i}, v_{2i}, ..., v_{ni}\}$, $S_{n-2} = \{v_2, v_n\}$ and $S_{n-1} = \{v_{n-1}\}$, then we can see that the partition set $\Omega = \{S_1, S_2, ..., S_{n-1}\}$ is a resolving partition of $C_n \circ \overline{K_m}$. Because according to the choice of each $S_i \in \Omega$, we can see that all the vertices in V_1, so that the representations of these vertices are identical with respect to the part $S_{n-2} = \{v_2, v_n\}$ of Ω, belong in distinct parts of Ω, and the representations of these vertices are not identical with respect to the part $S_{n-1} = \{v_{n-1}\}$ of Ω. On the other hand, there is exactly one vertex of each A_{ij} so that belongs in each part S_i, for $i \in \{3, ..., n-3\}$ so that is adjacent to v_i. Therefore, the representations of all the elements of each part S_i with respect to Ω are not identical, and hence, Ω is a resolving partition of $C_n \circ \overline{K_m}$. Thus $pd(C_n \circ \overline{K_m}) \leq n-1$.

□

Example 3.5. Let $C_5 \circ \overline{K_2}$ be a graph with vertex set $V_1 \cup V_2$, where $V_1 = V(C_5) = \{v_1, ..., v_5\}$ and

$$V_2 = \{v_{11}, v_{12}, v_{21}, v_{22}, v_{31}, v_{32}, v_{41}, v_{42}, v_{51}, v_{52}\},$$

so that every vertex $v_i \in V_1$ is adjacent to the vertices v_{i1}, v_{i2}, and $deg(v_{ij}) = 1$, for every vertex $v_{ij} \in V_2$. Then we can see that the partition set

$$\Delta = \{\{v_1, v_{11}, v_{21}, v_{31}, v_{41}, v_{51}\}, \{v_3, v_{12}, v_{22}, v_{32}, v_{42}, v_{52}\}, \{v_2, v_5\}, \{v_4\}\},$$

is a resolving partition of $C_5 \circ \overline{K_2}$, and hence $pd(C_5 \circ \overline{K_2}) \leq 4$.

The following proposition, is easy to obtain.

Proposition 3.6. *For any graph G of order $n \geq 1$, if $m > n$, then the cardinality of a minimum resolving partition of $G \circ \overline{K_m}$ is m.*

Now, we obtain the k-domination number of $C_n \circ \overline{K_m}$:

Theorem 3.7. *For $n \geq 3$ and $m \geq 2$,*

$$\gamma_k(C_n \circ \overline{K_m}) = \begin{cases} n & \text{if } k = 1, \\ nm & \text{if } 2 \leq k \leq m, \\ nm + \gamma_1(C_n) & \text{if } k = m+1, \\ nm + \gamma_2(C_n) & \text{if } k = m+2. \end{cases}$$

Proof. If $k = 1$ and we consider the subset $C = V(C_n) = \{v_1, ..., v_n\}$ of vertices of C_n, then the set C is a minimal 1-dominating set of $C_n \circ \overline{K_m}$, because every vertex of $V(C_n \circ \overline{K_m}) \setminus C$ has exactly one neighbor in C, and so $\gamma_1(C_n \circ \overline{K_m}) = n$. Now, if $2 \leq k \leq m$ and $D \subseteq V(C_n \circ \overline{K_m})$ is an arbitrary k-dominating set of $C_n \circ \overline{K_m}$, then any pendant vertex of $C_n \circ \overline{K_m}$ must lie in each arbitrary k-dominating set of $C_n \circ \overline{K_m}$, and so the size of any k-dominating set in $C_n \circ \overline{K_m}$ is greater than or equal to nm. If we now consider the subset $D = V_2 = \{v_{11}, ..., v_{1m}, ..., v_{n1}, ..., v_{nm}\}$, of vertices of $C_n \circ \overline{K_m}$, then we can see that this set is a k-dominating set of $C_n \circ \overline{K_m}$, because every vertex of $V(C_n \circ \overline{K_m}) \setminus D$ has at least k neighbors in D. In particular, we conclude that this set is a minimal k-dominating set of $C_n \circ \overline{K_m}$, and so $\gamma_k(C_n \circ \overline{K_m}) = nm$. Especially, if $k > m$ and $F \subseteq V(C_n \circ \overline{K_m})$ is an arbitrary k-dominating set of $C_n \circ \overline{K_m}$, then by similar way is done in the previous result, it can be shown that any k-dominating set in $C_n \circ \overline{K_m}$ has cardinality greater than nm, it follows that if $k = m + 1$, then $\gamma_k(C_n \circ \overline{K_m}) = nm + \gamma_1(C_n)$, also, if $k = m + 2$, then $\gamma_k(C_n \circ \overline{K_m}) = nm + \gamma_2(C_n)$. □

4. Conclusion

This paper considered the partition dimension of cocktail party graph $CP(m + 1)$ and corona product $G \circ \overline{K_m}$. Also the k-domination numbers of $CP(m + 1)$ and corona product $C_n \circ \overline{K_m}$ has computed for some cases.

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THE PARTITION DIMENSION AND k-DOMINATION NUMBER OF TWO SPECIFIC GRAPHS

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بُعد افرازی و عدد k-احاطه‌گری دو نوع از گراف‌های خاص

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برای یک $k-$ افراز مرتب $\Omega = \{S_1, S_2, ..., S_k\}$ از مجموعه رئوس یک گراف همبند G و یک رأس v از G، نمایش v نسبت به Ω به‌صورت $k-$ تایی

$$r(v|\Omega) = (d(v, S_1), d(v, S_2), ..., d(v, S_k)),$$

می باشد. افراز Ω یک افراز تفکیک‌کننده برای گراف G نامیده می‌شود، هرگاه برای هر دو رأس متمایز u و v از گراف G، $r(u|\Omega) \neq r(v|\Omega)$. اندازه کوچک‌ترین افراز تفکیک‌کننده در گراف G را با $pd(G)$ نشان می‌دهند. زیرمجموعه $D \subseteq V(G)$ را یک مجموعه $k-$ احاطه‌گر در G می‌نامند، هرگاه هر رأس $V(G) \setminus D$ با حداقل k رأس از D مجاور باشد. اندازه کوچک‌ترین مجموعه $k-$ احاطه‌گر در گراف G را با $\gamma_k(G)$ نشان می‌دهند. در این مقاله، بُعد افرازی گراف مهمانی کوکتل $CP(m+1)$ و حاصل‌ضرب کرونا $G \circ \overline{K_m}$ را تعیین می‌کنیم. به‌علاوه، اعداد $k-$ احاطه‌گر گراف‌های $CP(m+1)$ و $C_n \circ \overline{K_m}$ را محاسبه می‌کنیم.

کلمات کلیدی: مجموعه تفکیک‌کننده، بُعد افراز، عدد احاطه‌گر، گراف مهمانی کوکتل، حاصل‌ضرب کرونا.