Journal of Algebraic Systems, Vol. 14(No. 2): (2026), pp 343-367.
https://doi.org/10.22044/JAS.2024.14066.1800

CLASSIFICATION OF MONOIDS BY CONDITION $(GPWP_{sec})$ OF RIGHT ACTS

M. Shafiei, H. Mohammadzadeh Saany* and P. Rezaei

Abstract. In (Categories and General Algebraic Structures with Applications, 12(1):175-197 (2020)), Rashidi et al. introduced GPW-flatness of acts over monoids as a generalization of principal weak flatness. In this paper, we introduce Condition $(GPWP_{sec})$ of acts over monoids and compare it with GPW-flatness. Also, we obtain some general properties of Condition $(GPWP_{sec})$ and characterize those monoids for which this condition implies some other properties and vice versa.

1. Introduction

Throughout this paper, we use S to denote a monoid. We refer the reader to [6, 8] for the basic results, definitions and terminology related to semigroups and acts over monoids, and to [9, 10] for those definitions and results on flatness which are used in the paper.

We say that S is *right (left) reversible* if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'(su = s'v)$. A right ideal K of S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

An element s of S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$, that is, $ts = t's$, $t, t' \in S$, implies $te = t'e$. Also, S is called *left PP* if every $s \in S$ is right e-cancellable, for some idempotent $e \in S$.

It is easy to see that S is left PP if and only if for every $s \in S$, there exists $e \in E(S)$ such that $ker\rho_s = ker\rho_e$. This is equivalent to the condition that every principal left ideal of S is projective. Right PP monoids can be defined similarly. An element s of S is called *right semi-cancellative* if $ts = t's$, $t, t' \in S$, implies the existence of $r \in S$ such that $s = rs$ and $tr = t'r$. We say that S is *left PSF* if all principal left ideals of S are strongly flat. It is easy to see that S is left PSF if and only if every $s \in S$ is right semi-cancellable.

An element s of S is called *regular* if $sxs = s$ for some $x \in S$. We say that S is *regular* if all its elements are regular. An element s of S is called *left*

MSC(2020): 20M30.
Keywords: Condition $(GPWP_{sec})$; Eventually left PP; GPW-Left stabilizing.
Received: 24 January 2024, Accepted: 14 October 2024.
*Corresponding author.

almost regular if there exist $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable $c_1, c_2, ..., c_m \in S$ such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs. \end{aligned}$$

If all elements of S are left almost regular, then S is called left almost regular. It can be seen that every left almost regular monoid is left PP [8, Proposition 4.1.3].

A *right S-act* is a non-empty set A, usually denoted by A_S, on which S acts unitarian from the right, that is, $(as)t = a(st)$ and $a1 = a$ for every $a \in A$ and $s, t \in S$, where 1 is the identity of S.

We recall the following definitions from [2, 9, 10].

- An S-act A_S is *weakly pullback flat* (*WPF*) if the corresponding ϕ is bijective for every pullback diagram $P(S, S, f, g, S)$.
- An S-act A_S is *weakly kernel flat* (*WKF*) if the corresponding ϕ is bijective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.
- An S-act A_S is *principally weakly kernel flat* (*PWKF*) if for every pullback diagram $P(Ss, Ss, f, f, S)$ with $s \in S$, the corresponding ϕ is bijective.
- An S-act A_S is *translation kernel flat* (*TKF*) if the corresponding ϕ is bijective for every pullback diagram $P(S, S, f, f, S)$.
- An S-act A_S is *weakly homoflat* (*WP*) if for all $s, t \in S$, every homomorphism $f : {}_S(Ss \cup St) \to {}_SS$ and all $a, a' \in A_S$, if $af(s) = a'f(t)$, then there exist $a'' \in A_S$, $u, v \in S$ and $s', t' \in \{s, t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A \otimes_S (Ss \cup St)$ and $f(us') = f(vt')$.
- An S-act A_S is *principally weakly homoflat* (*PWP*) if $as = a's$ for $a, a' \in A_S$ and $s \in S$ implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

2. Main results

In this section, we introduce Condition ($GPWP_{sec}$) and present some of its general properties.

Definition 2.1. A monoid S is called *eventually left PP* if for every $s \in S$, a natural number $n \in \mathbb{N}$ exists such that s^n is right e-cancellable for some $e \in E(S)$. Equivalently, S is called eventually left PP if for every $s \in S$, a natural number $n \in \mathbb{N}$ can be found such that the principal left ideal Ss^n is projective.

It is clear that every left PP monoid is eventually left PP. The following example shows that the converse of this assertion is not true.

Example 2.2. Let $S = \{0, a, b, c\}$ be the monoid with the following table.

	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	b
c	0	c	b	a

It is clear that S is a commutative monoid. Also, S is not left PP, but it is eventually left PP.

Definition 2.3. An S-act A_S is strongly e-cancellative-($GPWP$) (or satisfies Condition ($GPWP_{sec}$)) if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for every $a, a' \in A_S$,

$$as^n = a's^n \Rightarrow (\exists e \in E(S))(ae = a'e, es^n = s^n).$$

Now, we establish some general properties.

Proposition 2.4. *The following statements are true.*

(1) Θ_S *satisfies Condition* ($GPWP_{sec}$).
(2) *If A_S satisfies Condition ($GPWP_{sec}$), then every subact of A_S satisfies the same condition.*
(3) *Any retract of a right S-act satisfying Condition ($GPWP_{sec}$) also satisfies Condition ($GPWP_{sec}$).*
(4) *If $A = \coprod_{i \in I} A_i$, where each A_i is a right S-act, satisfies Condition ($GPWP_{sec}$), then A_i satisfies Condition ($GPWP_{sec}$) for every $i \in I$.*

Proof. (1), (2) and (4) are obvious.

(3). Suppose that a right S-act B_S satisfies Condition ($GPWP_{sec}$) and $s \in S$. Then, according to our definition of ($GPWP_{sec}$), there exists $n \in \mathbb{N}$. Also, assume that A_S is a retract of B_S. Then, there exist homomorphisms $f : B_S \to A_S$ and $f' : A_S \to B_S$ such that $ff' = id_{A_S}$. Let $as^n = a's^n$ for $a, a' \in A_S$. Then $f'(as^n) = f'(a's^n)$ and so, $f'(a)s^n = f'(a')s^n$. Since $f'(a)$, $f'(a') \in B_S$ and B_S satisfies Condition ($GPWP_{sec}$), there exists

$e \in E(S)$ such that $f'(a)e = f'(a')e$ and $es^n = s^n$. Now, we obtain $f(f'(ae)) = f(f'(a'e))$. This means that A_S satisfies Condition $(GPWP_{sec})$. $\square$

A right S-act A_S is called *GPW-flat* if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$ such that for every $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S (Ss^n)$ (see [11]).

Proposition 2.5. *The following statements are true.*

(1) *Every right act satisfying Condition $(GPWP_{sec})$ is GPW-flat.*
(2) *If S is eventually left PP, then every GPW-flat right act satisfies Condition $(GPWP_{sec})$.*

Proof. (1). Suppose that A_S satisfies Condition $(GPWP_{sec})$ and $s \in S$. Then, there exists $n \in \mathbb{N}$. Let $as^n = a's^n$, for $a, a' \in A_S$. By the assumption, $e \in E(S)$ exists such that $es^n = s^n$ and $ae = a'e$. Thus,

$$a \otimes s^n = a \otimes es^n = ae \otimes s^n = a'e \otimes s^n = a' \otimes es^n = a' \otimes s^n$$

in $A \otimes_S Ss^n$. Hence, A_S is GPW-flat.

(2). Let S be eventually left PP. Then, $n \in \mathbb{N}$ exists such that s^n is right e-cancellable for some $e \in E(S)$. Also, assume that A_S is a GPW-flat right S-act and $s \in S$. Let $as^n = a's^n$ for $a, a' \in A_S$. By [11, Proposition 2.3],

$$\begin{array}{rlrl} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 s^n &= t_1 s^n \\ a_2 t_2 &= a_3 s_3 & s_2 s^n &= t_2 s^n \\ &\vdots & &\vdots \\ a_k t_k &= a' & s_k s^n &= t_k s^n, \end{array}$$

for $k \in \mathbb{N}$, $a_1, ..., a_k \in A_S$ and $s_1, t_1, ..., s_k, t_k \in S$. Since s^n is right e-cancellable,

$$\begin{array}{rlrl} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 e &= t_1 e \\ a_2 t_2 &= a_3 s_3 & s_2 e &= t_2 e \\ a_2 t_2 &= a_3 s_3 & s_2 e &= t_2 e \\ &\vdots & &\vdots \\ a_k t_k &= a' & s_k e &= t_k e. \end{array}$$

Therefore, $ae = a_1s_1e = a_1t_1e = a_2s_2e = a_2t_2e = \cdots = a_kt_ke = a'e$ and $es^n = s^n$. $\square$

Proposition 2.6. *For any family $\{A_i\}_{i\in I}$ of right S-acts, if $\prod_{i\in I} A_i$ satisfies Condition $(GPWP_{sec})$, then A_i satisfies Condition $(GPWP_{sec})$ for every $i \in I$.*

Proof. Let $s \in S$ and $i \in I$. By our assumption, $n \in \mathbb{N}$ exists such that $as^n = a's^n$, for $a, a' \in \prod_{i\in I} A_i$, implies $ae = a'e$ and $es^n = s^n$ for some $e \in E(S)$. Let $a_is^n = a_i's^n$ for every $a_i, a_i' \in A_i$. Consider a fixed element $a_k \in A_k$ for every $k \neq i$ and let

$$c_k = \begin{cases} a_i & if\ k = i \\ a_k & if\ k \neq i \end{cases},$$

$$c_k' = \begin{cases} a_i' & if\ k = i \\ a_k & if\ k \neq i \end{cases}.$$

Then $(c_k)_Is^n = (c_k')_Is^n$. Hence, the assumption allows us to write $(c_k)_Ie = (c_k')_Ie$ and $es^n = s^n$, for $e \in E(S)$. Now $a_ie = a_i'e$ and hence, A_i satisfies Condition $(GPWP_{sec})$. $\square$

Lemma 2.7. *Let $E(S) \subseteq C(S)$, where $C(S)$ is the center of S. Also, assume that A_S is a right S-act, $s \in S$ and $n \in \mathbb{N}$ exists such that $as^n = a's^n$, $a, a' \in A_S$, implies the existence of $f \in E(S)$ satisfying $af = a'f$ and $fs^n = s^n$. Let $m \in \mathbb{N}$ and $m > n$. If $as^m = a's^m$, then $ae = a'e$ and $es^m = s^m$ for some $e \in E(S)$.*

Proof. Since $m > n$, there exists $k \in \mathbb{N}$ such that $kn \leq m < (k+1)n$. Suppose that $as^m = a's^m$, for $a, a' \in A_S$. Then, $(as^{(m-n)})s^n = (a's^{(m-n)})s^n$. By the assumption, $e_1 \in E(S)$ exists such that $as^{(m-n)}e_1 = a's^{(m-n)}e_1$ and $e_1s^n = s^n$. By the equality $e_1s^n = s^n$ we obtain $e_1s^m = s^m$. Since $E(S) \subseteq C(S)$, it follows that $ae_1s^{(m-n)} = a'e_1s^{(m-n)}$ and so,

$$(ae_1s^{(m-2n)})s^n = (a'e_1s^{(m-2n)})s^n.$$

Again, the assumption implies the existence of $e_2 \in E(S)$ such that

$$ae_1s^{(m-2n)}e_2 = a'e_1s^{(m-2n)}e_2$$

and $e_2s^n = s^n$, and allows us to write $as^{(m-2n)}e_1e_2 = a's^{(m-2n)}e_1e_2$. Thus, by the equality $e_2s^n = s^n$ we obtain $e_2s^m = s^m$ and so, $e_1e_2s^m = s^m$. Continuing this procedure, we find $e_1, e_2, ..., e_k \in E(S)$ such that

$$as^{(m-kn)}e_1e_2...e_k = a's^{(m-kn)}e_1e_2...e_k \text{ and } e_1e_2...e_ks^m = s^m.$$

Now, we may consider two cases.

Case 1. If $m = kn$, then we deduce from

$$as^{(m-kn)}e_1e_2...e_k = a's^{(m-kn)}e_1e_2...e_k$$

that $ae_1e_2...e_k = a'e_1e_2...e_k$. Let $e = e_1e_2...e_k$. Then $ae = a'e$ and $es^m = s^m$, and we are done.

Case 2. If $m \neq kn$, then multiplying the equality

$$ae_1e_2...e_ks^{(m-kn)} = a'e_1e_2...e_ks^{(m-kn)}$$

by $s^{(k+1)n-m}$ we obtain

$$ae_1e_2...e_ks^n = a'e_1e_2...e_ks^n$$

and so, $e_{k+1} \in E(S)$ exists such that

$$ae_1e_2...e_ke_{k+1} = a'e_1e_2...e_ke_{k+1}$$

and $e_{k+1}s^n = s^n$. From the equality $e_{k+1}s^n = s^n$ it follows that $e_{k+1}s^m = s^m$. Also, $e_1e_2...e_{k+1}s^m = s^m$. Let $e = e_1e_2...e_ke_{k+1}$. Then $ae = a'e$ and $es^m = s^m$, and so, we are done. □

Proposition 2.8. *Let $E(S) \subseteq C(S)$. Also, assume that for each $1 \leq i \leq m$, A_i is a right S-act. Then, $\prod_{i=1}^{m} A_i$ satisfies Condition $(GPWP_{sec})$ if and only if A_i satisfies Condition $(GPWP_{sec})$ for every $1 \leq i \leq m$.*

Proof. Necessity. This is obvious by Proposition 2.6.

Sufficiency. Suppose that A_i satisfies Condition $(GPWP_{sec})$, for every $1 \leq i \leq m$, and let $s \in S$. Then, $n_i \in \mathbb{N}$ exists such that $a_is^{n_i} = a'_is^{n_i}$, for $a_i, a'_i \in A_i$, implies $a_ie = a'_ie$ and $es^{n_i} = s^{n_i}$ for some $e \in E(S)$. Let $n = \max\{n_1, n_2, ..., n_m\}$. Then $(a_1, a_2, ..., a_m)s^n = (a'_1, a'_2, ..., a'_m)s^n$ for $a_i, a'_i \in A_i$, $1 \leq i \leq m$. By Lemma 2.7, the equality $a_1s^n = a'_1s^n$ implies the existence of $e_1 \in E(S)$ such that $a_1e_1 = a'_1e_1$ and $e_1s^n = s^n$. Therefore, the equality $a_2s^n = a'_2s^n$ implies $a_2e_1s^n = a'_2e_1s^n$. Again, by Lemma 2.7, $e_2 \in E(S)$ exists such that $a_2e_1e_2 = a'_2e_1e_2$ and $e_2s^n = s^n$. Therefore, $a_1e_1e_2 = a'_1e_1e_2$, $a_2e_1e_2 = a'_2e_1e_2$ and $e_1e_2s^n = s^n$. Continuing this procedure, after m steps we find $e_1, e_2, ..., e_m \in E(S)$ such that for each i, $a_ie_1e_2...e_m = a'_ie_1e_2...e_m$, and $e_1e_2...e_ms^n = s^n$. Let $e = e_1e_2...e_m$. By the assumption, e is an idempotent. Thus, $a_ie = a'_ie$ for each i and $es^n = s^n$. Hence $(a_1, a_2, ..., a_m)e = (a'_1, a'_2, ..., a'_m)e$ and $es^n = s^n$, as required. □

Theorem 2.9. *Let $E(S)$ be a submonoid of S. Then, S_S satisfies Condition $(GPWP_{sec})$ if and only if the right S-act S^n_S satisfies Condition $(GPWP_{sec})$ for any $n \in \mathbb{N}$.*

Proof. Necessity. Suppose that S_S satisfies Condition $(GPWP_{sec})$. Let $s \in S$. Then, $n \in \mathbb{N}$ exists such that $ts^n = t's^n$, for every $t, t' \in S$, implies $te = t'e$ and $es^n = s^n$ for some $e \in E(S)$. If $(a_1, a_2, ..., a_m)s^n = (a'_1, a'_2, ..., a'_m)s^n$ for $a_i, a'_i \in S$ and $1 \leq i \leq m$, then $a_is^n = a'_is^n$ for any $1 \leq i \leq m$. By the assumption, there exists $e_1 \in E(S)$ such that $a_1e_1 = a'_1e_1$ and $e_1s^n = s^n$. The equalities $a_2s^n = a'_2s^n$ and $e_1s^n = s^n$ imply $a_2e_1s^n = a'_2e_1s^n$. Again, by the assumption, $e_2 \in E(S)$ exists such that $a_2e_1e_2 = a'_2e_1e_2$ and $e_2s^n = s^n$. Therefore, $a_1e_1e_2 = a'_1e_1e_2$, $a_2e_1e_2 = a'_2e_1e_2$ and $e_1e_2s^n = s^n$. Continuing this procedure, after m steps we find $e_1, e_2, ..., e_m \in E(S)$ such that for each i, $a_ie_1e_2...e_m = a'_ie_1e_2...e_m$, and $e_1e_2...e_ms^n = s^n$. Let $e = e_1e_2...e_m$. Since $E(S)$ is a submonoid of S, e is an idempotent. Thus, $a_ie = a'_ie$ for each i and $es^n = s^n$. Hence $(a_1, a_2, ..., a_m)e = (a'_1, a'_2, ..., a'_m)e$ and $es^n = s^n$, as required.

Sufficiency. If the right S-act S^n satisfies Condition $(GPWP_{sec})$, then by Proposition 2.6, S_S satisfies Condition $(GPWP_{sec})$. □

Recall from [1] that for a monoid S, the cartesian product $S \times S$ equipped with the right S-action $(s,t)u = (su, tu)$, $s, t, u \in S$, is called the *diagonal act* of S, which is denoted by $D(S)$.

In the following theorem, we obtain equivalent conditions for S^n_S to satisfy Condition $(GPWP_{sec})$.

Theorem 2.10. *Let $E(S)$ be a submonoid of S. The following statements are equivalent.*

(1) *For any $n \in \mathbb{N}$, S^n_S satisfies Condition $(GPWP_{sec})$.*
(2) *There exists $m \in \mathbb{N}$ such that S^m_S satisfies Condition $(GPWP_{sec})$.*
(3) *$D(S)$ satisfies Condition $(GPWP_{sec})$.*
(4) *S_S satisfies Condition $(GPWP_{sec})$.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are obvious.

(2) ⇒ (4). Define $\psi : S_S \to S^m_S$ by $\psi(s) = (s, s, ..., s)$. It is obvious that ψ is a monomorphism. Thus $S_S \cong Im\psi \leq S^m_S$ and so, by part (2) of Proposition 2.4, $Im\psi$ satisfies Condition $(GPWP_{sec})$. Hence S_S satisfies Condition $(GPWP_{sec})$.

(3) ⇒ (4). This easily follows from the proof of (2) ⇒ (4).

(4) ⇒ (1). By Theorem 2.9, the proof is straightforward. □

Definition 2.11. A right ideal K of S is called *GPW-left stabilizing* if for every $s \in S$, there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$, implies $ls^n = ks^n$ for some $k \in K$.

It is clear that every left stabilizing right ideal of S is GPW-left stabilizing.

Theorem 2.12. *Let K be a proper right ideal of S. Then, the following statements are true.*

1. *If $A_S = S\coprod^K S$ satisfies Condition $(GPWP_{sec})$, then K is GPW-left stabilizing and S_S satisfies Condition $(GPWP_{sec})$.*
2. *If K is GPW-left stabilizing and S is eventually left PP, then $A_S = S\coprod^K S$ satisfies Condition $(GPWP_{sec})$.*

Proof. (1). By part (1) of Theorem 2.5, $A_S = S\coprod^K S$ is GPW-flat and so, by [11, Theorem 2.10], K is GPW-left stabilizing. On the other hand, $A_S = S\coprod^K S = B_S \cup C_S$, where $B_S = \{(l,x) | l \in S \setminus K\} \dot{\cup} K$,

$$C_S = \{(t,y) | t \in S \setminus K\} \dot{\cup} K,$$

$B_S, C_S \leq A_S$ and $B_S \cong S_S \cong C_S$. By part (2) of Theorem 2.4, B_S satisfies Condition $(GPWP_{sec})$. Therefore, by the isomorphism $B_S \cong S_S$, S_S satisfies Condition $(GPWP_{sec})$.

(2). Since K is GPW-left stabilizing, it follows from [11, Theorem 2.10] that $A_S = S\coprod^K S$ is GPW-flat. On the other hand, since S is eventually left PP, GPW-flatness and Condition $(GPWP_{sec})$ are equivalent by part (2) of Theorem 2.5. Hence, $A_S = S\coprod^K S$ satisfies Condition $(GPWP_{sec})$. □

Every right cancellative monoid is left PP, and accordingly, eventually left PP. But, it is clear that no proper ideal of such a monoid can be GPW-left stabilizing.

Example 2.13. It is clear that $S = (\mathbb{N}, .)$ is a commutative and cancellative monoid, and $K = \mathbb{N} \setminus \{1\}$ is one of its ideals that is not GPW-left stabilizing.

The following result was obtained by Golchin in [3]. Let $S = G \dot{\cup} I$, where G is a group and I is an ideal of S, and assume that A is a right S-act that is ((principally) weakly) flat, torsion free, and satisfies Condition (P) or (P_E) as a right I^1-act. Then, it satisfies the same properties as a right S-act.

Similarly, we establish the following theorem for Condition $(GPWP_{sec})$.

Theorem 2.14. *Let $S = G \dot{\cup} I$, and A be a right S-act. Then, A satisfies Condition $(GPWP_{sec})$ as a right I^1-act if and only if it satisfies Condition $(GPWP_{sec})$ as a right S-act.*

Proof. Necessity. Suppose that A satisfies Condition $(GPWP_{sec})$ as a right I^1-act and $s \in S$. Then, we may consider two cases.

Case 1. $s \in G$. Then $Ss = S$ and so, for every $n \in \mathbb{N}$, $Ss^n = Ss = S$. If $as = a's$ for $a, a' \in A$, then $a = a'$. By putting $e = 1$, the desired result follows.

Case 2. $s \in I \subseteq I^1$. Since A satisfies Condition $(GPWP_{sec})$ as a right I^1-act, there exists a natural number $n \in \mathbb{N}$ such that for $a, a' \in A$, $as^n = a's^n$ implies $ae = a'e$ and $es^n = s^n$, for $e \in E(I^1) \subseteq E(S)$.

Therefore, by cases 1 and 2, A satisfies Condition $(GPWP_{sec})$ as a right S-act.

Sufficiency. Suppose that A satisfies Condition $(GPWP_{sec})$ as a right S-act. Since $E(S) = E(I) \cup \{1\} = E(I^1)$, A satisfies Condition $(GPWP_{sec})$ as a right I^1-act. □

Corollary 2.15. *Let $S = G \dot{\cup} I$, where G is a group and I is an ideal of S. If all right I^1-acts satisfy Condition $(GPWP_{sec})$, then all right S-acts satisfy Condition $(GPWP_{sec})$.*

Here, we present a criterion that allows us to determine whether a cyclic right S-act satisfies Condition $(GPWP_{sec})$.

Proposition 2.16. *Let ρ be a right congruence on S. Then, the right S-act S/ρ satisfies Condition $(GPWP_{sec})$ if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for $x, y \in S$, $(xs^n)\rho(ys^n)$ implies $(xe)\rho(ye)$ and $es^n = s^n$ for some $e \in E(S)$.*

Proof. By Definition 2.3, the proof is straightforward. □

Corollary 2.17. *For a monoid S, the principal right ideal zS satisfies Condition $(GPWP_{sec})$ if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for every $x, y \in S$, $zxs^n = zys^n$ implies $zxe = zye$ and $es^n = s^n$ for some $e \in E(S)$.*

Proof. Since $zS \cong S/ker\lambda_z$, we just need to apply Proposition 2.16 with $\rho = ker\lambda_z$. □

Now, we present a characterization of Rees factor S-acts that satisfy Condition $(GPWP_{sec})$.

Proposition 2.18. *Let K be a right ideal of S. The right Rees factor S-act S/K satisfies Condition $(GPWP_{sec})$ if and only if for every $s \in S$, a natural number $n \in \mathbb{N}$ exists such that,*

$$(\forall x, y \in S)[((xs^n = ys^n \in S \setminus K) \vee (xs^n, ys^n \in K))$$
$$\Rightarrow (\exists e \in E(S))(es^n = s^n \wedge (xe = ye \vee (xe, ye \in K)))].$$

Proof. Necessity. Suppose that the right Rees factor S-act S/K satisfies Condition $(GPWP_{sec})$ and $s \in S$. Then, a natural number $n \in \mathbb{N}$ exists such that Proposition 2.16 is satisfied. Let $xs^n = ys^n \in S \setminus K$ or $xs^n, ys^n \in K$, for $x, y \in S$. Then $(xs^n)\rho_K(ys^n)$ and by Proposition 2.16, there exists $e \in E(S)$ such that $(xe)\rho_K(ye)$ and $es^n = s^n$. Hence $xe = ye$ or $xe, ye \in K$, as required.

Sufficiency. Note that if $K = S$, then by Proposition 2.4, $S/K \cong \Theta_S$ satisfies Condition $(GPWP_{sec})$. Assume that K is a proper right ideal of S and $s \in S$. By the assumption, a natural number $n \in \mathbb{N}$ exists such the condition is satisfied. Let $(xs^n)\rho_K(ys^n)$ for $x, y \in S$. Then, $xs^n, ys^n \in K$ or $xs^n = ys^n$. By the condition, there exists $e \in E(S)$ such that $es^n = s^n$ and $(xe)\rho_K(ye)$. Therefore, by Proposition 2.16, S/K satisfies Condition $(GPWP_{sec})$, as required. □

In the following example, we show that the converse of part (1) of Proposition 2.5 is not true in general.

Example 2.19. Let $(I, \leq)$ be a totally ordered set with no successor for each element (as $\mathbb{R}$). Consider the commutative monoid

$$S = \{{x_i}^m | i \in I, m \in \mathbb{N}\} \cup \{1\} \cup \{0\},$$

in which

$${x_i}^m {x_j}^n = \begin{cases} {x_j}^n & if\ i < j \\ {x_i}^{m+n} & if\ i = j \end{cases}.$$

It is easy to prove that S is PSF. Since S does not have any idempotent except 0, 1, it is not left PP. Now, let $K = 0S = \{0\}$. Then $S/K \cong S_S$ is free, and so it is GPW-flat. But since 0, 1 are only idempotent elements of S, $S/K \cong S_S$ dose not satisfy Condition $(GPWP_{sec})$.

3. Classification by Condition $(GPWP_{sec})$ of Right Acts

In this section, we present a classification of monoids when acts with other properties satisfy Condition $(GPWP_{sec})$ and vice versa. We also provide a classification of monoids when all their acts satisfy Condition $(GPWP_{sec})$.

Recall from [11] that $s \in S$ is called *eventually regular* if s^n is regular for some $n \in \mathbb{N}$. This means that $s^n = s^n x s^n$ for some $n \in \mathbb{N}$ and $x \in S$. We say that S is *eventually regular* if every $s \in S$ is eventually regular. Obviously, every regular monoid is eventually regular. But, the converse is not true in general.

An element s of S is called *eventually left almost regular* if

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, $s_1, s_2, ..., s_m, r, r_1, ..., r_m \in S$ and right cancellable $c_1, c_2, ..., c_m \in S$. In other words, $s \in S$ is called eventually left almost regular if s^n is left almost regular for some $n \in \mathbb{N}$.

If every element of S is eventually left almost regular, then S is called eventually left almost regular. It is clear that every left almost regular monoid is eventually left almost regular, and every eventually regular monoid is eventually left almost regular.

Lemma 3.1. *Every eventually left almost regular monoid is eventually left PP.*

Proof. Let S be eventually left almost regular and $s \in S$. By the definition,

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, $s_1, s_2, ..., s_m, r, r_1, ..., r_m \in S$ and right cancellable $c_1, c_2, ..., c_m \in S$. Hence, we conclude that

$$\begin{aligned} &s^n r_1 = s_m r s^n r_1 \Rightarrow s_1 c_1 = s_m r s_1 c_1 \Rightarrow s_1 = s_m r s_1 \Rightarrow s_1 r_2 = s_m r s_1 r_2 \\ &\Rightarrow s_2 c_2 = s_m r s_2 c_2 \Rightarrow s_2 = s_m r s_2. \end{aligned}$$

Continuing this procedure, we finally obtain $s_i = s_m r s_i$ for every $1 \leq i \leq m$, which implies $s_m = s_m r s_m$. Hence, $e = s_m r$ is an idempotent such that $es^n = s^n$.

Now, let $l_1 s^n = l_2 s^n$ for $l_1, l_2 \in S$. Then,

$$l_1 s^n r_1 = l_2 s^n r_1 \Rightarrow l_1 s_1 c_1 = l_2 s_1 c_1 \Rightarrow l_1 s_1 = l_2 s_1 \Rightarrow l_1 s_1 r_2 = l_2 s_1 r_2$$

$$\Rightarrow l_1 s_2 c_2 = l_2 s_2 c_2 \Rightarrow l_1 s_2 = l_2 s_2.$$

Continuing this procedure, we obtain $l_1 s_i = l_2 s_i$ for every $1 \leq i \leq m$. Thus

$$l_1 s_m = l_2 s_m \Rightarrow l_1 s_m r = l_2 s_m r \Rightarrow l_1 e = l_2 e$$

and so, S is eventually left PP. □

An element a of A_S is called *divisible* by $s \in S$ if $b \in A_S$ exists such that $bs = a$. An act A_S is said to be *divisible* if $Ac = A$, for any left cancellable element c of S. It is clear that A_S is divisible if and only if every element of A_S is divisible by any left cancellable element of S.

Theorem 3.2. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All cyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(4) *All divisible right S-acts satisfy Condition $(GPWP_{sec})$.*
(5) *All principally weakly injective right S-acts satisfy Condition $(GPWP_{sec})$.*
(6) *All fg-weakly injective right S-acts satisfy Condition $(GPWP_{sec})$.*
(7) *All weakly injective right S-acts satisfy Condition $(GPWP_{sec})$.*
(8) *All injective right S-acts satisfy Condition $(GPWP_{sec})$.*
(9) *All cofree right S-acts satisfy Condition $(GPWP_{sec})$.*
(10) *S is eventually regular.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

Since cofree $\Rightarrow$ injective $\Rightarrow$ weakly injective $\Rightarrow$ fg-weakly injective $\Rightarrow$ principally weakly injective $\Rightarrow$ divisible, we immediately obtain the implications $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9)$.

$(3) \Rightarrow (10)$. By part (1) of Proposition 2.5, all right Rees factor acts of S are GPW-flat. It follows from [11, Theorem 4.5] that S is eventually regular.

$(9) \Rightarrow (10)$. Since every right S-act can be embedded into a cofree right S-act, by the assumption, every right S-act is a subact of a right S-act satisfying Condition $(GPWP_{sec})$. By part (2) of Proposition 2.4, all right S-acts satisfy Condition $(GPWP_{sec})$. It follows from Proposition 2.5 that all right S-acts are GPW-flat. Thus, by [11, Theorem 4.5], S is eventually regular.

$(10) \Rightarrow (1)$. By [11, Theorem 4.5], all right S-acts are GPW-flat. Since every eventually regular monoid is eventually left almost regular, and by Lemma 3.1, every eventually left almost regular monoid is eventually left PP, part (2) of Proposition 2.5 shows that all right S-acts satisfy Condition $(GPWP_{sec})$. □

A monoid S is called *right (left) generally regular* if for every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$ ($s^n = s^n xs$).

A monoid S for which all ($GPWP_{sec}$) right S-acts are divisible is not necessarily eventually regular. This is the content of the following example.

Example 3.3. Let $S = \mathbb{N} \cup G$, where $\mathbb{N}$ is the set of natural numbers and G is a non-trivial group with unit element e, and define the multiplication on S by $ng = gn = n$ for every $g \in G$ and $n \in \mathbb{N}$. Clearly, every left(right) cancellative element of S is left(right) invertible. Thus, all right S-acts are divisible by [11, Theorem 4.8]. Therefore, all ($GPWP_{sec}$) right S-acts are divisible. But, S is not right generally regular, and hence is not eventually regular.

Theorem 3.4. *Suppose that* (U) *is a property of S-acts which implies Condition* (PWP)*, and S_S satisfies the property* (U)*. Then, the following statements are equivalent.*

1. *All right S-acts satisfying the property* (U) *also satisfy Condition* ($GPWP_{sec}$)*.*
2. *All finitely generated right S-acts satisfying the property* (U) *also satisfy Condition* ($GPWP_{sec}$)*.*
3. *All cyclic right S-acts satisfying the property* (U) *also satisfy Condition* ($GPWP_{sec}$)*.*
4. *S_S satisfies Condition* ($GPWP_{sec}$)*.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Since S_S is a cyclic act satisfying the property (U), by the assumption, S_S satisfies Condition ($GPWP_{sec}$).

(4) ⇒ (1). Suppose that A_S is a right S-act satisfying the property (U). Let $s \in S$. By the assumption, there exists $n \in \mathbb{N}$ such that,

$$(\forall x, y \in S)(xs^n = ys^n) \Rightarrow (\exists e \in E(S))(xe = ye \wedge es^n = s^n).$$

Let $as^n = a's^n$, for $a, a' \in A_S$. Since A_S satisfies Condition (PWP), there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$. Therefore, we find $e \in E(S)$ such that $ue = ve$ and $es^n = s^n$. Thus $ae = a''ue = a''ve = a'e$ and so, A_S satisfies Condition ($GPWP_{sec}$). □

Note that the property (U) in the above theorem can be any property like the properties of being free, projective, projective generator, strongly flat, WPF, WKF, $PWKF$, TKF, (WP), and also Condition (P), Condition (P') and Condition (PWP).

Theorem 3.5. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All generator right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *$S \times A_S$ satisfies Condition $(GPWP_{sec})$ for every right S-act A_S.*
(4) *$S \times A_S$ satisfies Condition $(GPWP_{sec})$ for every generator right S-act A_S.*
(5) *A_S satisfies Condition $(GPWP_{sec})$ if $Hom(A_S, S_S) \neq \emptyset$.*
(6) *S is eventually regular.*

Proof. The implications $(1) \Rightarrow (2)$, $(3) \Rightarrow (4)$ and $(1) \Rightarrow (5)$ are obvious.

$(1) \Leftrightarrow (6)$. This follows from Theorem 3.2.

$(2) \Rightarrow (3)$. Suppose that A_S is a right S-act. Indeed, the mapping $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$ for $a \in A_S$ and $s \in S$, is an epimorphism in $Act - S$. Then, by [8, Theorem 2.3.16], $S \times A_S$ is a generator. Thus, by the assumption, $S \times A_S$ satisfies Condition $(GPWP_{sec})$.

$(3) \Rightarrow (1)$. This statement immediately follows from Proposition 2.6.

$(4) \Rightarrow (3)$. Suppose that A_S is a right S-act. By the proof of $(2) \Rightarrow (3)$, $S \times A_S$ is a generator right S-act and so, by the assumption, $S \times (S \times A_S)$ satisfies Condition $(GPWP_{sec})$. Then, Proposition 2.6 shows that $S \times A_S$ satisfies Condition $(GPWP_{sec})$.

$(5) \Rightarrow (3)$. Suppose that A_S is a right S-act. By the proof of $(2) \Rightarrow (3)$, $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$ for $a \in A_S$ and $s \in S$, is an epimorphism in $Act - S$. Then, $\mathrm{Hom}(S \times A_S, S_S) \neq \emptyset$. Thus, $S \times A_S$ satisfies Condition $(GPWP_{sec})$ by the assumption. □

Theorem 3.6. *The following statements are equivalent.*

(1) *All torsion free right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All torsion free finitely generated right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All torsion free cyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(4) *All torsion free right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(5) *S is eventually left almost regular.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. By part (1) of Proposition 2.5, all torsion free right Rees factor acts of S are GPW-flat. It follows from [11, Theorem 4.4] that S is eventually left almost regular.

$(5) \Rightarrow (1)$. By [11, Theorem 4.4], all torsion free right S-acts are GPW-flat. On the other hand, by Lemma 3.1, every eventually left almost regular monoid is eventually left PP. So, by part (2) of Proposition 2.5, all torsion free right S-acts satisfy Condition $(GPWP_{sec})$. □

The following example shows that Condition $(GPWP_{sec})$ does not imply Condition PWF.

Example 3.7. Let S=$\{0, 1, e, f, a\}$ be the monoid with the following table.

	0	1	e	f	a
0	0	0	0	0	0
1	0	1	e	f	a
e	0	e	e	a	a
f	0	f	0	f	0
a	0	a	0	a	0

As shown by Rashidi in [11, Example 4.3], S is eventually left almost regular, but fails to be left almost regular. Therefore, by Theorem 3.6, all torsion free right S-acts satisfy Condition $(GPWP_{sec})$, but by [8, Theorem 4.6.5], a torsion free right Rees factor S-act exists that does not satisfy Condition PWF.

Recall from [12] that a right S-act A_S is called *$\mathcal{R}$-torsion free* if for every $a, b \in A_S$ and for any right cancellable $c \in S$, $ac = bc$ and $a\mathcal{R}b$ imply $a = b$, where $\mathcal{R}$ is a Green relation, in the sense that for $a, b \in A_S$, $a\mathcal{R}b$ if and only if $aS = bS$.

Theorem 3.8. *The following statements are equivalent.*

(1) *All $\mathcal{R}$-torsion free right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All $\mathcal{R}$-torsion free finitely generated right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All $\mathcal{R}$-torsion free right S-acts generated by at most two elements satisfy Condition $(GPWP_{sec})$.*
(4) *S is eventually regular.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Let $s \in S$. Since S_S is $\mathcal{R}$-torsion free, by assumption it satisfies Condition $(GPWP_{sec})$. Then, there exists $n \in \mathbb{N}$ such that,

$$(\forall t, t' \in S)(ts^n = t's^n) \Rightarrow (\exists e \in E(S))(te = t'e \wedge es^n = s^n).$$

If $s^nS = S$, then $x \in S$ exists such that $s^nx = 1$ and so, $s^nxs^n = s^n$. Thus, s is an eventually regular element. Now, assume that $s^nS \neq S$. Set

$$A_S = S \coprod^{s^nS} S = \{(l, x) | l \in S \setminus s^nS\} \dot{\cup} s^nS \dot{\cup} \{(t, y) | t \in S \setminus s^nS\}.$$

Indeed,

$$B_S = \{(l,x) | l \in S \setminus s^nS\} \dot{\cup} s^nS \cong S_S \cong \{(t,y) | t \in S \setminus s^nS\} \dot{\cup} s^nS = C_S.$$

Since $A_S = B_S \cup C_S$, A_S is generated by two different elements, namely, $(1,x)$ and $(1,y)$. By the above isomorphism, B_S and C_S satisfy Condition (E) and so, A_S satisfies Condition (E). By [12, Proposition 1.2], A_S is $\mathcal{R}$-torsion free and so, by the assumption, A_S satisfies Condition $(GPWP_{sec})$. Therefore, the equality $(1,x)s^n = (1,y)s^n$ implies the existence of $e \in E(S)$ such that $es^n = s^n$ and $(1,x)e = (1,y)e$. The last equality implies that $e \in s^nS$ and so, $x \in S$ exists such that $e = s^nx$. Therefore, $s^n = es^n = s^nxs^n$. Hence, S is eventually regular.

(4) $\Rightarrow$ (1). The desired result follows from Theorem 3.2. □

Recall from [4, 5, 9] that a right S-act A_S satisfies Condition (E') if $as = as'$ and $sz = s'z$, for $a \in A_S$ and $s, s', z \in S$, imply the existence of $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S satisfies Condition (EP) if $as = at$, for $a \in A_S$ and $s, t \in S$, implies the existence of $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also, we say that A_S satisfies Condition $(E'P)$ if $as = at$ and $sz = tz$, for $a \in A_S$ and $s, t, z \in S$, imply the existence of $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(P) \Rightarrow (EP) \Rightarrow (E'P)$, $(E) \Rightarrow (E') \Rightarrow (E'P)$, $(E) \Rightarrow (EP)$ and $(P) \Rightarrow (P') \Rightarrow (E'P)$.

Theorem 3.9. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All right S-acts satisfying Condition $(E'P)$ also satisfy Condition $(GPWP_{sec})$.*
(3) *All right S-acts satisfying Condition (EP) also satisfy Condition $(GPWP_{sec})$.*
(4) *All right S-acts satisfying Condition (E') also satisfy Condition $(GPWP_{sec})$.*
(5) *All right S-acts satisfying Condition (E) also satisfy Condition $(GPWP_{sec})$.*
(6) *S is eventually regular.*

Proof. Since $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6). Since S_S satisfies Condition (E), the result can be obtained similar to Theorem 3.8.

(6) $\Rightarrow$ (1). This follows from Theorem 3.2. □

Similar to Theorem 3.8, it follows that Theorem 3.9 is true for finitely generated right S-acts and right S-acts generated by at most two elements.

We recall from [8] that a right S-act A_S is *(strongly) faithful* if for $s, t \in S$, the validity of $as = at$ for (some) all $a \in A$ implies the equality $s = t$.

Theorem 3.10. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All faithful right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All finitely generated faithful right S-acts satisfy Condition $(GPWP_{sec})$.*
(4) *All faithful right S-acts generated by at most two elements satisfy Condition $(GPWP_{sec})$.*
(5) *S is eventually regular.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). Since S_S is faithful, a reasoning similar to the proof of Theorem 3.8 allows us to obtain the desired result.

(5) ⇒ (1). This follows from Theorem 3.2. □

Theorem 3.11. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All indecomposable right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All finitely generated indecomposable right S-acts satisfy Condition $(GPWP_{sec})$.*
(4) *All indecomposable right S-acts generated by at most two elements satisfy Condition $(GPWP_{sec})$.*
(5) *All locally cyclic S-acts satisfy Condition $(GPWP_{sec})$.*
(6) *All finitely generated locally cyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(7) *All locally cyclic right S-acts generated by at most two elements satisfy Condition $(GPWP_{sec})$.*
(8) *S is eventually regular.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) and (1) ⇒ (5) ⇒ (6) ⇒ (7) are obvious.

(4) ⇒ (8). Since S_S is an indecomposable right S-act, $S\coprod^{I}S$ is also indecomposable, for every proper right ideal I of S. The desired result follows from a reasoning similar to the proof of Theorem 3.8.

(7) ⇒ (8). By the assumption, all cyclic right S-acts satisfy Condition $(GPWP_{sec})$ and so, S is eventually regular by Theorem 3.2.

(8) ⇒ (1). The desired result follows from Theorem 3.2.

□

Theorem 3.12. *The following statements are equivalent.*

1. *All right S-acts satisfying Condition $(GPWP_{sec})$ are (strongly) faithful.*
2. *All finitely generated right S-acts satisfying Condition $(GPWP_{sec})$ are (strongly) faithful.*
3. *All cyclic right S-acts satisfying Condition $(GPWP_{sec})$ are (strongly) faithful.*
4. *All right Rees factor acts of S satisfying Condition $(GPWP_{sec})$ are (strongly) faithful.*
5. $S = \{1\}$.

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). It follows from part (1) of Proposition 2.4 that $S/S_S \cong \Theta_S$ satisfies Condition $(GPWP_{sec})$. Thus, by the assumption, Θ_S is faithful. Let $s, t \in S$. Then $\theta s = \theta t$ implies $s = t$ and so, $S = \{1\}$.

(5) ⇒ (1). If $S = \{1\}$, then all right S-acts are strongly faithful. This proves (1). □

Theorem 3.13. *The following statements are equivalent.*

1. *All right S-acts satisfying Condition $(GPWP_{sec})$ are (projective-) generator.*
2. *All finitely generated right S-acts satisfying Condition $(GPWP_{sec})$ are (projective-) generator.*
3. *All cyclic right S-acts satisfying Condition $(GPWP_{sec})$ are (projective-) generator.*
4. *All right Rees factor acts of S satisfying Condition $(GPWP_{sec})$ are (projective-) generator.*
5. $S = \{1\}$.

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). By part (1) of Proposition 2.4, the right Rees factor S-act $S/S_S \cong \Theta_S$ satisfies Condition $(GPWP_{sec})$. Thus, by the assumption, Θ_S is a generator. By [8, Theorem 2.3.16], an epimorphism $\phi : \Theta_S \to S_S$ exists. Hence, $S = \{1\}$.

(5) ⇒ (1). If $S = \{1\}$, then any right S-act is a (projective-) generator and so, the desired result follows. □

Theorem 3.14. *The following statements are equivalent.*

1. *All right S-acts satisfying Condition $(GPWP_{sec})$ are free.*

(2) *All finitely generated right S-acts satisfying Condition $(GPWP_{sec})$ are free.*
(3) *All cyclic right S-acts satisfying Condition $(GPWP_{sec})$ are free.*
(4) *All right Rees factor acts of S satisfying Condition $(GPWP_{sec})$ are free.*
(5) $S = \{1\}$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By the assumption, any right Rees factor act of S satisfying Condition $(GPWP_{sec})$ is a generator. It follows from the previous theorem that $S = \{1\}$.

(5) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are free. This proves (1). □

Theorem 3.15. *The following statements are equivalent.*

(1) *All strongly faithful right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All finitely generated strongly faithful right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *All strongly faithful right S-acts generated by at most two elements satisfy Condition $(GPWP_{sec})$.*
(4) *S is not left cancellative or it is eventually regular.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let S be left cancellative and $s \in S$. Then, by [7, Lemma 3.7], S_S is strongly faithful. Now, by the assumption, the proof is similar to that of Theorem 3.8.

(4) $\Rightarrow$ (1). If S is not left cancellative, then by [7, Lemma 3.7], no strongly faithful right S-act exists and so, the desired result follows. If S is left cancellative, then by [7, Lemma 3.7], a strongly faithful right S-act exists. By the assumption, S is eventually regular. Then, by Theorem 3.2, all right S-acts satisfy Condition $(GPWP_{sec})$. □

Theorem 3.16. *The following statements are equivalent.*

(1) *All strongly faithful cyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *All strongly faithful monocyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(3) *S is not left cancellative or S_S satisfies Condition $(GPWP_{sec})$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Suppose that S is left cancellative. Then, by [7, Lemma 3.7], S_S is strongly faithful. Now, the isomorphisms $S/\rho(1,1) \cong S/\Delta_S \cong S_S$ and the assumption show that S_S satisfies Condition $(GPWP_{sec})$.

(3) $\Rightarrow$ (1). Suppose that S is not left cancellative. Then, by [7, Lemma 3.7], no strongly faithful right S-act exists and so, the desired result follows. Now, let S be left cancellative and hence, S_S satisfies Condition $(GPWP_{sec})$ by the assumption. If $A_S = aS$ is a cyclic strongly faithful right S-act, we define $f : aS \to S_S$ by $f(as) = s$. Then, f is an isomorphism of right S-acts. Now, by the isomorphism $aS \cong S_S$, A_S satisfies Condition $(GPWP_{sec})$ and so, the desired result follows. □

Theorem 3.17. *The following statements are equivalent.*

(1) *At least one strongly faithful cyclic right S-act exists that satisfies Condition $(GPWP_{sec})$.*
(2) *At least one strongly faithful monocyclic right S-act exists that satisfies Condition $(GPWP_{sec})$.*
(3) *S is left cancellative and each strongly faithful cyclic right S-act satisfies Condition $(GPWP_{sec})$.*
(4) *S is left cancellative and each strongly faithful monocyclic right S-act satisfies Condition $(GPWP_{sec})$.*
(5) *S is left cancellative and S_S satisfies Condition $(GPWP_{sec})$.*

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are obvious.

(1) $\Rightarrow$ (3). By the assumption and [7, Lemma 3.7], S is left cancellative. If S/ρ is a strongly faithful cyclic right S-act, then $\rho = \Delta_S$ by [7, Lemma 3.9]. Hence, $S/\rho = S/\Delta_S \cong S_S$. Then, S_S satisfies Condition $(GPWP_{sec})$. Finally, each strongly faithful cyclic right S-act satisfies Condition $(GPWP_{sec})$ by Theorem 3.16.

(4) $\Rightarrow$ (5). By Theorem 3.16, the proof is straightforward.

(5) $\Rightarrow$ (2). Since S is left cancellative, S_S is strongly faithful by [7, Lemma 3.7]. Since $S/\rho(1,1) \cong S_S$, at least one strongly faithful monocyclic right S-act exists such that satisfies Condition $(GPWP_{sec})$. □

Recall from [8] that, if ρ is a right congruence on S and $s \in S$, then by ρs we denote the right congruence on S defined by

$$x(\rho s)y \Leftrightarrow (sx)\rho(sy)$$

for $x, y \in S$.

If λ is a left congruence on S and $s \in S$, then by $s\lambda$ we denote the left congruence on S defined by

$$x(s\lambda)y \Leftrightarrow (xs)\lambda(ys)$$

for $x, y \in S$.

It is clear that if ρ is a right congruence, then ρs is also a right congruence, and if λ is a left congruence, then $s\lambda$ is also a left congruence, for $s \in S$.

Lemma 3.18. *Let $\rho \in Con(S_S)$. Then, the following statements are equivalent.*

(1) *The cyclic right S-act S/ρ is faithful.*
(2) *ρ does not contain any left congruence τ on S such that $\tau \neq \Delta_S$.*
(3) *$\bigcap_{u\in S} \rho u = \Delta_S$.*

Proof. (1) $\Rightarrow$ (2). By [8, Proposition 1.5.24], this is obvious.

(2) $\Rightarrow$ (3). Let $\sigma = \bigcap_{u\in S} \rho u$. Since for each $u \in S$, $\rho u \in Con(S_S)$, it is clear that $\sigma \in Con(S_S)$. Now, we show that σ is a left congruence on S. If $x, y \in S$, then

$$(x, y) \in \sigma \Leftrightarrow (\forall u \in S)(x, y) \in \rho u \Leftrightarrow (\forall u \in S)(ux, uy) \in \rho.$$

Now, if $l \in S$, then

$$\begin{aligned}(x, y) \in \sigma &\Leftrightarrow (\forall u \in S)(ux, uy) \in \rho \Rightarrow (\forall u \in S)(ulx, uly) \in \rho \\ &\Rightarrow (\forall u \in S)(lx, ly) \in \rho u \Rightarrow (lx, ly) \in \bigcap_{u\in S} \rho u = \sigma.\end{aligned}$$

Therefore, σ is a left congruence on S and clearly, $\bigcap_{u\in S} \rho u \subseteq \rho$. On the other hand, by the assumption, ρ does not contain any non-trivial left congruence on S. Hence, $\sigma = \Delta_S$.

(3) $\Rightarrow$ (1). Suppose that S/ρ is not faithful. Then,

$$\begin{aligned}&\exists x, y \in S, x \neq y, \forall u \in S, [u]_\rho x = [u]_\rho y \Rightarrow (\forall u \in S)(ux, uy) \in \rho \\ &\Rightarrow (\forall u \in S)(x, y) \in \rho u \Rightarrow (x, y) \in \bigcap_{u\in S} \rho u.\end{aligned}$$

Therefore, $\sigma = \bigcap_{u\in S} \rho u \neq \Delta_S$, which is a contradiction by the assumption. So, S/ρ is faithful. □

Theorem 3.19. *The following statements are equivalent.*

(1) *All faithful cyclic right S-acts satisfy Condition $(GPWP_{sec})$.*
(2) *For any $\rho \in Con(S_S)$, ρ contains non-trivial left congruence τ on S or the right act S/ρ satisfies Condition $(GPWP_{sec})$.*
(3) *For any $\rho \in Con(S_S)$, $\bigcap_{u\in S} \rho u \neq \Delta_S$ or the cyclic right S-act S/ρ satisfies Condition $(GPWP_{sec})$.*

Proof. (1) $\Rightarrow$ (2). Let ρ be a right congruence on S that does not contain any non-trivial left congruence on S. Then, by Lemma 3.18, S/ρ is faithful and so, S/ρ satisfies Condition $(GPWP_{sec})$ by the assumption.

(2) ⇒ (3). Let ρ be a right congruence on S such that $\bigcap_{u\in S}\rho u=\Delta_S$. Then, by Lemma 3.18 and the assumption, S/ρ satisfies Condition $(GPWP_{sec})$.

(3) ⇒ (1). Let ρ be a right congruence on S such that the cyclic right S-act S/ρ is faithful. Then, by Lemma 3.18, $\bigcap_{u\in S}\rho u=\Delta_S$ and so, S/ρ satisfies Condition $(GPWP_{sec})$ by the assumption. □

In the following theorem, we investigate those situations in which the rest of the properties imply Condition $(GPWP_{sec})$.

Theorem 3.20. *The following statements are equivalent.*

(1) *All right Rees factor acts of S satisfying Condition (P) satisfy Condition $(GPWP_{sec})$.*
(2) *All WPF right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(3) *All strongly flat right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(4) *All projective right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(5) *Any projective generator right Rees factor acts of S satisfies Condition $(GPWP_{sec})$.*
(6) *All free right Rees factor acts of S satisfy Condition $(GPWP_{sec})$.*
(7) *S does not contain a left zero or S_S satisfies Condition $(GPWP_{sec})$.*

Proof. Since

$$\text{free} \Rightarrow \text{projective generator} \Rightarrow \text{projective} \Rightarrow \text{strongly flat} \Rightarrow \text{WPF}$$
$$\Rightarrow \text{Condition (P)},$$

the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) are obvious.

(6) ⇒ (7). Suppose that S contains a left zero, say z. Let $K_S=zS=\{z\}$. So, K_S is a right ideal such that $|K_S|=1$. Therefore $S/K_S\cong S_S$ is free and so, by the assumption, $S/K_S\cong S_S$ satisfies Condition $(GPWP_{sec})$.

(7) ⇒ (1). Let K be a right ideal of S such that S/K satisfies Condition (P). If $K=S$, then $S/K=S/S_S\cong\Theta_S$. Hence, by Theorem 2.4, $S/K\cong\Theta_S$ satisfies Condition $(GPWP_{sec})$. If $K\neq S$, then by [8, Propositon 3.13.9], $|K|=1$. If $z\in K$, then $K=zS=\{z\}$. Therefore z is a left zero of S and so, S_S satisfies Condition $(GPWP_{sec})$. Hence, $S/K\cong S_S$ satisfies Condition $(GPWP_{sec})$. □

Notation: We use C_l (C_r) to denote the set of all left (right) cancellable elements of S.

Lemma 3.21. *Let $S\neq C_r$. Then, the following statements are true.*

(1) *$I = S \setminus C_r$ is a proper right ideal of S.*
(2) *$S/I (I = S \setminus C_r)$ is a torsion free S-act.*
(3) *If S is eventually left PP, then $I = S \setminus C_r$ is a GPW-left stabilizing right ideal and so, $A_S = S\coprod^{K} S$ satisfies Condition $(GPWP_{sec})$.*

Proof. For the proofs of (1) and (2), we refer the reader to [7, Lemma 3.12]. Let $s \in S$. Since S is eventually left PP, there exists $n \in \mathbb{N}$ such that s^n is right e-cancellable for some $e \in E(S)$. Let $r \in S \setminus I = C_r$ be such that $rs^n \in I$. Since $I = S \setminus C_r$, rs^n is not right cancellable. Thus, there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 rs^n = l_2 rs^n$. Now, by the assumption, $e \in E(S)$ exists such that $l_1 re = l_2 re$ and $es^n = s^n$. Therefore, $res^n = rs^n$. Since $l_1 \neq l_2$, the equality $l_1 re = l_2 re$ implies $re \in S \setminus C_r = I$. So, $I = S \setminus C_r$ is a GPW-left stabilizing right ideal. Hence, by part (2) of Theorem 2.12, $A_S = S\coprod^{I} S$ satisfies Condition $(GPWP_{sec})$. □

Lemma 3.22. [7, Lemma 3.13] *Let S be right cancellative. Then, for every right S-act,*

strongly torsion free $\Longleftrightarrow$ torsion free $\Longleftrightarrow$ GP-flat
$\Longleftrightarrow$ principally weakly flat $\Longleftrightarrow$ Condition (PWP)
$\Longleftrightarrow$ Condition (P') $\Longleftrightarrow$ Condition (PWP_E)
$\Longleftrightarrow$ TKF $\Longleftrightarrow$ Condition (PWP_{ssc}) $\Longleftrightarrow$ PWKF.

It is easy to verify that, if S is right cancellative in right S-acts, then Condition $(GPWP_{sec})$ is equivalent to every property of Lemma 3.22.

Theorem 3.23. *Let (*) be a property of S-acts such that*

$$\text{Condition } (GPWP_{sec}) \Rightarrow \text{Property } (*) \Rightarrow \text{torsion free.}$$

Then, the following statements are equivalent.

(1) *S is eventually left PP and the property (*) implies PWKF.*
(2) *S is eventually left PP and the property (*) implies TKF.*
(3) *S is eventually left PP and the property (*) implies Condition (PWP).*
(4) *S is eventually left PP and the property (*) implies Condition (P').*
(5) *S is right cancellative.*

Proof. The implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are obvious, because

$$\text{PWKF} \Rightarrow \text{TKF} \Rightarrow \text{Condition (PWP)}$$

and

$$\text{Condition } (P') \Rightarrow \text{Condition (PWP)}.$$

(3) $\Rightarrow$ (5). Suppose that S is not right cancellative and $I = S \setminus C_r$. Then, I is a GPW-left stabilizing right ideal of S by Lemma 3.21, and

$$A_S = S\coprod^{I} S = \{(l,x) | l \in S \setminus I\} \dot\cup \{(t,y) | t \in S \setminus I\} \dot\cup I = (1,x)S \cup (1,y)S$$

satisfies Condition $(GPWP_{sec})$, by part (3) of Lemma 3.21. By the assumption, A_S satisfies Condition (PWP). If $i \in I$, then the equality $(1,x)i = (1,y)i$ implies the existence of $a \in A_S$ and $u, v \in S$ such that $(1,x) = au$, $(1,y) = av$ and $ui = vi$. Therefore, $t, l \in S \setminus I$ exist such that $(l,x) = a = (t,y)$, which is a contradiction. Hence, S is right cancellative, as required.

(5) $\Rightarrow$ (1). Since S is right cancellative, it is eventually left PP. Also, by Lemma 3.22, for every right S-act, the properties of being torsion free and PWKF are equivalent to Condition $(GPWP_{sec})$. Thus, by the assumption, every right S-act satisfying the property (*) is PWKF.

(5) $\Rightarrow$ (4). Since S is right cancellative, S is eventually left PP. Also, by Lemma 3.22, the property of being torsion free is equivalent to Condition (P') and Condition $(GPWP_{sec})$. Thus, by the assumption, every right S-act satisfying the property (*) also satisfies Condition (P'). □

Note that the property (*) in the above theorem can be any property such as Condition $(GPWP_{sec})$, GP-flatness and GPW-flatness.

References

1. S. Bulman-Fleming and A. Gilmour, Flatness properties of diagonal acts over monoids, *Semigroup Forum*, **76** (2009), 298–314.
2. S. Bulman-Fleming, M. Kilp and V. Laan, Pullbacks and flatness properties of acts II, *Comm. Algebra*, **29** (2001), 851–878.
3. A. Golchin, On flatness of acts, *Semigroup Forum*, **67** (2003), 262–270.
4. A. Golchin and H. Mohammadzadeh Saany, Condition (EP), *Int. Math. Forum*, **2** (2007), 911–918.
5. A. Golchin and H. Mohammadzadeh Saany, Condition $(E'P)$, *J. Sci. Islam. Repub. Iran*, **7** (2006), 343–349.
6. J. M. Howie, *Fundamentals of Semigroup Theory*, London Mathematical Society Monographs, OUP, 1995.
7. P. Khamechi, H. Mohammadzadeh Saany and L. Nouri, Classification of monoids by Condition (PWP_{ssc}) of right acts, *Categ. Gen. Algebr. Struct. Appl.*, **12** (2020), 175–197.
8. M. Kilp, U. Knauer and A. Mikhalev, Monoids, *Acts and Categories*, W. De Gruyter, Berlin, 2000.
9. V. Laan, *Pullbacks and flatness properties of atcs*, PhD Thesis, Tartu, 1999.
10. V. Laan, Pullbacks and flatness properties of acts I, *Comm. Algebra*, **29** (2001), 829–850.
11. H. Rashidi, A. Golchin and H. Mohammadzadeh Saany, On GPW-flat acts, *Categ. Gen. Algebr. Struct. Appl.*, **12** (2020), 175–197.

12. A. Zare, A. Golchin and H. Mohammadzdeh, $\mathcal{R}$-torsion free acts over monoids, *J. Sci. Islam. Repub. Iran*, **24** (2013), 275–285.

Malihe Shafiei
Department of Mathematics, Sistan and Baluchestan University, P.O. Box 987-98155, Zahedan, Iran.
Email: mshafiei8660@yahoo.com

Hossein Mohammadzadeh Saany
Department of Mathematics, Sistan and Baluchestan University, P.O. Box 987-98155, Zahedan, Iran.
Email: hmsdm@math.usb.ac.ir

Parisa Rezaei
Department of Mathematics, Sistan and Baluchestan University, P.O. Box 987-98155, Zahedan, Iran.
Email: p_rezaei@math.usb.ac.ir

Journal of Algebraic Systems

CLASSIFICATION OF MONOIDS BY CONDITION $(GPWP_{sec})$ OF RIGHT ACTS

M. SHAFIEI, H. MOHAMMADZADEH SAANY AND P. REZAEI

دسته‌بندی تکواره‌ها بر اساس شرط $(GPWP_{sec})$ از سیستم های راست

ملیحه شفیعی[۱]، حسین محمدزاده‌ثانی[۲] و پریسا رضایی[۳]

[۱,۲,۳]دانشکده ریاضی، آمار و علوم کامپیوتر، دانشگاه سیستان و بلوچستان، زاهدان، ایران

در (رسته‌ها و ساختارهای جبری عمومی با کاربردها، (۲۰۲۰) ۱۹۷-۱۷۵ :(۱)(۱۲) رشیدی و همکاران خاصیت GPWـ همواری سیستم‌ها روی تکواره‌ها را به عنوان تعمیمی از به طور اساسی ضعیف هموای معرفی کردند. در این مقاله شرط $(GPWP_{sec})$ را معرفی و آن را با GPWـ هموای مقایسه می‌کنیم. برخی از خواص کلی شرط $(GPWP_{sec})$ را به دست آورده و به مشخص‌سازی تکواره‌هایی می‌پردازیم که این شرط برخی از خواص دیگر را روی سیستم‌هایشان نتیجه می‌دهد و برعکس.

کلمات کلیدی: شرط $(GPWP_{sec})$، نهایتاً PP چپ، GPWـ ثابت‌ساز چپ.