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t-PRIME SUBMODULES AND THEIR DECOMPOSITIONS

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Abstract. Let R be a commutative ring with identity. For $t \in \mathbb{N}$, a proper submodule N of an R-module M is called a t-prime submodule if $rm \in N$ $(r \in R, m \in M)$, then $m \in N$ or $r^t \in (N :_R M)$. We obtain some other characterizations of t-prime submodules. Also, by some other notions like t-secondary submodules, various properties of t-prime submodules are investigated. To this end, we deal with irreducible as well as reduced t-prime decompositions of a submodule. We provide several examples to illustrate our results.

1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote, respectively, the natural numbers, the ring of integers, and the field of rational numbers and $t \in \mathbb{N}$. If N is an R-submodule of M, annihilator of R-module $\dfrac{M}{N}$ is defined as $Ann_R(\frac{M}{N}) = (N :_R M) = \{r \in R : rM \subseteq N\}$. Thus, the annihilator of M, denoted by $Ann_R(M)$, is $(0 :_R M)$. Suppose that I is an ideal of R. We denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \quad for \quad some \quad n \in \mathbb{N}\}$.

Recall that a proper submodule N of M is called prime (*primary*) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ $(r^n \in (N :_R M)$, for some $n \in \mathbb{N})$(see [1], [4], [6]).

In [8] for $t \in \mathbb{N}$, we defined the concept t-prime submodule and found some basic properties of it. It is shown that for ring extension

$$f : R \to S,$$

such that S is a free R-module and N is a submodule of an R-module M, N is a t-prime submodule of M if and only if $N \bigotimes_R S$ is a t-prime R-submodule of $M \bigotimes_R S$. In this paper, the concepts t-secondary module and t-prime decomposition of a submodule was introduced. In section 2, we investigate some other properties of t-prime submodules. We find the relation between t-prime submodules with some other notions in module theory. We studied irreducible as well as reduced t-prime decompositions of a submodule in

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section 3. In particular, we apply our results to special finitely generated modules. In section 4, we provide several examples, for the notions and the necessary conditions of some propositions in the former sections.

2. t-Prime and t-Secondary Submodules

In this section, first, we recall the notion t-prime submodule, for $t \in \mathbb{N}$ and say some characterizations of it [8]. Moreover other various properties of t-prime submodules are considered.

Definition 2.1 ([8], Definition 2.1)**.** A proper submodule N of a module M over a commutative ring R is said to be a t-prime submodule, if for $a \in R$ and $x \in M$, $ax \in N$, then $x \in N$ or $a^t \in (N :_R M)$. Also a proper ideal I of R is called a t-prime ideal, if for $r, s \in R$, $rs \in I$, then $r^t \in I$ or $s^t \in I$.

Let I be a non-empty subset of R. We denote $\sqrt[t]{I} = \{r \in R | r^t \in I\}$.
So, a proper submodule N of an R-module M is a t-prime submodule, if for $a \in R$ and $x \in M$, $ax \in N$, then $x \in N$ or $a \in \sqrt[t]{(N :_R M)}$.

Recall that a proper ideal I of R is called semiprime, if whenever $a^n \in I$ for $a \in R$ and $n \in \mathbb{N}$, then $a \in I$ [7].

Lemma 2.2 ([8], Lemma 2.2)**.** *Let I and J be ideals of R and $t \in \mathbb{N}$. Then, the following statements hold.*

(i) $I \subseteq \sqrt[t]{I}$.

(ii) $I = R$ *if and only if* $\sqrt[t]{I} = R$.

(iii) If $I \subseteq J$, *then* $\sqrt[t]{I} \subseteq \sqrt[t]{J}$.

(iv) $\sqrt[t]{I \cap J} = \sqrt[t]{I} \cap \sqrt[t]{J}$.

(v) For all $s \geq t$, $\sqrt[t]{I} \subseteq \sqrt[s]{I}$.

(vi) $\sqrt[t]{\sqrt[s]{I}} = \sqrt[s]{\sqrt[t]{I}} = \sqrt[ts]{I}$.

(vii) $\sqrt[t]{I} \subseteq \sqrt{I}$.

(viii) $\sqrt[t]{\sqrt{I}} = \sqrt{I}$.

(ix) If I is a semiprime or radical ideal, then $\sqrt[t]{I} = \sqrt{I} = \sqrt[s]{I} = I$, *for any* $s \in \mathbb{N}$.

Proposition 2.3 ([8], Proposition 2.4)**.** *Let N be a t-prime submodule of an R-module M. Then*

(i) $(N :_R M)$ *is a t-prime ideal of R.*

(ii) $\sqrt{(N :_R M)}$ *is a prime ideal of R.*

Theorem 2.4 ([8], Theorem 2.1)**.** *Let M be an R-module, and N be a proper submodule of M. Then the following statements are equivalent:*

(i) N *is a t-prime submodule of* M*;*

(ii) $IL \subseteq N$*, for ideal* I *of* R *and submodule* L *of* M*, implies that* $L \subseteq N$ *or* $I \subseteq \sqrt[t]{(N :_R M)}$*;*

(iii) $N = (N :_M r)$ *or* $r \in \sqrt[t]{(N :_R M)}$*, for any* $r \in R$*;*

(iv) $Rx \subseteq N$ *or* $(N :_R x) \subseteq \sqrt[t]{(N :_R M)}$*, for any* $x \in M$*;*

(v) $N = \{m \in M | rm \in N\}$*, for all* $r \in R - \sqrt[t]{(N :_R M)}$*;*

(vi) $N = \{m \in M | Jm \subseteq N\}$*, for all ideal* J *of* R *such that* $J \not\subseteq \sqrt[t]{(N :_R M)}$*;*

(vii) $(N :_R m) \subseteq \sqrt[t]{(N :_R M)}$*, for all* $m \in M - N$*;*

(viii) $(N :_R L) \subseteq \sqrt[t]{(N :_R M)}$*, for any submodule* L *of* M *such that* $N \subset L$*;*

(ix) $ann_R(m + N) \subseteq \sqrt[t]{(N :_R M)}$*, for all* $m \in M - N$*;*

(x) $Z_R(\frac{M}{N}) \subseteq \sqrt[t]{(N :_R M)}$*;*

(xi) $N = \{m \in M | rm \in N, \text{ for some } r \in R - \sqrt[t]{(N :_R M)}\}$.

Proposition 2.5 ([8], Proposition 2.5)**.** *Let* N *be a submodule of an* R*-module* M *such that* $(N :_R M)$ *is a semiprime or radical ideal of* R*. Then* N *is a prime submodule if and only if* N *is a t-prime submodule.*

Corollary 2.6 ([8], Corollary 2.1)**.** *Let* N *be a t-prime submodule of an* R*-module* M*. Then, for any* $r \in R$*,* $(N :_M r) = M$ *or* $(N :_M r)$ *is a t-prime submodule of* M*.*

Proposition 2.7 ([8], Proposition 2.9)**.** *(i) Let* $\{N_i\}_{i \in I}$ *be a nonempty set of t-prime submodules of an* R*-module* M *such that* $(N_i :_R M) = (N_j :_R M)$*, for any* $i, j \in I$*. Then* $\bigcap_{i \in I} N_i$ *is a t-prime submodule.*

(ii) Let $\{N_i\}_{i \in I}$ *be a chain of t-prime submodules of a finitely generated* R*-module* M*. Then* $\bigcup_{i \in I} N_i$ *is a t-prime submodule of* M*.*

Definition 2.8 ([8], Definition 2.4)**.** An R-module M is said to be a t-torsion-free R-module, if $rx = 0$, for $r \in R$ and $x \in M$, then $x = 0$ or $r \in \sqrt[t]{Ann_R(M)}$.

Theorem 2.9 ([8], Theorem 2.3)**.** *Let* N *be a proper submodule of an* R*-module* M*. Then* N *is a t-prime submodule if and only if* $\frac{M}{N}$ *is a t-torsion-free* R*-module.*

Definition 2.10 ([8], Definition 2.5)**.** An R-module M is called t-prime module, if the zero is a t-prime submodule of it.

Recall that a proper submodule N of M is said to be an r-submodule, if for $a \in R$, $m \in M$ and whenever $am \in N$ with $ann_M(a) = 0$, then $m \in N$ [3].

Proposition 2.11 ([8], Proposition 2.12)**.** *Let M be an R-module. Then, the following conditions are equivalent.*

(i) M is a t-prime module.

(ii) $Z_R(M) = \sqrt[t]{Ann_R(M)}$.

(iii) Any r-module is a t-prime submodule.

Proposition 2.12 ([8], Proposition 2.13)**.** *Let N be a proper submodule of a torsion-free R-module M. Then the following conditions are equivalent.*

(i) N is a t-prime submodule;

(ii) $rN = N \cap rM$, for any $r \in R - \sqrt[t]{(N :_R M)}$;

(ii) $N = (N :_M r)$, for any $r \in R - \sqrt[t]{(N :_R M)}$.

Proposition 2.13 ([8], Proposition 2.16)**.** *(i) Let $\{P_i\}_{i \in I}$ be a nonempty set of prime submodules of an R-module M. If $\bigcap_{i \in I} P_i$ is a t-prime submodule, then $\bigcap_{i \in I} P_i$ is a prime submodule.*

(ii) Let $\{P_i\}_{i \in I}$ be a nonempty set of primary submodules of an R-module M. If $\bigcap_{i \in I} P_i$ is a t-prime submodule, then $\bigcap_{i \in I} P_i$ is a primary submodule.

At the follow of this section, we obtain some other characterizations of t-prime submodules.

Proposition 2.14. *Let M be an R-module, and S be a multiplicatively closed subset of R such that $R - Ann_R(M) \subseteq S$. If S^* is a S-closed subset of M and N is a submodule of M such that $N \cap S^* = \emptyset$, then there exists a t-prime submodule L of M such that $N \subseteq L$ and $L \cap S^* = \emptyset$.*

Proof. Put $\Omega = \{L | N \subseteq L \leq M; L \cap S^* = \emptyset\}$. Since $N \in \Omega$, Ω is a non-empty set and partially ordered with inclusion. By Zorn's Lemma, Ω has a maximal element like L. Since $L \cap S^* = \emptyset$, L is a proper submodule of M. Assume that L is not a t-prime submodule. So there exist $r \in R - \sqrt[t]{(L :_R M)}$ and $x \in M - L$ such that $rx \in L$. Thus $r \notin Ann_R(M)$ and by the maximality of L in Ω and since $L \subset (L :_M r)$, we have that $(L :_M r) \notin \Omega$. So there exists $y \in S^*$ such that $ry \in L$. Now as S^* is S-closed and $r \in S$, we have $ry \in L \cap S^*$, which is a contradition. Therefore L is a t-prime submodule. □

Theorem 2.15. *Let $f : M \longrightarrow M'$ be an R-homomorphism. Then the followings hold:*

(i) If f is an epimorphism, and N is a t-prime submodule of M containing $Ker(f)$, then $f(N)$ is a t-prime submodule of M'.

(ii) If L' is a t-prime submodule of M', then $f^{-1}(L') = M$ or $f^{-1}(L')$ is a t-prime submodule of M.

(iii) If N' is a t-prime submodule of M' and $(f^{-1}(N') :_R M) \not\subset \sqrt[t]{(N' :_R M')}$, then $f^{-1}(N') = M$.

Proof. (i) It is clear that $f(N)$ is a proper submodule of M'. Let $rx' \in f(N)$, for $r \in R$, $x' \in M'$. Since f is an epimorphism, there exists $x \in M$ such that $x' = f(x)$. Then $rx' = rf(x) = f(rx) \in f(N)$. As $Ker(f) \subseteq N$, we conclude that $rx \in N$. Also, note that $(N :_R M) \subseteq (f(N) :_R M')$. Since N is a t-prime submodule of M, we get the result that $x \in N$ and so $x' = f(x) \in f(N)$ or $r \in \sqrt[t]{(N :_R M)}$ which implies $r \in \sqrt[t]{(f(N) :_R M')}$. Therefore $f(N)$ is a t-prime submodule.

(ii) Let $f^{-1}(L') \neq M$ and $rx \in f^{-1}(L')$, for $r \in R$, $x \in M$. Then $f(rx) = rf(x) \in L'$. Since L' is a t-prime submodule of M', $f(x) \in L'$ and so $x \in f^{-1}(L')$ or $r \in \sqrt[t]{(L' :_R M')}$ which implies $r \in \sqrt[t]{(f^{-1}(L') :_R M)}$ (note that $(L' :_R M') \subseteq (f^{-1}(L') :_R M)$). Consequently, $f^{-1}(L')$ is a t-prime submodule of M.

(iii) Consider $N = f^{-1}(N')$, $r \in (N :_R M) - \sqrt[t]{(N' :_R M')}$ and let $x \in M$. Then $rx \in N$ and so $rf(x) \in N'$. Now as N' is t-prime, $f(x) \in N'$ and hence $x \in N$. Therefore $N = M$. □

Corollary 2.16. *Let M be an R-module and L, N be two submodules of M. Then the following statments hold:*

(i) If $L \subseteq N$, then N is a t-prime submodule of M if and only if $\frac{N}{L}$ is a t-prime submodule of $\frac{M}{L}$.

(ii) If N is a t-prime submodule of M and $L \not\subseteq N$, then $N \cap L$ is a t-prime submodule of L.

Let M be an R-module and S be a multiplicatively closed subset of R. Consider the natural homomorphism φ from M to M_S defined by $\varphi(m) = \dfrac{m}{1}$, for any $m \in M$. Then for each submodule L of M_S, we define L^c as an inverse image of L under the above homomorphism.

Proposition 2.17. *Let M be an R-module and S a multiplicatively closed subset of R.*

(i) If N is a t-prime submodule of M, then $N_S = M_S$ or N_S is a t-prime submodule of M_S.

(ii) If M is finitely generated, L is a t-prime submodule of M_S, then L^c is a t-prime submodule of M.

Proof. (i) Let $N_S \neq M_S$ and $\frac{a}{s}\frac{m}{l} \in N_S$, for $a \in R$, $s, l \in S$, $m \in M$. Then we have $uam \in N$, for some $u \in S$. Since N is a t-prime submodule of M, we conclude that $m \in N$ and so $\frac{m}{l} \in N_S$ or $ua \in \sqrt[t]{(N :_R M)}$ which implies $\dfrac{a}{s} \in \sqrt[t]{(N_S :_R M_S)}$. Therefore N_S is a t-prime submodule of M_S.

(ii) Let $am \in L^c$, for $a \in R$, $m \in M$. Then we have $\frac{a}{1}\frac{m}{1} \in L$. As L is a t-prime submodule of M_S, we conclude that $\frac{m}{1} \in L$ and so $m \in L^c$ or $\dfrac{r}{1} \in \sqrt[t]{(L :_{R_S} M_S)}$, which implies $r \in \sqrt[t]{(L^c :_R M)}$. Therefore L^c is a t-prime submodule of M. □

Proposition 2.18. *Let N be a t-prime submodule of an R-module M and S be a multiplicatively closed subset of R such that $S \cap \sqrt[t]{(N :_R M)} = \emptyset$. Then*

(i) $(N_S :_{R_S} M_S) = (N :_R M)_S$.

(ii) $N_S = \{\dfrac{x}{s} \in M_S | x \in N\}$.

Proof. (i) Let $\dfrac{r}{l} \in (N :_R M)_S$, for $r \in (N :_R M)$ and $l \in S$. So $rM \subseteq N$ and hence $\dfrac{r}{l}M_S \subseteq N_S$. Thus $\dfrac{r}{l} \in (N_S :_{R_S} M_S)$. Then $(N :_R M)_S \subseteq (N_S :_{R\ S} M_S)$. Now let $\dfrac{r}{l} \in (N_S :_{R_S} M_S)$ and $x \in M$. Then $\dfrac{r}{l}\dfrac{x}{1} \in N_S$ and so there exists $u \in S$ such that $urx \in N$. If $u^t \in (N :_R M)$, then $u \in S \cap \sqrt[t]{(N :_R M)}$, which yields a contradiction. So as N is t-prime, $rx \in N$. Therefore $r \in (N :_R M)$ and hence $\dfrac{r}{l} \in (N :_R M)_S$.

(ii) It is clear that $\{\dfrac{x}{s} \in M_S | x \in N\} \subseteq N_S$. Let $\dfrac{x}{l} \in N_S$. So, there exists $u \in S$ such that $ux \in N$. Thus $u \notin \sqrt[t]{(N :_R M)}$ and since N is t-prime, $x \in N$. Hence $\dfrac{x}{l} \in \{\dfrac{x}{s} \in M_S | x \in N\}$. □

Lemma 2.19. *Let M be a finitely generated R-module such that $Ann_R(M)$ is a radical ideal and for every multiplicatively closed set $S \subseteq R$, the kernel of $\varphi : M \longrightarrow M_S$ is either (0) or M. Then M is a t-prime module.*

Proof. Assume that $rx = 0$ where $r \in R - \sqrt[t]{Ann_R(M)}$ and $x \in M$. So $r \notin \sqrt{Ann_R(M)}$ and hence $r^n \neq 0$, for every $n \in \mathbb{N}$. We put

$$S = \{r^n : n \in \mathbb{N} \cup \{0\}\}.$$

Clearly S is a multiplicatively closed subset of R. If $Ker(\varphi) = 0$, then as $\varphi(x) = \frac{x}{1} = \frac{rx}{r} = 0$ we have $x = 0$. Let $Ker(\varphi) = M$. Since M is finitely generated, we can write $M = Rx_1+Rx_2+...+Rx_t$, for some $x_1, x_2, ..., x_t \in M$. Then $\varphi(x_i) = \frac{x_i}{1} = 0$ for any $1 \leq i \leq t$. Thus for any i, there exists $l_i \in \mathbb{N}$ such that $r^{l_i}x_i = 0$. Let us take $j = max\{l_1, l_2, ..., l_t\}$. Thus we have $r^jM = 0$

and so $r \in \sqrt{Ann_R(M)}$, which is a contradiction. Therefore M is a t-prime module. $\square$

Lemma 2.20. *Let M be a t-prime module. Then for every multiplicatively closed subset S of R, the kernel of $\varphi : M \longrightarrow M_S$ is either (0) or M.*

Proof. Let $0 \neq y \in Ker\varphi$. Thus $\varphi(y) = \dfrac{y}{1} = 0$ and so there exists $u \in S$ such that $sy = 0$. As $y \neq 0$ and (0) is t-prime, $s^t \in Ann_R(M)$. Hence $s^t \in S \cap Ann_R(M)$. Therefore $Ker\varphi = M$. $\square$

As a consequence of Lemmas 2.19 and 2.20, we have the following result.

Theorem 2.21. *Let M be a finitely generated R-module such that $Ann_R(M)$ is a radical ideal. Then M is a t-prime module if and only if for every multiplicatively closed subset S of R, the kernel of $\varphi : M \longrightarrow M_S$ is either (0) or M.*

Lemma 2.22. *Let $\{L_i\}_{i\in I}$ be a family of R-submodules of $\{M_i\}_{i\in I}$. If $\Pi_{i\in I}L_i$ is a t-prime submodule of $\Pi_{i\in I}M_i$, then for every $i \in I$, $L_i = M_i$ or L_i is a t-prime submodule of M_i. If for all $i, j \in I$, $(L_i :_R M_i) = (L_j :_R M_j)$, then the converse hold.*

Proof. Let $\Pi_{i\in I}L_i$ be a t-prime submodule of $\Pi_{i\in I}M_i$ and i be an arbitrary in I. We will prove $L_i(\neq M_i)$ is a t-prime submodule of M_i. Suppose that $rx \in L_i$, for $r \in R$ and $x \in M_i$. Put $x_i := x$ and $x_j := 0$ for all $j \neq i$. Then we have $r(x_j)_{j\in I} \in \Pi_{j\in I}L_j$. Since $\Pi_{j\in I}L_j$ is a t-prime submodule of $\Pi_{j\in I}M_j$, so $(x_j)_{j\in I} \in \Pi_{j\in I}L_j$ and hence $x_i \in L_i$ or $r^t \in (\Pi_{j\in I}L_j :_R \Pi_{j\in I}M_j) \subseteq (L_i :_R M_i)$. Therefore L_i is a t-prime submodule of M_i. Now assume that for all $i, j \in I$, $(L_i :_R M_i) = (L_j :_R M_j)$ and for every $i \in I$, L_i is a t-prime submodule of M_i. Let $r(x_j)_{j\in I} \in \Pi_{j\in I}L_j$, for $r \in R$ and $(x_j)_{j\in I} \in \Pi_{j\in I}M_j$. So for every $i \in I$, $rx_i \in L_i$. If there exists $l \in I$ such that $x_l \notin L_l$, then

$$r^t \in (L_l :_R M_l) = (\Pi_{i\in I}L_i :_R \Pi_{i\in I}M_i).$$

Therefore $\Pi_{i\in I}L_i$ is a t-prime submodule of $\Pi_{i\in I}M_i$. $\square$

Lemma 2.23. *Let $\{L_i\}_{i\in I}$ be a family of R-submodules of $\{M_i\}_{i\in I}$. Then $L_j \times \prod_{i\in I, i\neq j} M_i$ is a t-prime submodule of $\prod_{i\in I} M_i$, if and only if L_j is a t-prime submodule of M_j , for any $j \in I$.*

Proposition 2.24. *Let N be a proper R-submodule of M. Then N is a t-prime submodule of M if and only if for each $a \in R - \sqrt[t]{(N :_R M)}$, the homothety $\lambda_a : \frac{M}{N} \longrightarrow \frac{M}{N}$ is injective.*

Proof. Suppose that N is a t-prime submodule and $\lambda_a(x+N) = 0_{\frac{M}{N}}$ for $a \in R - \sqrt[t]{(N :_R M)}$, $x \in M$. Then $ax \in N$ and since N is a t-prime submodule, so $x \in N$ and $x + N = 0$. Hence λ_a is injective. Conversely, suppose that $rx \in N$ where $r \notin \sqrt[t]{(N :_R M)}$, for $r \in R$, $x \in M$. It follows that $\lambda_r(x+N) = 0$. Since λ_r is injective, $x + N = 0$ and so $x \in N$. □

Theorem 2.25. *Let N be a proper R-submodule of M. Then N is a t-prime submodule of M if and only if for each $a \in R$, the homothety $\lambda_a : \frac{M}{N} \longrightarrow \frac{M}{N}$ is injective or $\lambda_a^t = 0$.*

Proof. Let N be t-prime and for $a \in R$, λ_a does not be injective. So there exists $x \in M - N$ such that $\lambda_a(x+N) = \overline{0}$. Hence $ax \in N$ and then $a \in \sqrt[t]{Ann_R(\dfrac{M}{N})}$. Thus $a^t(\dfrac{M}{N}) = \overline{0}$ and therefore $\lambda_a^t = 0$. Now assume that for each $a \in R$, the homothety $\lambda_a : \frac{M}{N} \longrightarrow \frac{M}{N}$ is injective or $\lambda_a^t = 0$ and $ax \in N$, for $a \in R$ and $x \in M$. Hence $\lambda_a(x+N) = ax + N = \overline{0}$. If λ_a is injective, then $x \in N$; else $\lambda_a^t = 0$. So $a^t(\dfrac{M}{N}) = \overline{0}$ and therefore $a \in \sqrt[t]{(N :_R M)}$. □

In [5], I.G. Macdonald introduced the notion of secondary modules. A nonzero R-module M is said to be secondary, if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent. We call a nonzero R-module M t-secondary, if for each $a \in R$ the homothety φ_a of M given by multiplication by a is either surjective or $(\varphi_a)^t = 0$.

Proposition 2.26. *If M is a t-secondary R-module such that every ascending chain of cyclic submodules of it stops, then every proper submodule of M is a t-prime submodule.*

Proof. Let N be a proper submodule of M and $rx \in N$, for $r \in R$ and $x \in M$. Assume that φ_r is the homothety $M \to M$, for r. If $(\varphi_r)^t = 0$, then $r^t \in Ann_R(M)$ and so $r \in \sqrt[t]{(N :_R M)}$. If φ_r is surjective, then we have

$$\begin{aligned} x &= rx_1 \\ x_1 &= rx_2 \\ x_2 &= rx_3 \\ &\vdots \\ x_n &= rx_{n+1} \\ &\vdots \end{aligned}$$

for some $x_i \in M$. Then $< x > \subseteq < x_1 > \subseteq < x_2 > ... \subseteq < x_n > \subseteq ....$ Since M is complete, there exists $n \in \mathbb{N}$ such that $< x_n >=< x_i >$, for every $i \geq n$. Hence there exists $s \in R$ such that $x_{n+1} = sx_n$. It follows that $x_n = rsx_n$. So we have $x = srx$. As $rx \in N$, so $x \in N$. Therefore N is a t-prime submodule. □

Proposition 2.26, gives that once the following corollary on Noetherian t-secondary modules.

Corollary 2.27. *Let M be a Noetherian t-secondary module. Then every proper submodule is a t-prime submodule.*

3. t-prime decomposition

In this section, we deal with irreducible as well as reduced t-prime decompositions of a submodule. To this end, we give the definitions of the former concepts.

Definition 3.1. Let N be a submodule of an R-module M. A decomposition $N = \cap_{i=1}^{n} N_i$, where N_i $(1 \leq i \leq n)$ are t-prime submodules of M is called a t-prime decomposition of N in M. The t- pime decomposition is said to be reduced if there does not exist j, $(1 \leq j \leq n)$ such that $\cap_{i=1,\ i\neq j}^{n} N_i \subseteq N_j$ and all P_i distinct $(1 \leq i \leq n)$, where $P_i = \sqrt{(N_i :_R M)}$.

By Proposition 2.7 (i), the intersection of any t-prime submodule which the colon ideals of module into them are equal, is a t-prime submodule. So it is clear that any t-prime decomposition can be changed to reduced one.

Example 3.2. By Lemma 2.1 (ii) in [8], $6\mathbb{Z}$ has a t-prime decompostion.

Remark 3.3. (i) By Proposition 2.2(i) in [8], any t-prime submodule is a primary submodule. Then any t-prime decomposition, is a primary decomposition.

(ii) By Remark 3.1 in [8], $N = 8\mathbb{Z} \bigoplus 4\mathbb{Z}$ is a primary submodule, which is not a 2-prime submodule of $\mathbb{Z}$-module $\mathbb{Z} \bigoplus \mathbb{Z}$. Similarly, $L = 4\mathbb{Z} \bigoplus 8\mathbb{Z}$ is a primary submodule, which is not a 2-prime submodule. Therefore $N \cap L$ is a primary decomposition of $\mathbb{Z}$-module $\mathbb{Z} \bigoplus \mathbb{Z}$, which is not a 2-prime decomposition. (Note that in this example, ring is Noetherian and module is finitely generated.)

(iii) By Proposition 2.2(iii) in [8], a t-prime submodule is an s- prime submodule, for any $s \geq t$. Then any t-prime decomposition is an s- prime decomposition, for any $s \geq t$. But the converse is not true in general.

Recall that a proper submodule N of M is called irreducible, if N can not be expressed as an intersection of two submodules of M properly containing N [2].

Theorem 3.4. *Let M be a Noetherian R-module and N be an irreducible submodule of M such that $(N :_R M)$ is a radical ideal. Then N is a t-prime submodule.*

Proof. Assume that there exists $a \in R$ such that $\lambda_a : \dfrac{M}{N} \to \dfrac{M}{N}$ is not injective and $\lambda_a^t \neq 0$. Since $Ker\lambda_a^t \subseteq Ker\lambda_a^{2t} \subseteq \cdots$ and $\dfrac{M}{N}$ is Noetherian, there exists i such that $Ker\lambda_a^{ti} = Ker\lambda_a^{t(i+1)} = \cdots$. Put $\phi = \lambda_a^{ti}$. Then $Ker\phi = Ker\phi^2$. It is easy to see that $Ker\phi \cap Im\phi = (0)$. If $Ker\phi = (0)$, then ϕ is injective and so λ_a is injective too, which is a contradiction. If $Im\phi = (0)$, then $\lambda_{a^{ti}} = \lambda_a^{ti} = 0$ and so $a^{ti} \in (N :_R M)$. Hence $a^t \in \sqrt{(N :_R M)} = (N :_R M)$. Thus $\lambda_a^t = 0$, which is a contradiction. Now put $N_1 = \pi^{-1}(Ker\phi)$ and $N_2 = \pi^{-1}(Im\phi)$, where $\pi : M \to \dfrac{M}{N}$ is the natural projection. Then $N = N_1 \cap N_2$, such that N_1 and N_2 are submodules of M properly containig N, that is a contradiction (as N is an irreducible submodule). Therefore N is a t-prime submodule. □

Recall that a prime ideal P is said to be associated with M, if P is the annihilator of some non-zero element of M. The set $Ass(M)$ denotes the set of prime ideals associated to M.

Theorem 3.5. *Let R be a Noetherian ring, M be a finitely generated R-module, $(0) = \cap_{i=1}^{n} N_i$ be a reduced t-prime decomposition and $P_i = \sqrt{(N_i :_R M)}$. Then $Ass(M) = \{P_1, ..., P_n\}$.*

Proof. Let $P \in Ass(M)$. So there exists a non-zero $x \in M$ such that $P = Ann(x)$. As $x \neq 0$, by rearranging the N_i, there exists j such that $x \notin \cup_{i=1}^{j} N_i$ and $x \in \cap_{i=j+1}^{n} N_i$. For any i and any $a \in P_i$, as $P_i = \sqrt{(N_i :_R M)}$, there exists the smallest $s \in \mathbb{N}$ such that $a^s M \subseteq N_i$. So $\lambda_a : \dfrac{M}{N_i} \to \dfrac{M}{N_i}$ is not injective. Hence as N_i is t-prime, by Theorem 2.25, $\lambda_a^t = 0$. It means $a^t \in (N_i :_R M)$. As R is Noetherian, P_i is finitely generated and so there exists $n_i \in \mathbb{N}$ such that $P_i^{n_i} \subseteq (N_i :_R M)$. Thus $\prod_{i=1}^{j} P_i^{n_i} \subseteq (N_i :_R M)$. Then $\prod_{i=1}^{j} P_i^{n_i} x \subseteq \cap_{i=1}^{n} N_i = (0)$. Therefore $\prod_{i=1}^{j} P_i^{n_i} \subseteq Ann(x) = P$ and so there exists $l \in \mathbb{N}$, $(1 \leq l \leq j)$ such that $P_l \subseteq P$. Now assume that $a \in P$. So $ax = 0$ and since $x \notin N_l$, $\lambda_a : \dfrac{M}{N_l} \to \dfrac{M}{N_l}$ is not injective. Hence as N_l is t-prime, by Theorem 2.25, $\lambda_a^t = 0$. It means $a^t \in (N_l :_R M)$. So $a \in \sqrt{(N_l :_R M)} = P_l$. Thus $P = P_l \in \{P_1, ..., P_n\}$. Now we show that for any j, $(1 \leq j \leq n)$, $P_j \in Ass(M)$. As $(0) = \cap_{i=1}^n N_i$ is a reduced t-prime decomposition, there exists $x \in \cap_{i=1,\ i \neq j}^n N_i - N_j$. On the other hand, as R is Noetherian, there exists the smallest $n \in \mathbb{N}$ such that $P_j^n x \subseteq N_j$ and $P_j^{n-1} x \not\subseteq N_j$. Let $y \in P_j^{n-1} x - N_j$. Then $P_j y \subseteq P_j^n x \subseteq \cap_{i=1}^n N_i = (0)$. Hence $P_j \subseteq Ann(y)$. If $a \in Ann(y)$, then $ay = 0$ and since $y \notin N_j$, $\lambda_a : \frac{M}{N_j} \to \frac{M}{N_j}$ is not injective. Hence as N_j is t-prime, by Theorem 2.25, $\lambda_a^t = 0$. It means $a^t \in (N_i :_R M)$. So $a \in \sqrt{(N_j :_R M)} = P_j$ and thus $Ann(y) \subseteq P_j$. Therefore $P_j = Ann(y) \in Ass(M)$. □

Corollary 3.6. *Let R be a Noetherian ring and M be a finitely generated R-module. If $N = \cap_{i=1}^n N_i$ is a reduced t-prime decomposition, then $P_i (= \sqrt{(N_i :_R M)})$ are uniquely determined by N.*

Proof. By Corollary 2.16, $(0) = \cap_{i=1}^n \frac{N_i}{N}$ is a reduced t-prime decomposition in $\frac{M}{N}$. Therefore $Ass(\frac{M}{N}) = \{P_1, ..., P_n\}$ is uniquely determined by N. □

Corollary 3.7. *With assumption of Theorem 3.5, $Ass(M)$ is a finite set and $M = 0$ if and only if $Ass(M) = \emptyset$.*

4. Examples

This section is devoted to examples. We provide several examples which is illustrated throughout our results mentioned in previous sections.

Remark 4.1. If $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 8\mathbb{Z} \oplus 4\mathbb{Z}$, then as
$$\sqrt{(N :_R M)} = 2\mathbb{Z},$$
N is a primary submodule of M, but it is not a 2-prime submodule, as $2(4, 2) \in N$, but $2 \notin \sqrt[2]{(N :_R M)}$ and $(4, 2) \notin N$.

Hence every primary decomposition is not a t-prime decomposition.

In the following examples we show that the condition $Ker(f) \subseteq N$ in Theorem 2.15(i) and the condition monomorphism in Theorem 2.15(ii), are necessary.

Example 4.2. Consider the $\mathbb{Z}$-epimorphism
$$\psi : \mathbb{Z} \longrightarrow \mathbb{Z}_6; \qquad a \longmapsto \bar{a}$$

Clearly $\psi(0) = \bar{0}$ and $Ker(\psi) = 6\mathbb{Z} \nsubseteq (0)$. $(\bar{0})$ is not a t-prime submodule of $\mathbb{Z}_6$. Since $2 \,.\, \overline{3} = \overline{0}$ but $2^t \notin (\overline{0} :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$.

Example 4.3. Consider the zero homomorphism

$$g : \mathbb{Q} \longrightarrow \mathbb{Z};$$

clearly $Ker(g) = \mathbb{Q}$. So g is not a monomorphism. By Example 3.6 in [8], $g^{-1}(0)$ is not a t-prime submodule.

Example 4.4. (i) $M = \mathbb{Z}_4$ is a 2-secondary $\mathbb{Z}$-module.
(ii) $M = \mathbb{Z} \oplus 3\mathbb{Z}$ is not a 2-secondary $\mathbb{Z}$-module (Consider φ_2).

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t-PRIME SUBMODULES AND THEIR DECOMPOSITIONS

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زیرمدول‌های t-اول و تجزیه آن‌ها

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فرض کنید R یک حلقه جابجایی و یکدار باشد. برای $t \in \mathbb{N}$، زیرمدول سره‌ی N از R-مدول M را زیرمدول t-اول گوییم؛ اگر (برای $r \in R$ و $m \in M$) از $rm \in N$، نتیجه بگیریم $m \in N$ یا $r^t \in (N :_R M)$. ما در این مقاله، ویژگی‌های دیگری از زیرمدول‌های t-اول را بدست می‌آوریم. همچنین، با کمک برخی مفاهیم دیگر مانند زیرمدول‌های t-ثانویه، خواص گوناگونی از زیرمدول‌های t-اول را بدست می‌آوریم. در پایان، مفهوم تجزیه (کاهش یافته) t-اول از یک مدول را بیان خواهیم کرد. مثال‌های متعددی برای مفاهیم بدست آمده نیز ذکر می‌کنیم.

کلمات کلیدی: زیرمدول t-اول، زیرمدول اول، زیرمدول اولیه.