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ON ZAGREB ENERGIES OF SOME GRAPH OPERATIONS

I. J. Gogoi*, S. Borah and A. Bharali

Abstract. Recently, Zagreb energies, a graph invariant based on the eigenvalues of the Zagreb matrices have been proposed as an analogous to graph energy. In this communication, the Zagreb energies and Zagreb spectral radius are examined in relation to a number of graph operations, such as m-splitting graphs, m-shadow graphs, m-duplicate graphs, and extended bipartite double graphs. Further, we explore these generalised graphs within the context of specific graph types such as complete graphs, complete bipartite graphs, cycle graphs, and the complements of cycle graphs. Furthermore, we report an error present in [20] that contradicts the claim of hyperenergetic behaviour for the splitting graph of regular graphs, and also establish the non-hyperenergetic behaviour of the m-shadow graph.

1. Introduction

Throughout the paper, we consider connected graph with at least two vertices. Let $G \coloneqq (V, E)$ be a graph with n vertices and m edges, the vertex set $V(G) \coloneqq \{v_1, \dots, v_n\}$ and edge set $E(G) \coloneqq \{e_1, \dots, e_m\}$. The degree of a vertex v_i is denoted by d_i, $d_i \coloneqq d_G(v_i)$. The adjacency matrix $A(G) \coloneqq (a_{ij})$ of a graph G is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda_1, \dots, \lambda_p$ are distinct eigenvalues of $A(G)$, then the spectrum of $A(G)$ can be written as

$$\operatorname{spec}(A(G)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}$$

where m_i is the respective multiplicity of the eigenvalue λ_i.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A(G)$. Then the energy [6] of G is

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

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*Corresponding author.

In recent times, graph energy has become an important graph invariant that has been discovered to have a significant influence on chemical graph theory. The concept originated from Hückel molecular orbital theory, where it established a good correlation with the total π-electron energy of conjugated hydrocarbon molecules. One may recall [7] for the origin and chemical interpretation of graph energy.

In 2005, Rodríguez [18] investigated the spectral properties of graph invariants, revealing the interrelationship between the weighted adjacency matrix and topological indices. Since then, researchers have made significant progress in this area and published many papers on graph energy and its variants [1, 2, 3, 15, 17]. In this paper, we consider the first Zagreb and the second Zagreb matrix and their energies.

The first Zagreb matrix [15] $Z^{(1)}(G) := (z_{ij}^{(1)})$ of a graph G is defined as

$$z_{ij}^{(1)} = \begin{cases} d_i + d_j, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

The second Zagreb matrix [15] $Z^{(2)}(G) := (z_{ij}^{(2)})$ of a graph G is defined as

$$z_{ij}^{(2)} = \begin{cases} d_i d_j, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

Based on the eigenvalues of the Zagreb matrices, the Zagreb energies [15] have been proposed. If $\mu_1^{(1)}, \dots, \mu_n^{(1)}$ and $\mu_1^{(2)}, \dots, \mu_n^{(2)}$ are eigenvalues of $Z^{(1)}(G)$ and $Z^{(2)}(G)$ respectively, then the first Zagreb energy of a graph G, denoted by $ZE_1(G)$, and the second Zagreb energy of a graph G, denoted by $ZE_2(G)$ are respectively defined as

$$ZE_1(G) = \sum_{i=1}^{n} |\mu_i^{(1)}| \quad \text{and} \quad ZE_2(G) = \sum_{i=1}^{n} |\mu_i^{(2)}|.$$

A graph G is first Zagreb hyperenergetic if $ZE_1(G) > ZE_1(K_n)$ and first Zagreb non-hyperenergetic if $ZE_1(G) < ZE_1(K_n)$. Similarly G is second Zagreb hyperenergetic if $ZE_2(G) > ZE_2(K_n)$ and second Zagreb non-hyperenergetic if $ZE_2(G) < ZE_2(K_n)$.

Graph operations indeed offer a powerful toolkit for exploring the properties and behaviours of graphs. By manipulating the structure of a graph through operations such as addition, deletion, contraction, or transformation, researchers can analyze how these changes affect various graph metrics, including energy. In 2017, Ma and Liu [10] determined the energy of the resulting graphs from other unary operations on a graph or other binary operations on two graphs. Chu et al. [4] in 2019 have shown that the different graph energies of the regular splitting graph are a multiple of the corresponding energy of a given graph. In 2020, Mondal et al. [12] presented some novel graph energies, and several QSPR models have been utilised to show their applicability as molecular descriptors. Sheikholeslami et al. [20] determined certain Zagreb hyperenergetic, borderenergetic, and equienergetic graphs in 2021. For more works in these areas, see [11, 13, 14, 22, 23].

2. Some Preliminaries

We begin with the definition of the Kronecker product of graphs and a related lemma that will be needed to obtain our findings.

Definition 2.1. [5] Let $A \coloneqq (a_{ij})$ and $B \coloneqq (b_{ij})$ be two matrices of order $m \times n$ and $p \times q$, respectively. Then, their Kronecker product (or tensor product), $A \otimes B$ is obtained from A when every element a_{ij} is replaced by the block $a_{ij}B$ and is of order $mp \times nq$. i.e.,

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Lemma 2.2. [5] *Let $A \in M^p$ and $B \in M^q$. If α is an eigenvalue of the matrix A with corresponding eigenvector y and β is an eigenvalue of the matrix B with corresponding eigenvector z then, $\alpha\beta$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector yz.*

Now we define some special graphs that are constructed or derived from an original undirected graph, such as the m-splitting graph, the m-shadow graph, the m-duplicate graph, and the extended bipartite double graph.

Definition 2.3. [8] For a graph G, the m-splitting graph $Spl_m(G)$ is obtained by adding new m vertices say $v_1^{(i)}, \ldots, v_m^{(i)}$ corresponding to each vertex v_i of G such that each $v_k^{(i)}$, $1 \leq k \leq m$ is adjacent to each vertices that is adjacent to v_i in G.

If $m = 1$, then the m-splitting graph is known as the splitting graph of G. The m-splitting graph of K_3 is shown in Figure 1.

The number of vertices in $Spl_m(G)$ is $(m+1) \times |V(G)|$.

Definition 2.4. [8] Let G be a simple graph with n vertices. Then the m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m

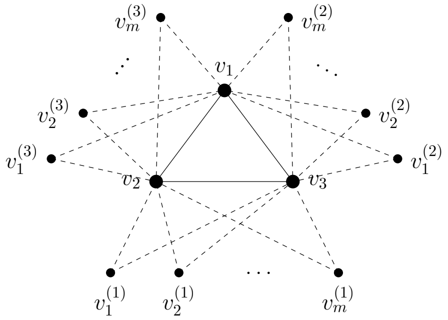

FIGURE 1. m-splitting graph of K_3

copies of G, say $G_1, \ldots, G_m$ then join each vertex v_i in G_i to the neighbours of the corresponding vertex v_j in G_j, for $1 \leq j \leq m$.

If $m = 2$, the m-shadow graph is known as the shadow graph of G. The 4-shadow graph of P_3 is shown in Figure 2. The number of vertices in the m-shadow graph is $m \times |V(G)|$.

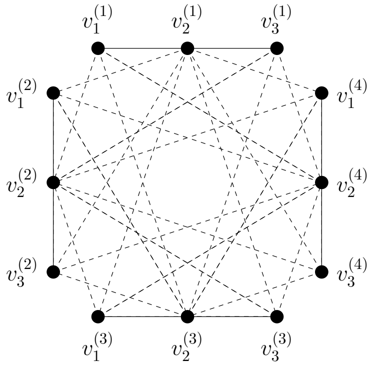

FIGURE 2. 4-shadow graph of P_3

Definition 2.5. [8] Let G be a simple graph of order n with vertex set V and edge set E. Let V' be a set such that $|V| = |V'|$ and $V \cap V' = \phi$ and $f : V \longrightarrow V'$ be bijective. A duplicate graph of G is $D(G) = (V_1, E_1)$, where the vertex set $V_1 = V \cup V'$ and the edge set E_1 of $D(G)$ is defined as the edge ab is in E if and only if both ab' and $a'b$ are in E_1.

In general, the m-duplicate graph $D^m(G)$ is defined as

$$D^m(G) = D^{m-1}(D(G)).$$

The duplicate graph of P_3 is shown in Figure 3. The number of vertices in the m-duplicate graph is $2^m \times |V(G)|$.

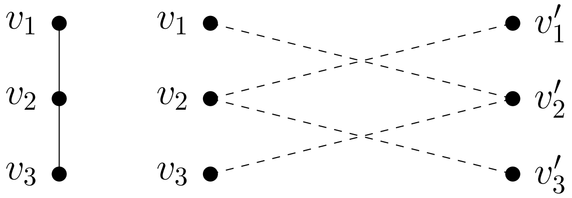

FIGURE 3. Duplicate graph of P_3

Definition 2.6. [16] Let G be a graph with the vertex set

$$V(G) \coloneqq \{v_1, \ldots, v_n\}.$$

The extended bipartite double graph $Ebd(G)$ of a graph G is the bipartite graph with its partite sets $X \coloneqq \{x_1, \ldots, x_n\}$ and $Y \coloneqq \{y_1, \ldots, y_n\}$ in which two vertices x_i and y_j are adjacent if $i = j$ or v_i and v_j are adjacent in G.

The extended bipartite double graph of P_3 is shown in Figure 4.

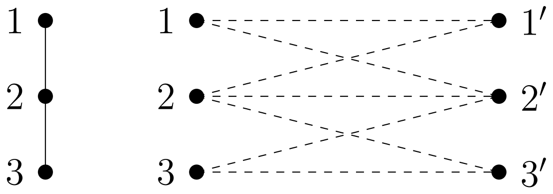

FIGURE 4. Extended bipartite double graph of P_3

These graphs are useful in different areas of network analysis and applied mathematics. Recently, Kumar and Sinha [9] mentioned that a variation of the concept of splitting graph has been applied in the analysis of online social networks, where newly added vertices $v_k^{(i)}$ are also connected to v_k, which is termed the "clone" of v_k. Also, creating classes of equienergetic graphs, hyperenergetic graphs, non-hyperenergetic graphs, etc. are significant mathematical applications for these graph operations. Shanthakumari and Lokesha [19] in 2022, computed a pair of equienergetic graph using the m-duplicate and the m-shadow graph. In the same year, Ramane et al. [16] determined the class of equienergetic graphs by using extended bipartite double graphs. In 2024, Vaidya and Popat [21] estimated the energy of an extended shadow graph and utilised it to create a new family of non-complete borderenergetic graphs and non-cospectral equienergetic graphs.

3. ZAGREB ENERGIES OF THE GENERALISED SPLITTING GRAPH

Theorem 3.1. *Let G be a regular graph with n vertices. Then the first Zagreb energy of the m-splitting graph of G is*

$$ZE_1(Spl_m(G)) = \sqrt{(1+m)^2 + m(2+m)^2}\ \ ZE_1(G),$$

where $ZE_1(G)$ is the first Zagreb energy of the graph G.

Proof. Suppose G be a k-regular graph with vertices $v_1, \dots, v_n$. Let $v_1^{(i)}, \dots, v_m^{(i)}$ be the vertices corresponding to each vertex v_i of G that are added in G to create $Spl_m(G)$ such that $d(v_1^{(i)}) = \cdots = d(v_m^{(i)})$. From Definition 2.3, with appropriate labelling of the vertices, the first Zagreb matrix, $Z^{(1)}(Spl_m(G))$ of $Spl_m(G)$ is

$$\begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1(m+1)} \\ D_{21} & D_{22} & \cdots & D_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ D_{(m+1)1} & D_{(m+1)2} & \cdots & D_{(m+1)(m+1)} \end{bmatrix},$$

where $D_{11} = (1+m)Z^{(1)}(G)$, $D_{1k} = D_{k1} = \frac{2+m}{2}Z^{(1)}(G)$ for $1 < k \leq m+1$ and all other blocks are 0. In tensor form, we may write it as

$$Z^{(1)}(Spl_m(G)) = X \otimes Z^{(1)}(G),$$

where $X \coloneqq (x_{ij})$ has entries

$$x_{ij} = \begin{cases} (1+m), & i = j = 1; \\ \frac{2+m}{2}, & i = 1,\, j \geq 2 \ \&\ j = 1,\, i \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\operatorname{rank}(X) = 2$ and $\operatorname{trace}(X) = (1+m)$. Let the two nonzero eigenvalues are ξ_1 and ξ_2 with

$$\xi_1 + \xi_2 = (1+m). \tag{3.1}$$

Again, $\operatorname{trace}(X^2) = \frac{2(1+m)^2 + m(2+m)^2}{2}$. Therefore

$$\xi_1^2 + \xi_2^2 = \frac{2(1+m)^2 + m(2+m)^2}{2}. \tag{3.2}$$

Solving Equations (3.1) and (3.2), we get $\xi_1 = \frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}$ and $\xi_2 = \frac{(1+m)-\sqrt{(1+m)^2+m(2+m)^2}}{2}$. By Lemma 2.2,

$$\operatorname{spec}(Z^{(1)}(Spl_m(G))) = \begin{pmatrix} 0 & \xi_1\mu_1^{(1)} & \cdots & \xi_1\mu_n^{(1)} & \xi_2\mu_1^{(1)} & \cdots & \xi_2\mu_n^{(1)} \\ n(m-1) & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $\mu_i^{(1)}$, $1 \leq i \leq n$ are eigenvalues of the matrix $Z^{(1)}(G)$.
Also, in the eigenvalue ξ_2 we have $(1+m) \leq \sqrt{(1+m)^2+m(2+m)^2}$, this implies $\left|\frac{(1+m)-\sqrt{(1+m)^2+m(2+m)^2}}{2}\right| = \frac{\sqrt{(1+m)^2+m(2+m)^2}-(1+m)}{2}$. Therefore,

$$\begin{aligned} ZE_1(Spl_m(G)) &= \sum_{i=1}^{n}\left(\left|\xi_1\mu_i^{(1)}\right| + \left|\xi_2\mu_i^{(1)}\right|\right) \\ &= \sum_{i=1}^{n}\Bigg(\left|\frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\right| \\ &+ \left|\frac{(1+m)-\sqrt{(1+m)^2+m(2+m)^2}}{2}\right|\Bigg)\mu_i^{(1)} \\ &= \sum_{i=1}^{n} \sqrt{(1+m)^2+m(2+m)^2}\left|\mu_i^{(1)}\right| \\ &= \sqrt{(1+m)^2+m(2+m)^2}\ \ ZE_1(G). \quad \square \end{aligned}$$

Corollary 3.2. *Let G_1 and G_2 be two non-cospectral first Zagreb equienergetic regular graphs, then $Spl_m(G_1)$ and $Spl_m(G_2)$ are non-cospectral first Zagreb equienergetic.*

Let $\rho^{Z^{(1)}(G)}$ be the first Zagreb spectral radius, i.e., the largest absolute value of the eigenvalues of $Z^{(1)}(G)$. Now, we can compute the first Zagreb spectral radius of the generalised splitting graph of a graph G.

Theorem 3.3. *Let G be a regular graph of order n. Then the first Zagreb spectral radius, $\rho^{Z^{(1)}(Spl_m(G))}$ of the m-splitting graph of G is*

$$\rho^{Z^{(1)}(Spl_m(G))} = \frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\rho^{Z^{(1)}(G)}$$

Proof. From Theorem 3.1, the nonzero eigenvalues of $Z^{(1)}(Spl_m(G))$ are $\frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\mu_i^{(1)}$ and $\frac{(1+m)-\sqrt{(1+m)^2+m(2+m)^2}}{2}\mu_i^{(1)}$ where $\mu_i^{(1)}$, $1 \leq i \leq n$ are the first Zagreb eigenvalues of the graph G. Note that

$$\left|\frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\right| > \left|\frac{(1+m)-\sqrt{(1+m)^2+m(2+m)^2}}{2}\right|,$$

which implies

$$\begin{aligned} \rho^{Z^{(1)}(Spl_m(G))} &= \max_{i=1}^{n}\left|\frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\mu_i^{(1)}\right| \\ &= \frac{(1+m)+\sqrt{(1+m)^2+m(2+m)^2}}{2}\rho^{Z^{(1)}(G)}. \end{aligned}$$

$\square$

Let K_n and $K_{n,n}$ denote the complete graph with n vertices and the complete bipartite graph with each partition having n vertices respectively. In 2021, Sheikholeslami et al. [20] computed the first Zagreb energy of the complete graph and complete bipartite graph as $4(n-1)^2$ and $4n^2$ respectively. Combining these with Theorem 3.1 yields the following two propositions.

Proposition 3.4. *The first Zagreb energy of the m-splitting graph of K_n is*

$$ZE_1(Spl_m(K_n)) = 4(n-1)^2\sqrt{(1+m)^2 + m(2+m)^2}.$$

Proposition 3.5. *The first Zagreb energy of the m-splitting graph of $K_{n,n}$ is*

$$ZE_1(Spl_m(K_{n,n})) = 4n^2\sqrt{(1+m)^2 + m(2+m)^2}.$$

In [20], Sheikholeslami et al. makes an interesting observation about the splitting graph of regular graphs. Specifically, they state that for a regular graph G, the splitting graph of G is first Zagreb hyperenergetic i.e., $ZE_1(Spl(G)) > 4(2n-1)^2$. However, it seems that some errors in the evidence presented by Sheikholeslami et al. contradict the claim of hyperenergetic behaviour for the splitting graph of regular graphs. To illustrate this inconsistency, we present an example that contradicts the claim.

Example 3.6. Consider the complete graph K_3. The computed first Zagreb energy of its splitting graph $Spl(K_3)$ is 57.6888205. If the claim of hyperenergetic behaviour for the splitting graph of regular graphs is true, we would expect this value to be greater than the first Zagreb energy of the complete graph with 6 vertices, K_6, which is 100. However, as we can see, K_6 has a higher first Zagreb energy than $Spl(K_3)$, which contradicts the claim of hyperenergetic behaviour.

Lemma 3.7. [5] *Let C_n and $\overline{C_n}$ be the cycle graph and the complement of the cycle graph with n vertices. Then* $\operatorname{spec}(C_n) = \begin{pmatrix} 2 & 2\cos\frac{2\pi k}{n} \\ 1 & 1 \end{pmatrix}$ *and* $\operatorname{spec}(\overline{C_n}) = \begin{pmatrix} n-3 & -1-2\cos\frac{2\pi k}{n} \\ 1 & 1 \end{pmatrix}$, $k = 1, \dots, n-1$.

Theorem 3.8. *Let C_n be the cycle graph with n vertices. Then the first Zagreb energy of the m-splitting graph of C_n is*

$$ZE_1(Spl_m(C_n)) = \begin{cases} \frac{16\sqrt{(1+m)^2+m(2+m)^2}\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \textit{when } n \equiv 0 \textit{ mod(4);} \\ \frac{8\sqrt{(1+m)^2+m(2+m)^2}}{\sin\frac{\pi}{2n}}, & \textit{when } n \equiv 1 \textit{ mod(4);} \\ \frac{16\sqrt{(1+m)^2+m(2+m)^2}}{\sin\frac{\pi}{n}}, & \textit{when } n \equiv 2 \textit{ mod(4).} \end{cases}$$

Proof. Since there are precisely two neighbours for every vertex in the cycle, the first Zagreb matrix of the graph C_n is

$$\begin{bmatrix} 0 & 4 & 0 & \cdots & 4 \\ 4 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

From Lemma 3.7, the spectrum of $Z^{(1)}(C_n)$ is $8\cos\frac{2\pi k}{n}$, where $k \in \{0,\ldots,n-1\}$. Now we shall consider three cases: $n \equiv 0 \bmod(4)$, $n \equiv 1 \bmod(4)$ and $n \equiv 2 \bmod(4)$. First, suppose that $n \equiv 0 \bmod(4)$. Then $\cos\frac{2\pi k}{n} > 0$ only if $k < \frac{n}{4}$ and $\frac{3n}{4} \leq k < n$. Therefore, the positive eigenvalues of $Z^{(1)}(C_n)$ are

$$\begin{aligned} &8\cos\left(\frac{2\pi}{n}\times 0\right), 8\cos\left(\frac{2\pi}{n}\times 1\right), \ldots, 8\cos\left(\frac{2\pi}{n}\times(\frac{n}{4}-1)\right), \\ &8\cos\left(\frac{2\pi}{n}\times\frac{3n}{4}\right), \ldots, 8\cos\left(\frac{2\pi}{n}\times(n-1)\right). \end{aligned}$$

Consider C be the sum of the positive eigenvalues of $Z^{(1)}(C_n)$. Let us choose S in such a way that

$$\begin{aligned} C+iS =& 8e^{\left(\frac{2\pi}{n}\times 0\right)i} + \ldots + 8e^{\left(\frac{2\pi}{n}\times(\frac{n}{4}-1)\right)i} + 8e^{\left(\frac{2\pi}{n}\times\frac{3n}{4}\right)i} + \ldots + 8e^{\left(\frac{2\pi}{n}\times(n-1)\right)i} \\ =& 8\sum_{k=0}^{\frac{n}{4}-1} e^{i\frac{2\pi k}{n}} + 8\sum_{k=\frac{3n}{4}}^{n-1} e^{i\frac{2\pi k}{n}} \\ =& 8\frac{1-\left(e^{i\frac{2\pi}{n}}\right)^{\frac{n}{4}}}{1-e^{i\frac{2\pi}{n}}} + 8e^{i\left(\frac{2\pi}{n}\times\frac{3n}{4}\right)}\frac{1-\left(e^{i\frac{2\pi}{n}}\right)^{\frac{n}{4}}}{1-e^{i\frac{2\pi}{n}}} \\ =& (8-8i)\frac{1-i}{1-e^{i\frac{2\pi}{n}}}. \end{aligned}$$

Comparing real & imaginary part, we get $C = 8\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$.
Similarly, we can obtain the energy for $n \equiv 1 \bmod(4)$ and $n \equiv 2 \bmod(4)$. Therefore,

$$ZE_1(C_n) = \begin{cases} \frac{16\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \bmod(4); \\ \frac{8}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \bmod(4); \\ \frac{16}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \bmod(4). \end{cases} \tag{3.3}$$

The rest of the proof is obtained from Theorem 3.1 and Equation (3.3). □

Theorem 3.9. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. If $\Upsilon_1 = 4(n-3)\sqrt{(1+m)^2 + m(2+m)^2}$, then the first Zagreb energy of the m-splitting graph of $\overline{C_n}$ is*

$$ZE_1(Spl_m(\overline{C_n})) = \begin{cases} \Upsilon_1\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \ mod(3);\\ \Upsilon_1\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \ mod(3);\\ \Upsilon_1\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \ mod(3).\end{cases}$$

Proof. In $\overline{C_n}$ (complement of the cycle graph C_n), each vertex is connected to all other vertices except its two immediate neighbours in C_n, i.e., $\overline{C_n}$ is $(n-3)$-regular. Therefore, the matrix $Z^{(1)}(\overline{C_n}) := (z_{ij})$ can be described as

$$z_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } i \text{ is adjacent to } j \text{ in } C_n;\\ 2(n-3), & \text{otherwise.}\end{cases}$$

By Lemma 3.7, the spectrum of $Z^{(1)}(\overline{C_n})$ is

$$\operatorname{spec}(Z^{(1)}(\overline{C_n})) = \begin{pmatrix} 2(n-3)^2 & 2(n-3)\left(-1-2\cos\frac{2\pi k}{n}\right)\\ 1 & 1\end{pmatrix},$$

$k \in \{1, \ldots, n-1\}$. Now we shall consider three cases: $n \equiv 0$ mod(3), $n \equiv 1$ mod(3) and $n \equiv 2$ mod(3). First, suppose that $n \equiv 0$ mod(3). Then $2(n-3)\left(-1-2\cos\frac{2\pi k}{n}\right) > 0$ only if $\cos\frac{2\pi k}{n} \leq -\frac{1}{2}$. i.e., $\frac{n}{3} \leq k \leq \frac{2n}{3}$. Consider $\zeta_1, \ldots, \zeta_n$ are eigenvalues of $Z^{(1)}(\overline{C_n})$. Then

$$\begin{aligned}\sum_{+}\zeta_i =& 2(n-3)^2 + \sum_{k=\frac{n}{3}}^{\frac{2n}{3}} -2(n-3)\left(-1-2\cos\frac{2\pi k}{n}\right)\\ =& 2(n-3)^2 - 2(n-3)\frac{n+3}{3} - 4(n-3)\sum_{k=\frac{n}{3}}^{\frac{2n}{3}}\cos\frac{2\pi k}{n}.\end{aligned}$$

Let $C = \sum_{k=\frac{n}{3}}^{\frac{2n}{3}}\cos\frac{2\pi k}{n}$, we choose S such that $C + iS = \sum_{k=\frac{n}{3}}^{\frac{2n}{3}} e^{i\frac{2\pi k}{n}}$. Comparing real & imaginary part, we get $C = -\frac{1}{2} - \frac{\sqrt(3)}{2}\cot\frac{\pi}{n}$. Which gives

$$\sum_{+}\zeta_i = 2(n-3)\left(\frac{2n-9}{3} + \sqrt(3)\cot\frac{\pi}{n}\right).$$

Similarly, we can compute the energy for the cases $n \equiv 1 \bmod(3)$ and $n \equiv 2 \bmod(3)$. Thus,

$$ZE_1(\overline{C_n}) = \begin{cases} 4(n-3)\left(\frac{2n-9}{3} + \sqrt(3)\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \bmod(3); \\ 4(n-3)\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n}}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \bmod(3); \\ 4(n-3)\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n}}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \bmod(3). \end{cases} \tag{3.4}$$

Combining Equation (3.4) and Theorem 3.1 we get the required result. $\square$

Theorem 3.10. *Let G be a simple graph with n vertices. Then the second Zagreb energy of the m-splitting graph of G is*

$$ZE_2(Spl_m(G)) = (1+m)\sqrt{(1+m)^2+4m}\ \ ZE_2(G).$$

Proof. Let G be a graph with vertices $v_1, \dots, v_n$ and $v_1^{(i)}, \dots, v_m^{(i)}$ be the vertices corresponding to each vertex v_i of G that are added in G to create $Spl_m(G)$ such that $d(v_1^{(i)}) = \cdots = d(v_m^{(i)})$. Accordingly, the second Zagreb matrix, $Z^{(2)}(Spl_m(G))$ is a block matrix, with each block being of order n with entries

$$\begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1(m+1)} \\ D_{21} & D_{22} & \cdots & D_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ D_{(m+1)1} & D_{(m+1)2} & \cdots & D_{(m+1)(m+1)} \end{bmatrix},$$

where $D_{11} = (1+m)^2 Z^{(2)}(G)$, $D_{1k} = D_{k1} = (1+m)Z^{(2)}(G)$ for $1 < k \leq m+1$ and all other blocks are 0. In tensor form, we may write it as

$$Z^{(2)}(Spl_m(G)) = Y \otimes (1+m)Z^{(2)}(G),$$

where $Y := (y_{ij})$ has entries

$$y_{ij} = \begin{cases} (1+m), & i = j = 1; \\ 1, & i = 1,\ j \geq 2\ \&\ j = 1,\ i \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Let ζ_1 and ζ_2 are nonzero eigenvalues of Y. Now,

$$\operatorname{rank}(Y) = 2,\ \operatorname{trace}(Y) = 1+m,$$

and $\operatorname{trace}(Y^2) = (1+m)^2 + 2m$. Hence,

$$\zeta_1 + \zeta_2 = 1+m. \tag{3.5}$$

$$\zeta_1^2 + \zeta_2^2 = (1+m)^2 + 2m. \tag{3.6}$$

Solving Equations (3.5) and (3.6), we get $\zeta_1 = \frac{(1+m)+\sqrt{(1+m)^2+4m}}{2}$ and $\zeta_2 = \frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}$. By Lemma 2.2, spec$(Z^{(2)}(Spl_m(G)))$ is

$$\begin{pmatrix} 0 & (1+m)\zeta_1\mu_1^{(2)} & \cdots & (1+m)\zeta_1\mu_n^{(2)} & (1+m)\zeta_2\mu_1^{(2)} & \cdots & (1+m)\zeta_2\mu_n^{(2)} \\ n(m-1) & 1 & \cdots & 1 & \cdots & 1 \end{pmatrix},$$

where $\mu_i^{(2)}$, $1 \le i \le n$ are eigenvalues of $Z^{(2)}(G)$.

Also, in the eigenvalue ζ_2, $\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}$ is always negative, it implies $\left|\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}\right| = \frac{\sqrt{(1+m)^2+4m}-(1+m)}{2}$. Therefore,

$$\begin{aligned} ZE_2(Spl_m(G)) &= \sum_{i=1}^{n}\left(\left|\zeta_1(1+m)\mu_i^{(2)}\right| + \left|\zeta_2(1+m)\mu_i^{(2)}\right|\right) \\ &= \sum_{i=1}^{n}\left(\frac{(1+m)+\sqrt{(1+m)^2+4m}}{2} + \frac{\sqrt{(1+m)^2+4m}-(1+m)}{2}\right)(1+m)\left|\mu_i^{(2)}\right| \\ &= \sum_{i=1}^{n}(1+m)\sqrt{(1+m)^2+4m}\ \left|\mu_i^{(2)}\right| \\ &= (1+m)\sqrt{(1+m)^2+4m}\ \ ZE_2(G). \quad \square \end{aligned}$$

Corollary 3.11. *Let G_1 and G_2 be two non-cospectral second Zagreb equienergetic graphs, then $Spl_m(G_1)$ and $Spl_m(G_2)$ are non-cospectral second Zagreb equienergetic.*

Proposition 3.12. *The second Zagreb energy of the m-splitting graph of K_n is*

$$ZE_2(Spl_m(K_n)) = 2(n-1)^3(1+m)\sqrt{(1+m)^2+4m}.$$

Proof. In K_n, each vertex is of degree $n-1$, implying that the second Zagreb matrix of K_n is of the form $(n-1)^2[J_n - I_n]$ where J_n is the $n \times n$ matrix with all entries are 1 and I_n is the identity matrix of order n. This gives the eigenvalues of $Z^{(2)}(K_n)$ and are $(n-1)^3$ and $-(n-1)^2$ with multiplicity 1 and $n-1$ respectively, and hence

$$ZE_2(K_n) = 2(n-1)^3, \tag{3.7}$$

which gives the required result. $\square$

Proposition 3.13. *The second Zagreb energy of the m-splitting graph of $K_{n,n}$ is*

$$ZE_2(Spl_m(K_{n,n})) = 2n^3(1+m)\sqrt{(1+m)^2+4m}.$$

Proof. The second Zagreb matrix of $K_{n,n}$ is a block matrix with the diagonal blocks being 0 and the other blocks being n^2 times of the matrix J_n, where J_n is the matrix with all entries 1. In tensor form, we may write it as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes n^2 J_n$. By using Lemma 2.2, we have

$$ZE_2(K_{n,n}) = 2n^3, \tag{3.8}$$

which gives the required result. □

Theorem 3.14. *Let G be a simple graph of order n. Then the second Zagreb spectral radius $\rho^{Z^{(2)}(Spl_m(G))}$ of the m-splitting graph is*

$$\rho^{Z^{(2)}(Spl_m(G))} = (1+m)\frac{(1+m) - \sqrt{(1+m)^2+4m}}{2}\rho^{Z^{(2)}(G)},$$

where $\rho^{Z^{(2)}(G)}$ is the spectral radius of $Z^{(2)}(G)$.

Proof. From Theorem 3.10, the nonzero eigenvalues of $Z^{(2)}(Spl_m(G))$ are $(1+m)\frac{(1+m)+\sqrt{(1+m)^2+4m}}{2}\mu_i^{(2)}$ and $(1+m)\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}\mu_i^{(2)}$, where $\mu_i^{(2)}$, $1 \leq i \leq n$ are eigenvalues of $Z^{(2)}(G)$. Also, $|\frac{(1+m)+\sqrt{(1+m)^2+4m}}{2}| > |\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}|$, which implies

$$\begin{aligned}\rho^{Z^{(1)}(Spl_m(G))} =& \max_{i=1}^{n}\left|(1+m)\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}\mu_i^{(2)}\right| \\ =&(1+m)\frac{(1+m)-\sqrt{(1+m)^2+4m}}{2}\rho^{Z^{(2)}(G)}.\end{aligned}$$ □

Theorem 3.15. *Let C_n be the cycle graph with n vertices. Then the second Zagreb energy of the m-splitting graph of C_n is*

$$ZE_2(Spl_m(C_n)) = \begin{cases} \frac{16(1+m)\sqrt{(1+m)^2+4m}\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \bmod(4); \\ \frac{8(1+m)\sqrt{(1+m)^2+4m}}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \bmod(4); \\ \frac{16(1+m)\sqrt{(1+m)^2+4m}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \bmod(4). \end{cases}$$

Proof. Proof is similar to the proof of Theorem 3.8 because C_n is 2-regular, and hence $d_i + d_j = d_i d_j$. From Lemma 3.7, the spectrum of $Z^{(1)}(C_n)$ is

$8\cos\frac{2\pi k}{n}$, where $k = 0, \ldots, n-1$. Which gives

$$ZE_2(C_n) = \begin{cases} \frac{16\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \text{ mod}(4); \\ \frac{8}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \text{ mod}(4); \\ \frac{16}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \text{ mod}(4). \end{cases} \tag{3.9}$$

The rest of the proof is obtained directly from Theorem 3.10. □

Theorem 3.16. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. If $\Upsilon_2 = 2(n-3)^2(1+m)\sqrt{(1+m)^2+4m}$ then the first Zagreb energy of the m-splitting graph of $\overline{C_n}$ is*

$$ZE_2(Spl_m(\overline{C_n})) = \begin{cases} \Upsilon_2\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \textit{when } n \equiv 0 \textit{ mod(3);} \\ \Upsilon_2\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \textit{when } n \equiv 1 \textit{ mod(3);} \\ \Upsilon_2\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \textit{when } n \equiv 2 \textit{ mod(3).} \end{cases}$$

Proof. Recall that $\overline{C_n}$ is $(n-3)$-regular, and hence the matrix $Z^{(2)}(\overline{C_n}) \coloneqq (z_{ij})$ can be described as

$$z_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } i \text{ is adjacent to } j \text{ in } C_n; \\ (n-3)^2, & \text{otherwise.} \end{cases}$$

Therefore, the second Zagreb spectrum of $\overline{C_n}$ is

$$\operatorname{spec}(Z^{(2)}(\overline{C_n})) = \begin{pmatrix} (n-3)^3 & (n-3)^2\left(-1-2\cos\frac{2\pi k}{n}\right) \\ 1 & 1 \end{pmatrix},$$

$k \in \{1, \ldots, n-1\}$. Now we shall consider three cases: $n \equiv 0 \text{ mod}(3)$, $n \equiv 1 \text{ mod}(3)$ and $n \equiv 2 \text{ mod}(3)$. First, suppose that $n \equiv 0 \text{ mod}(3)$. Then $(n-3)^2\left(-1-2\cos\frac{2\pi k}{n}\right) > 0$ only if $\frac{n}{3} \leq k \leq \frac{2n}{3}$. Consider $\zeta_1, \ldots, \zeta_n$ are eigenvalues of $Z^{(2)}(\overline{C_n})$. Then

$$\sum_{+} \zeta_i = (n-3)^2\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right).$$

Similarly, we can compute the energy for $n \equiv 1 \text{ mod}(3)$ and $n \equiv 2 \text{ mod}(3)$. Thus

$$ZE_2(Spl_m(\overline{C_n})) = \begin{cases} 2(n-3)^2\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \text{ mod}(3); \\ 2(n-3)^2\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \text{ mod}(3); \\ 2(n-3)^2\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \text{ mod}(3). \end{cases} \tag{3.10}$$

The remaining portions of the proof are taken straight from Theorem 3.10. □

4. ZAGREB ENERGIES OF THE GENERALISED SHADOW GRAPH

Theorem 4.1. *Let G be a simple graph with n vertices. Then the first Zagreb energy of the m-shadow graph of G is*

$$ZE_1(D_m(G)) = m^2 ZE_1(G).$$

Proof. Suppose G be a graph with vertices $v_1, \dots, v_n$ having degree $d_1, \dots, d_n$. Accordingly, to construct the m-shadow graph, we consider m copies of G with vertices $v_1^{(i)}, \dots, v_m^{(i)}$ for $1 \leq i \leq m$, so that each vertex $v_k^{(i)}$ in G_i is adjacent to the neighbours of the corresponding vertex $v_l^{(j)}$ in G_j, for $1 \leq j \leq m$. Note that the degree of each vertex is $d(v_k^{(i)}) = md_k$. Finally, we obtain the first Zagreb matrix, $Z^{(1)}(D_m(G))$ of the m-shadow graph which is a block matrix, with each block being m times of the original matrix $Z^{(1)}(G)$ of the graph G.

Now using the properties of tensor product, $Z^{(1)}(D_m(G)) = J_m \otimes mZ^{(1)}(G)$, where J_m is the $m \times m$ matrix with all entries 1 and has eigenvalues m and 0 with multiplicity 1 and $m-1$, respectively. Let $\mu_1^{(1)}, \dots, \mu_n^{(1)}$ are the eigenvalues of $Z^{(1)}(G)$. By Lemma 2.2

$$\operatorname{spec}(Z^{(1)}(D_m(G))) = \begin{pmatrix} 0 & m^2\mu_1^{(1)} & \cdots & m^2\mu_n^{(1)} \\ n(m-1) & 1 & \cdots & 1 \end{pmatrix},$$

which follows that

$$ZE_1(D_m(G)) = \sum_{i=1}^{n} \left| m^2 \mu_i^{(1)} \right| = m^2 ZE_1(G). \quad \square$$

Corollary 4.2. *Let G_1 and G_2 be two non-cospectral first Zagreb equienergetic graphs, then $D_m(G_1)$ and $D_m(G_2)$ are non-cospectral first Zagreb equienergetic.*

Since $ZE_1(K_n) = 4(n-1)^2$ and $ZE_1(K_{n,n}) = 4n^2$, we have the following two propositions from Theorem 4.1.

Proposition 4.3. *The first Zagreb energy of the m-shadow graph of K_n is*

$$ZE_1(D_m(K_n)) = 4(n-1)^2 m^2.$$

Proposition 4.4. *The first Zagreb energy of the m-shadow graph of $K_{n,n}$ is*

$$ZE_1(D_m(K_{n,n})) = 4n^2 m^2.$$

Theorem 4.5. *Let G be a graph of order n. Then the first Zagreb spectral radius $\rho^{Z^{(1)}(D_m(G))}$ of the m-shadow graph of G is*

$$\rho^{Z^{(1)}(D_m(G))} = m^2 \rho^{Z^{(1)}(G)},$$

where $\rho^{Z^{(1)}(G)}$ is the spectral radius of $Z^{(1)}(D_m(G))$.

Proof. The nonzero eigenvalues of $Z^{(1)}(D_m(G))$ are $m^2\mu_i^{(1)}$ where $\mu_i^{(1)}$, $1 \leq i \leq n$ are eigenvalues of $Z^{(1)}(G)$. This implies

$$\rho^{Z^{(1)}(D_m(G))} = \max_{i=1}^{n} |m^2\mu_i^{(1)}| = m^2 \rho^{Z^{(1)}(G)}. \qquad \square$$

Using the fact that a graph of order n is first Zagreb non-hyperenergetic if $ZE_1(G) < 4(n-1)^2$, we have the following result from Theorem 4.1.

Theorem 4.6. *Let G be a first Zagreb non-hyperenergetic graph then the m-shadow graph of G is first Zagreb non-hyperenergetic.*

Proof. If the original graph has n vertices, the corresponding m-shadow graph will have mn vertices. Which gives that the m-shadow graph is first Zagreb non-hyperenergetic if $ZE_1(D_m(G)) < 4(mn-1)^2$. Now,

$$ZE_1(D_m(G)) = m^2 ZE_1(G) < m^2 4(n-1)^2 = 4(mn-m)^2 < 4(mn-1)^2.$$

$\square$

Theorem 4.7. *Let C_n be the cycle graph with n vertices. Then the first Zagreb energy of the m-shadow graph of C_n is*

$$ZE_1(D_m(C_n)) = \begin{cases} \frac{16m^2 \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \text{ mod(4)}; \\ \frac{8m^2}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \text{ mod(4)}; \\ \frac{16m^2}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \text{ mod(4)}. \end{cases}$$

Proof. The proof is based on Theorem 4.1 that can be obtained using Equation (3.3). $\square$

Theorem 4.8. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. Then the first Zagreb energy of the m-shadow graph of $\overline{C_n}$ is*

$$ZE_1(D_m(\overline{C_n})) = \begin{cases} 4(n-3)m^2\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \text{ mod(3)}; \\ 4(n-3)m^2\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \text{ mod(3)}; \\ 4(n-3)m^2\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \text{ mod(3)}. \end{cases}$$

Proof. Theorem 4.1 and Equation (3.4) provide concrete evidence for the proof of the theorem. $\square$

Theorem 4.9. *Let G be a simple graph with n vertices. Then the second Zagreb energy of the m-shadow graph of G is $ZE_2(D_m(G)) = m^3 ZE_2(G)$.*

Proof. The second Zagreb matrix of the m-shadow graph, which is a block matrix, with each block being m^2 times of the original matrix $Z^{(2)}(G)$. By using this matrix, the second Zagreb energy of the m-shadow graph can be determined. We avoid the proof to prevent repetitions. □

Corollary 4.10. *Let G_1 and G_2 be two non-cospectral second Zagreb equienergetic graphs, then $D_m(G_1)$ and $D_m(G_2)$ are non-cospectral second Zagreb equienergetic.*

Equations (3.7) and (3.8) can be used to get the next two propositions from Theorem 4.9.

Proposition 4.11. *The second Zagreb energy of the m-shadow graph of K_n is $ZE_2(D_m(K_n)) = 2m^3(n-1)^3$.*

Proposition 4.12. *The second Zagreb energy of the m-shadow graph of $K_{n,n}$ is $ZE_2(D_m(K_{n,n})) = 2n^3m^3$.*

Theorem 4.13. *Let G be a graph of order n. Then the second Zagreb spectral radius $\rho^{Z^{(2)}(D_m(G))}$ of the m-shadow graph is $\rho^{Z^{(2)}(D_m(G))} = m^3\rho^{Z^{(2)}(G)}$, where $\rho^{Z^{(2)}(G)}$ is the spectral radius of the $Z^{(2)}(D_m(G))$.*

Proof. The nonzero eigenvalues of $Z^{(2)}(D_m(G))$ are $m^3\mu_i^{(2)}$, where $\mu_i^{(2)}$, $1 \le i \le n$ are eigenvalues of $Z^{(2)}(G)$, which implies

$$\rho^{Z^{(2)}(D_m(G))} = \max_{i=1}^{n} |m^3\mu_i^{(2)}| = m^3\rho^{Z^{(2)}(G)}.$$

□

Again, using the fact that a graph of order n is second Zagreb non-hyperenergetic if $ZE_2(G) < 2(n-1)^3$, we get the following result from Theorem 4.9.

Theorem 4.14. *Let G be a second Zagreb non-hyperenergetic graph, then the m-shadow graph of G is second Zagreb non-hyperenergetic.*

Proof. The m-shadow graph is second Zagreb non-hyperenergetic if $ZE_2(D_m(G)) < 2(mn-1)^3$. Which gives

$$ZE_2(D_m(G)) = m^3ZE_2(G) < m^3 2(n-1)^3 = 2(mn-m)^3 < 2(mn-1)^3.$$

□

Theorem 4.15. *Let C_n be the cycle graph with n vertices. Then the second Zagreb energy of the m-shadow graph of C_n is*

$$ZE_2(D_m(C_n)) = \begin{cases} \frac{16m^3 \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \text{ mod}(4); \\ \frac{8m^3}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \text{ mod}(4); \\ \frac{16m^3}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \text{ mod}(4). \end{cases}$$

Proof. The proof is straightforward from Theorem 4.9 and Equation (3.9). □

Theorem 4.16. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. Then the first Zagreb energy of the m-shadow graph of $\overline{C_n}$ is*

$$ZE_2(D_m(\overline{C_n})) = \begin{cases} 2(n-3)^2m^3\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \text{ mod}(3); \\ 2(n-3)^2m^3\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \text{ mod}(3); \\ 2(n-3)^2m^3\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \text{ mod}(3). \end{cases}$$

Proof. The proof is straightforward from Theorem 4.9 and Equation (3.10). □

5. ZAGREB ENERGIES OF THE GENERALISED DUPLICATE GRAPH

Theorem 5.1. *Let G be a simple graph with n vertices. Then the first Zagreb energy of the m-duplicate graph of G is*

$$ZE_1(D^m(G)) = 2^m ZE_1(G).$$

Proof. Suppose G be a graph with vertices $v_1, \ldots, v_n$ having degree $d_1, \ldots, d_n$. From Definition 2.5, the first Zagreb matrix of $D^m(G)$ is a block matrix with anti-diagonal blocks $Z^{(1)}(G)$ and all other blocks are 0. In tensor form, we may write it as

$$\begin{aligned} Z^{(1)}(D^m(G)) &= \left\{ \begin{array}{ll} 1, & i+j=n+1; \\ 0, & \text{otherwise.} \end{array} \right\} \otimes Z^{(1)}(G) \\ &= W \otimes Z^{(1)}(G). \end{aligned}$$

The eigenvalues of W are 1 and -1 with multiplicities of 2^{m-1}. By Lemma 2.2,

$$\text{spec}(Z^{(1)}(D^m(G))) = \begin{pmatrix} 0 & \mu_1^{(1)} & \cdots & \mu_n^{(1)} & -\mu_1^{(1)} & \cdots & -\mu_n^{(1)} \\ n(m-1) & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $\mu_i^{(1)}$, $1 \leq i \leq n$ are eigenvalues of $Z^{(1)}(G)$. Therefore,

$$ZE_1(D^m(G)) = \sum_{i=1}^{n} 2^{m-1}\left(\left|\mu_i^{(1)}\right| + \left|-\mu_i^{(1)}\right|\right) = 2^m ZE_1(G).$$

□

Corollary 5.2. *Let G_1 and G_2 be two non-cospectral first Zagreb equienergetic graphs, then $D^m(G_1)$ and $D^m(G_2)$ are non-cospectral first Zagreb equienergetic.*

Proposition 5.3. *The first Zagreb energy of the m-duplicate graph of K_n is*

$$ZE_1(D^m(K_n)) = 2^{m+2}(n-1)^2.$$

Proposition 5.4. *The first Zagreb energy of the m-duplicate graph of $K_{n,n}$ is $ZE_1(D^m(K_{n,n})) = 2^{m+2}n^2$.*

Theorem 5.5. *Let G be a graph of order n. Then the first Zagreb spectral radius $\rho^{Z^{(1)}(D^m(G))}$ of the m-duplicate graph is $\rho^{Z^{(1)}(D^m(G))} = \rho^{Z^{(1)}(G)}$, where $\rho^{Z^{(1)}(G)}$ is the spectral radius of the $Z^{(1)}(G)$.*

Proof. The nonzero eigenvalues of $Z^{(1)}(D^m(G))$ are $\pm\mu_i^{(1)}$, where $\mu_i^{(1)}$, $1 \leq i \leq n$ are eigenvalues of $Z^{(1)}(G)$, implies

$$\rho^{Z^{(1)}(D^m(G))} = \max_{i=1}^{n} | \pm \mu_i^{(1)}| = \rho^{Z^{(1)}(G)}.$$

□

Theorem 5.6. *Let C_n be the cycle graph with n vertices. Then the first Zagreb energy of the m-duplicate graph of C_n is*

$$ZE_1(D^m(C_n)) = \begin{cases} \frac{162^m \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \textit{when } n \equiv 0 \textit{ mod(4);} \\ \frac{82^m}{\sin\frac{\pi}{2n}}, & \textit{when } n \equiv 1 \textit{ mod(4);} \\ \frac{162^m}{\sin\frac{\pi}{n}}, & \textit{when } n \equiv 2 \textit{ mod(4).} \end{cases}$$

Proof. The proof is based on Theorem 5.1 and Equation (3.3). □

Theorem 5.7. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. Then the first Zagreb energy of the m-Duplicate graph of $\overline{C_n}$ is*

$$ZE_1(D^m(\overline{C_n})) = \begin{cases} 4(n-3)2^m\left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \textit{when } n \equiv 0 \textit{ mod(3);} \\ 4(n-3)2^m\left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \textit{when } n \equiv 1 \textit{ mod(3);} \\ 4(n-3)2^m\left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \textit{when } n \equiv 2 \textit{ mod(3).} \end{cases}$$

Proof. The proof is straightforward from Theorem 5.1 and Equation (3.4). □

Theorem 5.8. *Let G be a simple graph with n vertices. Then the second Zagreb energy of the m-duplicate graph of G is*

$$ZE_2(D^m(G)) = 2^m ZE_2(G).$$

Proof. The second Zagreb matrix of the m-duplicate graph is a block matrix, with anti-diagonal blocks $Z^{(2)}(G)$ and all other blocks being 0. We avoid the proof to prevent repetitions. □

Corollary 5.9. *Let G_1 and G_2 be two non-cospectral second Zagreb equienergetic graphs, then $D^m(G_1)$ and $D^m(G_2)$ are non-cospectral second Zagreb equienergetic.*

Proposition 5.10. *The second Zagreb energy of the m-duplicate graph of K_n is $ZE_2(D^m(K_n)) = 2^{m+1}(n-1)^3$.*

Proposition 5.11. *The second Zagreb energy of the m-duplicate graph of $K_{n,n}$ is $ZE_2(D^m(K_{n,n})) = 2^{m+1}n^3$.*

Theorem 5.12. *Let G be a graph of order n. Then the second Zagreb spectral radius $\rho^{Z^{(2)}(D^m(G))}$ of the m-duplicate graph is $\rho^{Z^{(2)}(D^m(G))} = \rho^{Z^{(2)}(G)}$, where $\rho^{Z^{(2)}(G)}$ is the spectral radius of $Z^{(2)}(G)$.*

Proof. The nonzero eigenvalues of $Z^{(2)}(D^m(G))$ are $\pm\mu_i^{(2)}$ where $\mu_i^{(2)}$, $1 \le i \le n$ are eigenvalues of $Z^{(2)}(G)$, implies

$$\rho^{Z^{(2)}(D^m(G))} = \max_{i=1}^{n} | \pm \mu_i^{(2)} | = \rho^{Z^{(2)}(G)}.$$

□

Theorem 5.13. *Let C_n be the cycle graph with n vertices. Then the second Zagreb energy of the m-duplicate graph of C_n is*

$$ZE_2(D^m(C_n)) = \begin{cases} \frac{162^m \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 0 \ mod(4); \\ \frac{82^m}{\sin\frac{\pi}{2n}}, & \text{when } n \equiv 1 \ mod(4); \\ \frac{162^m}{\sin\frac{\pi}{n}}, & \text{when } n \equiv 2 \ mod(4). \end{cases}$$

Proof. The proof is straightforward from Theorem 5.8 and Equation (3.9). □

Theorem 5.14. *Let $\overline{C_n}$ be the complement of the cycle graph C_n with n vertices. Then the first Zagreb energy of the m-duplicate graph of $\overline{C_n}$ is*

$$ZE_2(D^m(\overline{C_n})) = \begin{cases} 2(n-3)^2 2^m \left(\frac{2n-9}{3} + \sqrt{3}\cot\frac{\pi}{n}\right), & \text{when } n \equiv 0 \ mod(3); \\ 2(n-3)^2 2^m \left(\frac{2n-8}{3} + \frac{2\sin\frac{\pi}{3}(1-\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 1 \ mod(3); \\ 2(n-3)^2 2^m \left(\frac{2n-10}{3} + \frac{2\sin\frac{\pi}{3}(1+\frac{1}{n})}{\sin\frac{\pi}{n}}\right), & \text{when } n \equiv 2 \ mod(3). \end{cases}$$

Proof. The proof is straightforward from Theorem 5.8 and Equation (3.10). □

6. ZAGREB ENERGIES OF THE EXTENDED BIPARTITE DOUBLE GRAPHS

Recently, Ramane et al. [16] established the energy of $Ebd(G)$ in terms of other graph parameters. Here we extend the energy of $Ebd(G)$ in case of Zagreb matrices.

Theorem 6.1. *Let G be a simple k-regular graph with n vertices. Then the first Zagreb energy of the $Ebd(G)$ is*

$$ZE_1(Ebd(G)) = 4(k+1)\left(E(G) + n - 2n^- - 2\sum_{\lambda_i\in(-1,0)}(|\lambda_i| - 1)\right),$$

where $\lambda_1,\dots,\lambda_n$ are eigenvalues of the graph G and n^- is the number of negative eigenvalues.

Proof. The $Ebd(G)$ of a k-regular graph is $(k+1)$-regular, which follows the first Zagreb matrix of $Ebd(G)$ as

$$\begin{aligned} Z^{(1)}(Ebd(G)) =&2(k+1)\begin{bmatrix} 0 & A(G)+I(G) \\ A(G)+I(G) & 0 \end{bmatrix}_{2n} \\ =&2(k+1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G)+I(G). \end{aligned}$$

If $\lambda_1,\dots,\lambda_n$ are eigenvalues of G, then by Lemma 2.2, the eigenvalues of $Ebd(G)$ are $\pm 2(k+1)(\lambda_i+1)$. Therefore,

$$\begin{aligned} ZE_1(Ebd(G)) =&\sum_{i=1}^{n}|\pm 2(k+1)(\lambda_i+1)| \\ =&4(k+1)\sum_{i=1}^{n}|\lambda_i+1| \\ =&4(k+1)\left(\sum_{\lambda_i\le -1}(-\lambda_i-1) + \sum_{\lambda_i>-1}(\lambda_i+1)\right) \\ =&4(k+1)\left(\sum_{\lambda_i\le -1}|\lambda_i| - \sum_{\lambda_i\le -1}1 + \sum_{\lambda_i\in(-1,0)}\lambda_i + \sum_{\lambda_i\ge 0}\lambda_i + \sum_{\lambda_i\in(-1,0)}1 + \sum_{\lambda_i\ge 0}1\right) \end{aligned}$$

Now, $\sum_{\lambda_i\le -1} 1 = n^- - \sum_{\lambda_i\in(-1,0)} 1$, $\sum_{\lambda_i\ge 0} 1 = n - n^-$ and also using the fact that $\sum_{\lambda_i\in(-1,0)} \lambda_i = -\sum_{\lambda_i\in(-1,0)} |\lambda_i|$, we have

$$\begin{aligned} & ZE_1(Ebd(G)) \\ & = 4(k+1)\left(\left(E(G) - \sum_{\lambda_i \in (-1,0)} |\lambda_i|\right) - n^- + \sum_{\lambda_i \in (-1,0)} 1 - \sum_{\lambda_i \in (-1,0)} |\lambda_i| + \sum_{\lambda_i \in (-1,0)} 1 + n - n^-\right) \\ & = 4(k+1)\left(E(G) + n - 2n^- - 2\sum_{\lambda_i \in (-1,0)} (|\lambda_i| - 1)\right). \qquad \square \end{aligned}$$

Similarly, we can compute the second Zagreb energy of $Ebd(G)$.

Theorem 6.2. *Let G be a simple k-regular graph with n vertices. Then the second Zagreb energy of the $Ebd(G)$ is*

$$ZE_2(Ebd(G)) = 2(k+1)^2\left(E(G) + n - 2n^- - 2\sum_{\lambda_i \in (-1,0)} (|\lambda_i| - 1)\right),$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the graph G and n^- is the number of negative eigenvalues.

7. CONCLUSION

The study of graph energies and transformations is inherently interdisciplinary because it combines mathematical graph theory with a wide range of application domains. These findings contribute to the understanding of graph energies and their behaviour under graph transformations. They have implications for various applications where graph theory and structural analysis are relevant, such as network analysis and machine learning, etc. Furthermore, these studies often provide theoretical insights into graph structural properties and how these properties relate to graph energies. Here, the first and second Zagreb energies and spectral radii of the m-splitting, m-shadow, m-duplicate, and extended bipartite double graphs are obtained in terms to the energy and spectral radii of the original graph. Additionally, we observe some graphs that are Zagreb equienergetic. Also, we report an error present in [20] that contradicts the claim of hyperenergetic behaviour for the splitting graph of regular graphs. Further, we establish the non-hyperenergetic behaviour of the m-shadow graph.

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Idweep Jyoti Gogoi
Department of Mathematics, Dibrugarh University, Assam, India.
Email: `igogoi1995@gmail.com`

Sumanta Borah
Department of Mathematics, Dibrugarh University, Assam, India.
Email: `sumantaborah9@gmail.com`

Ankur Bharali
Department of Mathematics, Dibrugarh University, Assam, India.
Email: `a.bharali@dibru.ac.in`

Journal of Algebraic Systems

ON ZAGREB ENERGIES OF SOME GRAPH OPERATIONS

I. J. GOGOI, S. BORAH AND A. BHARALI

دربارهی انرژیهای زاگربِ برخی عملیات گرافی

ایدویپ جیوتی گوگوی[۱]، سومانتا بورا[۲] و انکور بهارالی[۳]

[۱,۲,۳]گروه ریاضی، دانشگاه دیبروگره، آسام، هند

اخیراً انرژی زاگرب، یک ثابت گرافی مبتنی بر مقادیر ویژهی ماتریسهای زاگرب، بهعنوان مفهومی مشابه انرژی گراف معرفی شده است. در این مقاله، انرژیهای زاگرب و شعاع طیفی زاگرب در ارتباط با چندین عملیات گرافی، از جمله گرافهای m-شکافتی، گرافهای m-سایه، گرافهای m-تکراری و گرافهای دوگانه بسطیافته بررسی میشوند. همچنین، این گرافهای تعمیمیافته را در چارچوب برخی کلاسهای خاص گراف، مانند گرافهای کامل، گرافهای دوبخشی کامل، گرافهای دوری و متمم گرافهای دوری مطالعه میکنیم. علاوه بر این، خطایی را در مرجع [۲۰] گزارش میدهیم که ادعای هایپرانرژیک بودن گراف شکافتیِ گرافهای منظم را نقض میکند. همچنین نشان میدهیم که گراف m-سایه دارای رفتار هایپرانرژیک نیست.

کلمات کلیدی: انرژی گراف، انرژی زاگرب، عملیات گراف.