Journal of Algebraic Systems, Vol. 14(No. 3): (2026), pp 407-425.
https://doi.org/10.22044/JAS.2024.14545.1841

NON-PARALLEL KRULL DIMENSION OF MODULES

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Abstract. In this paper, the concept of non-parallel Krull (resp., Noetherian) dimension of an R-module is introduced, and some related properties are investigated. Using these concepts, we generalize some of the results of our previous work and try to obtain some appropriate new results about np-Artinian (resp., np-Noetherian) modules. We give a characterization for modules with non-parallel Krull (resp., Noetherian) dimension and show that these modules have finite type dimension. It is shown that any R-module M with non-parallel Krull dimension at most α is either atomic or λ-f.e. for some $\lambda \leq \alpha$. Also, it is proved that any np-Noetherian R-module has non-parallel Krull dimension. In particular, for semiprime right non-atomic rings, we show that Krull dimension and non-parallel Krull dimension coincide.

1. Introduction

The notion of Krull dimension of a module M, denoted by k-dim M and quantifying its deviation from being Artinian, was first proposed by Gabriel and Rentschler (for finite ordinals) in 1967. Subsequently, in 1970 this definition was extended to infinite ordinals by Krause [4, 10]. The dual of this dimension, which measures the deviation of an R-module from Noetherianness was first introduced and studied by Lemonnier [11] and called by him codeviation. Karamzadeh also extensively studied this dimension in his Ph.D. thesis and called it Noetherian dimension (denoted by n-dim (M)), see [6].
In 2004, Smith and Vedadi [15], investigated modules with chain conditions on non-essential submodules. We say that a module M is ne-Noetherian (resp., ne-Artinian), if M satisfies ACC (resp., DCC) on non-essential submodules, i.e., for every ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of non-essential submodules of M, there exists an index $k \geq 1$ such that $N_k = N_i$ for every $i \geq k$. Later, Davoudian [2], generalized the concept of ne-Artinian (resp., ne-Noetherian) modules and introduced the notion of non-essential Krull (resp., Noetherian) dimension of an R-module M, denoted by *nek*-dim M (resp., *nen*-dim M). The notion of

MSC(2020): 16P60, 16P20, 16P40.
Keywords: Non-parallel submodules; Atomic modules; Non-parallel Krull dimension; Non-parallel Noetherian dimension; Type dimension.
Received: 18 May 2024, Accepted: 14 October 2024.
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parallel submodule (resp., type dimension) of an R-module is a special generalization of essential submodule (resp., Goldie dimension), see Remark 2.2 (resp., Definition 2.7). For more details about type dimension we refer to [1, Chapter 4]. The authors of [14] (resp., [12]), introduced the concept of parallel Krull dimension (resp., type Krull dimension) of an R-module M, denoted by pk-dim M (resp., tk-dim M) which is Krull-like dimension extension of the concept of DCC on poset of parallel submodules (resp., type submodules). In [13], we studied chain conditions on non-parallel submodules, and we said that a module M is np-Noetherian (resp., np-Artinian), if M satisfies ACC (resp., DCC) on non-parallel submodules.

In this paper, motivated by the above works, we introduce and study the concept of non-parallel Krull (resp., Noetherian) dimension of an R-module M which is denoted by npk-dim M (resp., npn-dim M). We generalize some properties of results of our previous work ([13]) and sometimes obtain new results about np-Artinian (resp., np-Noetherian) modules. In Section 2, we first recall some basic definitions and terminologies of type theory and Krull dimension which we need in the sequel. In Section 3, we investigate some properties of non-parallel Krull dimension. We observe that any R-module with non-essential Krull dimension has non-parallel Krull dimension but by an example we show that the converse is not true in general case. It is proved that any R-module with non-parallel Krull dimension has finite type dimension. We give a characterization theorem for modules with non-parallel Krull dimension. Also, as another main result of this section, we show that any R-module M with npk-dim $M \leq \alpha$, is either atomic or λ-f.e. for some $\lambda \leq \alpha$ and using this, we show that every non-atomic module with non-parallel Krull dimension (in particular, any np-Artinian module) has finite Goldie dimension. Moreover, we show that np-Noetherian modules have non-parallel Krull dimension and as an application of this result, we are able to prove that any non-atomic np-Noetherian module is λ-f.e. for some ordinal number λ. Next, we show that for a semiprime right non-atomic ring R, the existence of Krull dimension and non-parallel Krull dimension is equivalent and these two dimensions coincide. In particular, we prove that for a semiprime right duo ring R, R has non-parallel Krull dimension and npk-dim $R = \alpha$ if and only if it has non-essential Krull dimension and nek-dim $R = \alpha$. In Section 4, we introduce and study the dual of non-parallel Krull dimension and call it non-parallel Noetherian dimension. We give a structure theorem for modules with this dimension. We see that almost the dual of all results in the previous section are also true. However, we present some new results. For

example, we show that if M is an R-module with non-parallel Noetherian dimension such that $T(M) \nsubseteq \text{rad}(M)$, where $T(M)$ denotes the intersection of all non-parallel submodules, then M has Noetherian dimension and $n\text{-dim}\, M = nen\text{-dim}\, M = npn\text{-dim}\, M$. Also, it is shown that an R-module M has non-parallel Krull dimension if and only if it has non-parallel Noetherian dimension. In particular, we show that for any semiprime right non-atomic ring, the existence of Krull (resp., Noetherian) dimension, non-essential Krull (resp., Noetherian) dimension and non-parallel Krull (resp., Noetherian) dimension are equivalent. Moreover, for semiprime right non-atomic rings, we show that Krull dimension and non-parallel Krull dimension coincide.
Throughout this article, all rings are associative with $1 \neq 0$ and all modules are unital right modules. For an R-module M, we write $\text{rad}(M)$, $Z(M)$ for the intersection of all maximal submodules and singular submodule of M, respectively. The notations $N \subseteq M$, $N \subseteq_e M$, $N \parallel M$ and $N \subseteq_t M$ will denote that N is a submodule, an essential submodule, a parallel submodule and a type submodule of a module M, respectively. For any subset S of a ring R we set $\mathbf{r}(S) = \{r \in R \mid Sr = 0\}$, i.e., $\mathbf{r}(S)$ is the right annihilator of S in R. In particular, if $S = \{s\}$, then we write $\mathbf{r}(s)$ instead of $\mathbf{r}(\{s\})$. Also, $\mathbf{l}(S)$ (resp., $\mathbf{l}(s)$) denotes the left annihilator of S (resp., $\{s\}$). It is convenient that, when we are dealing with Krull-like (resp., Noetherian-like) dimension, we may begin our list of ordinals with -1. Also, when we talk about the dimension of a ring, we mean the dimension of that ring as a right module over itself.

2. Preliminaries

This section contains some preliminary results that are needed in the sequel. First, we recall the following definition which is a particular case of [1, Definition 4.1.1].

Definition 2.1. Let M be an R-module and N, K are submodules of M. Then N and K are orthogonal, written as $N \perp K$, if they do not have nonzero isomorphic submodules. Now, N is called a parallel submodule of M, denoted by $N \parallel M$, if for any non-zero submodule L of M we have $L \not\perp N$. If N is not a parallel submodule of M, then we say that N is a non-parallel submodule of M, that is, there exists a non-zero submodule L of M such that for any submodule L' of L and any submodule N' of N, we have $L' \ncong N'$. We denote by $\text{NP}(M)$ the set of all non-parallel submodules of M.

A non-zero module M is called atomic, if every submodule of M is a parallel submodule. We should remind the reader that these atomic modules are different from those defined in [8].

Remark 2.2. [13, Remark 2.3] Let M be a module. It is clear that N is an essential submodule of M if and only if for any non-zero submodule K of M, N and K have a non-zero equal submodule. It is easy to see that N is a parallel submodule of M if and only if for any non-zero submodule K of M, N and K have a non-zero isomorphic submodule. Hence, if K is an essential submodules of M, then K is a parallel submodule of M. In particular, any uniform module is atomic. However, the converse of these facts is not true in general, for instance, see [13, Example 3.3(3)].

Definition 2.3. [1, Definition 4.1.2] A submodule P of a module M is called a type submodule, denoted as $P \subseteq_t M$, if the following equivalent conditions hold:

(1) If $P \subseteq Y \subseteq M$ with $P \parallel Y$, then $P = Y$.
(2) If $P \subseteq Y \subseteq M$, then $P \perp X$ for some $0 \neq X \subseteq Y$.
(3) P is a complement submodule of M such $P \oplus D \subseteq_e M$ and $P \perp D$ for some $D \subseteq M$.

The following lemma is easy to prove.

Lemma 2.4. [1, Lemma 4.1.5] Let M be a module and N, K, L are submodules of M.

(1) If $N \nparallel K$, then $N \nsubseteq_e K$.
(2) If $N \subseteq_t K$, then $N \nparallel K$.
(3) Let $N \subseteq K \subseteq L$. Then $N \nparallel L$ if and only if $N \nparallel K$ or $K \nparallel L$.
(4) If $N \subseteq_t M$ and $N \subseteq_e K \subseteq M$, then $N = K$.

Lemma 2.5. [13, Lemma 2.6] Let M be an R-module and let N be any submodule of M. Then there exists a type submodule P of M such that it is maximal with respect to $N \subseteq P$ and $N \parallel P$.

Lemma 2.6. [1, Proposition 4.1.6] Let M be a module and N a submodule of M. Let $N \subseteq_t M$ and $N \subseteq K \subseteq M$. Then $\frac{K}{N} \subseteq_t \frac{M}{N}$ if and only if $K \subseteq_t M$.

Definition 2.7. An R-module M has finite type dimension n, denoted by $t.\dim M = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. If no such n exists, we say that the type dimension of M is infinite, and write $t.\dim M = \infty$. If $M = 0$, then $t.\dim M = 0$.

Remark 2.8. For a module M, $t.\dim M = \infty$ if and only if there exists an infinite number of pairwise orthogonal non-zero submodules of M.

Proposition 2.9. [1, Proposition 4.1.12(2)] The following statements are equivalent for a module M.

(1) $t.\dim M < \infty$.
(2) M has ACC on type submodules.
(3) M has DCC on type submodules.

Remark 2.10. Note that an R-module M has finite Goldie dimension if and only if it has ACC (resp., DCC) on its complement submodules, see [9, Proposition (6.30)′]. Now, by the previous proposition and Definition 2.3, it is clear that if M has finite Goldie dimension, then it has finite type dimension, however, the converse is not true in general case, see Example 4.3(1).

Next, we give some results about Krull dimension of modules which are needed. For more information about Krull dimension and Noetherian dimension we refer the reader to [4, 6, 7].

Proposition 2.11. [4, Proposition 1.4] A module with Krull dimension has finite Goldie dimension.

Proposition 2.12. [4, Proposition 1.3] Any Noetherian R-module has Krull dimension.

Proposition 2.13. [4, Lemma 1.1(i)] Let M be an R-module and N a submodule of M. Then $k\text{-}\dim M = \sup\{k\text{-}\dim N,\ k\text{-}\dim\frac{M}{N}\}$, if either side exists.

Remark 2.14. The dual of all previous results are also true for modules with Noetherian dimension, see [8, Section 1].

Finally, let us recall the following well-known result of Lemonnier, see also [8, Proposition 1.1].

Proposition 2.15. [10, Corollary 6] Let M be an R-module. Then M has Krull dimension if and only if it has Noetherian dimension.

3. Non-parallel Krull dimension and its properties

In this section, we introduce and study the concept of non-parallel Krull dimension of a module, which is a Krull-like dimension extension of the concept of np-Artinian module. Using this dimension, we generalize some of the properties of np-Artinian modules.

Definition 3.1. Let M be an R-module. The non-parallel Krull dimension of M, denoted by npk-dim M is defined by transfinite recursion as follows: If $M = 0$, npk-dim $M = -1$. If α is an ordinal number and npk-dim $M \nless \alpha$, then npk-dim $M = \alpha$ provided there is no infinite descending chain of non-parallel submodules of M such as $N_1 \supseteq N_2 \supseteq \cdots$ such that for each $i = 1, 2, \cdots$, npk-dim $\frac{N_i}{N_{i+1}} \nless \alpha$. Otherwise, npk-dim $M = \alpha$, if

$$npk\text{-dim}\, M \nless \alpha$$

and for each chain of non-parallel submodules of M such as $N_1 \supseteq N_2 \supseteq \cdots$ there exists a positive integer t, such that for each $i \geq t$,

$$npk\text{-dim}\, \frac{N_i}{N_{i+1}} < \alpha.$$

A ring R has non-parallel Krull dimension, if as an R-module has non-parallel Krull dimension. It is possible that there is no ordinal α such that npk-dim $M = \alpha$, in this case we say M has no non-parallel Krull dimension.

Clearly, npk-dim $M = 0$ if and only if M satisfies DCC on its non-parallel submodules. So, npk-dim $M = 0$ if and only if M is np-Artinian.

First, let us investigate the relation between non-essential Krull dimension, type Krull dimension and non-parallel Krull dimension of modules.

Proposition 3.2. Let M be an R-module.

(1) If M has Krull dimension, then it has non-essential Krull dimension and nek-dim $M \leq k$-dim M.
(2) If M has non-essential Krull dimension, then it has non-parallel Krull dimension and npk-dim $M \leq nek$-dim M.
(3) If M has non-parallel Krull dimension, then it has type Krull dimension and tk-dim $M \leq npk$-dim M.
(4) If M has Krull dimension, then it has non-parallel Krull dimension and npk-dim $M \leq k$-dim M.

Proof. (1) See [2, Lemma 2.1].

(2) The proof is by transfinite induction on nek-dim $M = \alpha$. If $\alpha = 0$, then M has DCC on non-essential submodules, so npk-dim $M = 0$. Let $\alpha > 0$ and for every ordinal $\gamma < \alpha$, M has non-parallel Krull dimension $\leq \alpha$. Let $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of non-parallel submodules of M. By Lemma 2.4(1), this is a chain of non-essential submodules of M. Since nek-dim $M = \alpha$, there exists $n \in \mathbb{N}$ such that for every $i \geq n$, nek-dim $\frac{N_i}{N_{i+1}} = \beta < \alpha$, so by the induction hypothesis, $\frac{N_i}{N_{i+1}}$ has non-parallel

Krull dimension and $npk\text{-dim}\,\frac{N_i}{N_{i+1}} \leq nek\text{-dim}\,\frac{N_i}{N_{i+1}} = \beta < \alpha$ for every $i \geq n$. Thus, M has non-parallel Krull dimension and $npk\text{-dim}\,N \leq \alpha$.

(3) Similar to (2).

(4) It follows from parts (1) and (2). □

In Example 4.3, we show that the converse of part (2) of the previous proposition is not true in general. Moreover, with another example, we show that in part (3) equality does not necessarily hold.
By applying Lemma 2.4(3), the proof of the following fact is straightforward.

Lemma 3.3. Let M be an R-module with non-parallel Krull dimension. Then, for any submodule N of M, N has non-parallel Krull dimension and $npk\text{-dim}\,N \leq npk\text{-dim}\,M$.

Recall form [12] that an R-module M has homogeneous type Krull dimension, if every submodule of M has type Krull dimension.

Corollary 3.4. Let M be an R-module with non-parallel Krull dimension. Then, it has homogeneous type Krull dimension.

Proof. It follows from Proposition 3.2(3) and Lemma 3.3. □

Proposition 3.5. Let M be an R-module and $\frac{M}{N}$ has non-parallel Krull dimension for all $0 \neq N \in \mathrm{NP}(M)$. Then M has non-parallel Krull dimension and

$$npk\text{-dim}\,M \leq \sup\left\{npk\text{-dim}\,\frac{M}{N} : 0 \neq N \in \mathrm{NP}(M)\right\} + 1.$$

Proof. Put

$$\alpha = \sup\left\{npk\text{-dim}\,\tfrac{M}{N} : 0 \neq N \in \mathrm{NP}(M)\right\}.$$

Suppose that $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is an arbitrary descending chain of non-parallel submodules of M. Then for each i, we have $\frac{N_i}{N_{i+1}} \subsetneq \frac{M}{N_{i+1}}$. Now, by Lemma 3.3, we have $npk\text{-dim}\,\frac{N_i}{N_{i+1}} \leq \alpha$. It follows that $npk\text{-dim}\,M \leq \alpha+1$. □

Following [13], we say that a module M is orthogonal decomposable, if $M = N_1 \oplus N_2$ for some submodules N_1, N_2 of M with $N_1 \perp N_2$. Next, we give a structure theorem for modules with non-parallel Krull dimension. This result can be considered as a generalization of [13, Theorem 3.4] in the case $\alpha = 0$.

Theorem 3.6. Let M be an R-module. The following statements are equivalent.

(1) M has non-parallel Krull dimension and npk-dim $M \leq \alpha$.
(2) Every non-parallel submodule N of M has Krull dimension and k-dim $N \leq \alpha$.
(3) Every proper type submodule N of M has Krull dimension and k-dim $N \leq \alpha$.
(4) Every orthogonal decomposable submodule N of M has Krull dimension and k-dim $N \leq \alpha$.

Proof. ($1 \Rightarrow 2$) Let N be a non-parallel submodule of M and $N_1 \supseteq N_2 \supseteq \cdots$ be any descending chain of non-parallel submodules of N. By Lemma 2.4(3), for each $i \geq 0$ we have $N_i \nparallel M$. Hence, by (1), $k\text{-dim}\,\frac{N_i}{N_{i+1}} = npk\text{-dim}\,\frac{N_i}{N_{i+1}} < \alpha$ for each $i \geq 0$. Therefore, k-dim $N \leq \alpha$.

($2 \Rightarrow 3$) Follows from Lemma 2.4(2).

($3 \Rightarrow 4$) Let N be an orthogonal decomposable submodule of M. Then $N = N_1 \oplus N_2$ for some non-zero submodules N_1, N_2 with $N_1 \perp N_2$. By Lemma 2.5, for $i = 1, 2$, there exists a type submodule K_i of M such that $N_i \subseteq K_i$ and $N_i \parallel K_i$. We claim that $K_1 \perp K_2$. If not, there exist nonzero submodules K_1' and K_2' of K_1 and K_2, respectively, with $K_1' \cong K_2'$. Since $N_1 \parallel K_1$, there exists a nonzero submodule N_1' of N_1 such that $K_1' \cong N_1'$. Similarly, there exists a nonzero submodule N_2' of N_2 such that $K_2' \cong N_2'$. Therefore, we get $N_1' \cong N_2'$ which is a contradiction. Now, it is clear that K_1, K_2 are proper type submodules of M. Hence, by (2), k-dim $N_i \leq k$-dim $K_i \leq \alpha$, for $i = 1, 2$. Therefore, by Proposition 2.13, $k\text{-dim}\,N = \sup\{k\text{-dim}\,N_i \;:\; i = 1, 2\} \leq \alpha$.

($4 \Rightarrow 1$) Let $N_1 \supseteq N_2 \supseteq \cdots$ be any descending chain of non-zero non-parallel submodules of M. Since $N_1 \nparallel M$, there exists a non-zero submodule K of M such that $K \perp N_1$. Then $K \oplus N_1$ is an orthogonal decomposable submodule of M and so, by (4), it has Krull dimension and

$$k\text{-dim}\,K \oplus N_1 = \sup\{k\text{-dim}\,K,\; k\text{-dim}\,N_1\} \leq \alpha,$$

see Proposition 2.13. Hence, there exists a positive integer k such that $k\text{-dim}\,\frac{N_i}{N_{i+1}} < \alpha$, for each $i \geq k$. Now, Proposition 3.2(4) implies that $npk\text{-dim}\,\frac{N_i}{N_{i+1}} \leq k\text{-dim}\,\frac{N_i}{N_{i+1}} < \alpha$. □

Proposition 3.7. Let M be an R-module with non-parallel Krull dimension. Then M has finite type dimension.

Proof. Suppose that M has non-parallel Krull dimension. By Corollary 3.4, M has homogeneous type Krull dimension. Now, [12, Theorem 3.14] implies that tk-dim $M = 0$, so M satisfies DCC on type submodules. Therefore, M has finite type dimension, see Proposition 2.9. □

Theorem 3.8. Let M be an R-module with non-parallel Krull dimension α. Then, for each type submodule N of M, $\frac{M}{N}$ has non-parallel Krull dimension and $npk\text{-dim}\,\frac{M}{N} \leq \alpha$.

Proof. By Theorem 3.6, it suffices to show that every type submodule of $\frac{M}{N}$ has non-parallel Krull dimension $\leq \alpha$. Note that type submodules of $\frac{M}{N}$ are exactly of the form $\frac{K}{N}$, where K is a type submodule of M which contains N, see Lemma 2.6. Suppose that $\frac{K}{N}$ is an arbitrary type submodule of $\frac{M}{N}$. By Theorem 3.6, K has Krull dimension less than or equal to α. It follows that $\frac{K}{N}$ has Krull dimension $\leq \alpha$ and we are done. □

In the following example, we show that the condition $N \subseteq_t M$ in the previous theorem can not be omitted.

Example 3.9. It is clear that $\mathbb{Q}$ is an atomic $\mathbb{Z}$-module, so $npk\text{-dim}\,\mathbb{Q} = 0$. By [13, Example 3.15], $\frac{\mathbb{Q}}{\mathbb{Z}}$ has infinite type dimension, so Proposition 3.7 implies that $\frac{\mathbb{Q}}{\mathbb{Z}}$ does not have non-parallel Krull dimension.

Recall that an R-module M is called α-critical, if $k\text{-dim}\,M = \alpha$ and $k\text{-dim}\,\frac{M}{N} < \alpha$, for each nonzero submodule N of M. Moreover, M is called critical, if it is α-critical for some ordinal number α. We need to the following well-known observation, see [4, Corollary 2.2].

Lemma 3.10. Let M be an R-module with Krull dimension. Then M has an essential submodule which is a finite direct sum of critical submodules.

To state the next result, we first need to recall some concepts from [7]. Let M be an R-module. For each ordinal α, we define $S_\alpha = \bigoplus_{i\in I} C_i$, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of M. S_α is called an α-critical socle of M. Now a critical socle of M is defined to be a submodule S of M with $S = \sum_{\alpha<\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \leq \lambda$. If for some ordinal α, there is no α-critical submodule, then we put $S_\alpha = (0)$. Now, M is called λ-finitely embedded (briefly, λ-f.e.), if λ is the least ordinal such that each critical submodule of M is α-critical for some $\alpha \leq \lambda$ and M contains a finitely generated essential critical socle (equivalently, M contains an essential critical socle of Krull dimension λ). Note that 0-f.e. modules are precisely f.e. modules. For more study about critical (resp., λ-f.e.) modules, we refer the reader to [4, Chapter 2] (resp., [7]). The following interesting result is an extension of [13, Theorem 3.7].

Theorem 3.11. Let M be an R-module with non-parallel Krull dimension and *npk*-dim $M \leq \alpha$. Then M is either atomic or λ-f.e. for some $\lambda \leq \alpha$.

Proof. Suppose that M is not atomic. By Proposition 3.7, M has finite type dimension. Hence, there exists a positive integer $n \geq 2$ such that M contains an essential submodule of the form $N = N_1 \oplus \cdots \oplus N_n$, where N_i's are pairwise orthogonal atomic submodules of M. It is clear that N_i is a non-parallel submodule of M, so by Theorem 3.6, N_i has Krull dimension and *k*-dim $N_i \leq \alpha$ for each $1 \leq i \leq n$. Hence, N has Krull dimension and *k*-dim $N \leq \alpha$. Now, using Lemma 3.10, it is easy to see that N is λ-f.e. for some $\lambda \leq \alpha$. But since N is essential in M, it follows that M is λ-finitely embedded for some $\lambda \leq \alpha$ and we are done. □

Corollary 3.12. Every non-atomic module with non-parallel Krull dimension has finite Goldie dimension.

Proof. Let M be a non-atomic module with *npk*-dim $M = \alpha$. Then, by the previous theorem, M is λ-f.e. and hence it has an essential critical socle of Krull dimension λ, say S, for some $\lambda \leq \alpha$. Note that any module with Krull dimension has finite Goldie dimension, see 2.11. It follows that S, so M has finite Goldie dimension. □

Corollary 3.13. Every non-atomic np-Artinian module has finite Goldie dimension.

Remark 3.14. In Proposition 3.7, we have shown that all modules with non-parallel Krull dimension have finite type dimension. But Corollary 3.12 states that if these modules have at least one (non-zero) non-parallel submodule, then they have finite Goldie dimension. Also, Example 4.3(1) shows that the condition "being non-atomic" in Corollary 3.12 cannot be omitted.

Recall that any module with Krull dimension has a critical submodule, see [4, Theorem 2.1]. In the following result, we generalize this fact for non-atomic modules with non-parallel Krull dimension.

Proposition 3.15. Let M be a module with *npk*-dim $M \leq \alpha$.

(1) Any non-atomic (non-zero) submodule of M contains an β-critical submodule for some $\beta \leq \alpha$.
(2) Any non-zero submodule of M contains an atomic submodule.
(3) The intersection of all maximal type submodules of M is equal to 0.

Proof. (1) Let N be a non-atomic submodule of M. Then N has non-parallel Krull dimension and *npk*-dim $(N) \leq \alpha$, by Lemma 3.3. Hence, the previous theorem implies that M has an essential critical socle. It follows that N contains an α-critical submodule for some $\beta \leq \alpha$.

(2) If M is atomic, then it is easy to see that every non-zero submodule of M is atomic. Now, suppose that M is not atomic. Then M contains a critical submodule, by (1). Note that critical modules are uniform (see [4, Proposition 2.6]). Also, by Remark 2.2, uniform modules are atomic, so we are done.

(3) Follows by (2) and [16, Lemma 3.1]. □

The following result is an analogue of Proposition 2.12.

Theorem 3.16. Every np-Noetherian module has non-parallel Krull dimension.

Proof. Suppose that M is an np-Noetherian module. In view of Theorem 3.6, it suffices to show that every non-parallel submodule of M has Krull dimension $\leq \alpha$ for some ordinal number α. Let N be any non-parallel submodule of M. Then, by our assumption and [13, Theorem 3.12], N is a Noetherian submodule of M. Hence, N has Krull dimension, by Proposition 2.12. Set $\alpha = \sup\{k\text{-dim}\, N \mid N \in \mathrm{NP}(M)\}$. Now, M has non-parallel Krull dimension at most α. □

In [13, Theorem 3.7], we proved that any np-Artinian module is either atomic or finitely embedded. Now, using the previous theorem and Theorem 3.11, we are able to prove the following result.

Corollary 3.17. Let M be a non-atomic R-module. If M is np-Noetherian, then it is λ-f.e. for some ordinal number λ.

Proposition 3.18. Let R be a Noetherian ring which is not atomic as a right R-module. If one of the following conditions holds, then $k\text{-dim}\, R \leq \alpha$ if and only if $npk\text{-dim}\, R \leq \alpha$.

(1) $\mathbf{r}(S) = \mathbf{l}(S)$ for some critical socle S of R.

(2) Every minimal prime ideal is non-essential as a right ideal.

Proof. (1) Suppose that $npk\text{-dim}\, R \leq \alpha$. Then R is λ-f.e. for some $\lambda \leq \alpha$, by Theorem 3.11. Now, [7, Theorem 2.14] implies that $k\text{-dim}\, R \leq \alpha$. The converse follows from Proposition 3.2.

(2) Similar to (1), see [7, Proposition 2.15]. □

Recall that a ring R is right bounded, if every essential right ideal of R contains an ideal which is essential as a right ideal. Next, a ring R is right fully bounded provided every prime factor ring of R is right bounded. Also, a right FBN ring is a right fully bounded right Noetherian ring. Now, in view of [7, Corollary 2.16] and Theorem 3.11, we have the following result.

Proposition 3.19. Let R be a left Noetherian right FBN ring such that either $\mathbf{r}(S) = \mathbf{l}(S)$ for some critical socle S, or no minimal prime ideal is essential as a right ideal and let M be a finitely generated faithful R-module which is not atomic. Then $k\text{-dim}\, M \leq \alpha$ if and only if $npk\text{-dim}\, M \leq \alpha$.

Next, we show that for a semiprime right non-atomic ring R, the existence of Krull dimension and non-parallel Krull dimension are equivalent and these two dimensions coincide.

Theorem 3.20. Let R be a semiprime ring. Then R has non-parallel Krull dimension α if and only if either R is right atomic or it has Krull dimension α.

Proof. First, suppose that $npk\text{-dim}\, R = \alpha$ and R is not right atomic. Then R is λ-f.e. for some $\lambda \leq \alpha$, by Theorem 3.11. Now, [7, Theorem 2.18] implies that R has Krull dimension and $k\text{-dim}\, R = \lambda$ for some $\lambda \leq \alpha$. On the other hand, in view of Proposition 3.2(4), we have $\alpha = npk\text{-dim}\, R \leq k\text{-dim}\, R = \lambda$. It follows that $k\text{-dim}\, R = \alpha$. Conversely, if R is right atomic, then we are done. Thus, we suppose that R is not right atomic, and it has Krull dimension α. Then R has non-parallel Krull dimension and $npk\text{-dim}\, R = \beta \leq \alpha$, by Proposition 3.2(4). Now, using the above argument, we can conclude that $\alpha = k\text{-dim}\, R \leq npk\text{-dim}\, R = \beta$. Therefore, $npk\text{-dim}\, R = \alpha$. □

As an application of the previous theorem (in particular case $\alpha = 0$), we have the following result.

Corollary 3.21. A semiprime ring R is right np-Artinian if and only if either R is right atomic or right Artinian.

Using the same proof as we used in Theorem 3.20, we can prove the following result.

Theorem 3.22. Let R be a semiprime ring. Then R has non-essential Krull dimension α if and only if either R is right uniform or R has Krull dimension α.

Lemma 3.23. [13, Lemma 5.12] Let R be any semiprime ring and I, J two ideals of R. Then the following are equivalent.

(1) $I \perp J$.
(2) $I \cap J = (0)$.

(3) $IJ = (0)$.

Recall that a ring R is called right duo, if any right ideal of R is two sided.

Proposition 3.24. Let R be a semiprime right duo ring. Then the following are equivalent.

(1) R has non-parallel Krull dimension and $npk\text{-dim}\, R = \alpha$.
(2) R has non-essential Krull dimension and $nek\text{-dim}\, R = \alpha$.

Proof. Since R is right duo, in view of the previous lemma, it is not hard to see that if I is any right ideal of R which is non-essential as a right ideal, then I is non-parallel as a right ideal. By this fact, the desired result is obtained. □

Theorem 3.25. Let R be a commutative ring and M be an R-module such that $npk\text{-dim}\, M = \alpha$. Then M is atomic or $MI \subseteq Z(M)$ for some semiprime ideal I of R.

Proof. Suppose that M is not atomic. In view of Theorem 3.11, we infer that M is λ-f.e. for some $\lambda \leq \alpha$. Hence, there exists an independent family $\{N_i\}_{i=1}^n$ of critical submodules of M such that $\bigoplus_{i=1}^n N_i$ is an essential submodule of M. Note that the annihilator of any critical R-module is a prime ideal of R, see [3, Lemma 2.12]. It follows that for each $1 \leq i \leq n$, $N_iP_i = 0$ for some prime ideal P_i of R. Let $m \in M$. There exists an essential ideal E of R such that $mE \subseteq \bigoplus_{i=1}^n N_i$. Therefore,

$$m(\textstyle\bigcap_{j=1}^n P_j)E = mE(\bigcap_{j=1}^n P_j) \subseteq (\bigoplus_{i=1}^n N_i)(\bigcap_{j=1}^n P_j) = 0.$$

Put $I = \bigcap_{j=1}^n P_j$. In this case, $MI \subseteq Z(M)$. Now, it is clear that I is a semiprime ideal of R and $MI \subseteq Z(M)$. □

Proposition 3.26. Let M be an R-module. Then M has Krull dimension if and only if it has parallel Krull dimension and non-parallel Krull dimension.

Proof. Suppose that $pk\text{-dim}\, M = \alpha$ and $npk\text{-dim}\, M = \beta$. Let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules of M. If each N_i, $i = 1, 2, \ldots$, is a parallel submodule of M, then clearly $k\text{-dim}\, M = \alpha$. Hence, assume that there exists $k \geq 1$ such that $N_k \nparallel M$. It follows that for each $t \geq k$ we have $N_t \nparallel M$, by Lemma 2.4(3). Now, $N_t \supseteq N_{t+1} \supseteq \cdots$ is a chain of non-parallel submodules of M and hence $k\text{-dim}\, M = \beta$. Therefore, M has Krull dimension and $k\text{-dim}\, M = \sup\{\alpha, \beta\}$. □

4. Non-parallel Noetherian dimension

In this section, we introduce and study the dual of non-parallel Krull dimension.

Definition 4.1. Let M be an R-module. The non-parallel Noetherian dimension of M, denoted by $npn\text{-}\dim M$ is defined by transfinite recursion as follows: If $M = 0$, $npn\text{-}\dim M = -1$. If α is an ordinal number and $npn\text{-}\dim M \not< \alpha$, then $npn\text{-}\dim M = \alpha$ provided there is no infinite ascending chain of non-parallel submodules of M such as $N_1 \subseteq N_2 \subseteq \cdots$ such that for each $i = 1, 2, \cdots$, $npn\text{-}\dim \frac{N_{i+1}}{N_i} \not< \alpha$. Otherwise, $npn\text{-}\dim M = \alpha$, if $npn\text{-}\dim M \not< \alpha$ and for each chain of non-parallel submodules of M such as $N_1 \subseteq N_2 \subseteq \cdots$ there exists an integer t, such that for each $i \geq t$, $npn\text{-}\dim \frac{N_{i+1}}{N_i} < \alpha$. A ring R has non-parallel Noetherian dimension, if as an R-module it has non-parallel Noetherian dimension. It is possible that there is no ordinal α such that $npn\text{-}\dim M = \alpha$, in this case we say M has no non-parallel Noetherian dimension.

Clearly, $npn\text{-}\dim M = 0$ if and only if M is np-Noetherian.

The following result is an analogue of Proposition 3.2 and its proof is omitted.

Proposition 4.2. Let M be an R-module.

(1) If M has Noetherian dimension, then it has non-essential Noetherian dimension and $nen\text{-}\dim M \leq n\text{-}\dim M$.
(2) If M has non-essential Noetherian dimension, then it has non-parallel Noetherian dimension and $npn\text{-}\dim M \leq nen\text{-}\dim M$.
(3) If M has non-parallel Noetherian dimension, then it has type Noetherian dimension and $tn\text{-}\dim M \leq npn\text{-}\dim M$.
(4) If M has Noetherian dimension, then it has non-parallel Noetherian dimension and $npk\text{-}\dim M \leq n\text{-}\dim M$.

The following example shows that the converse of part (2) of the previous proposition is not true in general and also, the relation "$\leq$" in part (3) can be strict.

Example 4.3. (1) Consider $M = \bigoplus_{i>0} \mathbb{Z}_{p^i}$ as a $\mathbb{Z}$-module. Then it is clear that M is atomic, so $npk\text{-}\dim M = npn\text{-}\dim M = 0$. But we note that M has infinite Goldie dimension. Hence, [2, Proposition 2.2] (resp., [2, Corollary 3.1]) implies that M has not non-essential Krull (resp., Noetherian) dimension.

(2) Suppose that $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. By [13, Example 3.13], M is not np-Artinian nor np-Noetherian. Hence, *npk*-dim $M \geq 1$ and *npn*-dim $M \geq 1$. However, we note that M has Krull dimension, so it has type Krull (resp., Noetherian) dimension equal to zero, see [12, Remark 3.3].

Next, we give a characterization theorem for modules with non-parallel Noetherian dimension. This result can be considered as an extension of [13, Theorem 3.12] in the case $\alpha = 0$.

Theorem 4.4. Let M be an R-module. The following statements are equivalent.

(1) M has non-parallel Noetherian dimension and *npn*-dim $M \leq \alpha$.
(2) Every non-parallel submodule N of M has Noetherian dimension and *npn*-dim $N \leq \alpha$.
(3) Every proper closed submodule N of M has Noetherian dimension and *npn*-dim $N \leq \alpha$.
(4) Every orthogonal decomposable submodule N of M has Noetherian dimension and *npn*-dim $N \leq \alpha$.

In this case, M has finite type dimension.

Proof. ($1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$) Similar to the proof of Theorem 3.6.

($4 \Rightarrow 1$) We have to show that each non-parallel submodule of M has Noetherian dimension. First, applying similar arguments as used in the proof of Proposition 3.7, we conclude that M has finite type dimension. Let N be a non-parallel submodule of M. There exists a nonzero submodule H of M such that $N \perp H$. By (4), the orthogonal decomposable submodule $N \oplus H$ has Noetherian dimension and *npn*-dim $N \oplus H \leq \alpha$. Hence N has Noetherian dimension and *n*-dim $N \leq \alpha$. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ be any ascending chain of non-parallel submodules of M. Now, we can apply the proof ($5 \Rightarrow 1$) of [13, Theorem 3.12] to obtain an integer number t such that $\bigcup_{i \geq t} K_i$ is a non-parallel submodule of M. By what we have shown at the beginning of the proof, *n*-dim $\cup_{i \geq t} K_i \leq \alpha$. Thus, there exists an integer number l such that for each $i \geq l$, *n*-dim $\frac{K_{i+1}}{K_i} < \alpha$. In view of the previous proposition, *npn*-dim $\frac{K_{i+1}}{K_i} \leq$ *n*-dim $\frac{K_{i+1}}{K_i}$ for each $i \geq l$. This implies that *npn*-dim $M \leq \alpha$. □

The following result is an analogue of Proposition 3.3.

Lemma 4.5. Let M be an R-module with non-parallel Noetherian dimension. Then, for any submodule N of M, N has non-parallel Noetherian dimension and $nek\text{-}\dim N \leq nek\text{-}\dim M$.

Corollary 4.6. Let M be an R-module with non-parallel Noetherian dimension. Then it has homogeneous type Noetherian dimension.

Proposition 4.7. Let M be an R-module with non-parallel Noetherian dimension α. Then, for each type submodule N of M, $\frac{M}{N}$ has non-parallel Noetherian dimension and $npn\text{-}\dim \frac{M}{N} \leq \alpha$.

Let M be an R-module. It is defined that $T(M) = \bigcap_{i \in I} N_i$, where the N_i's are parallel submodules of M, see [14]. In the following result, we find a condition under which any module with non-parallel Noetherian dimension, has Noetherian dimension and these dimensions coincide. One can show that the dual of this result is also true for modules with non-parallel Krull dimension.

Theorem 4.8. Let M be an R-module such that $T(M) \nsubseteq \text{rad}(M)$. If M has non-parallel Noetherian dimension, then it has Noetherian dimension and $n\text{-}\dim M = nen\text{-}\dim M = npn\text{-}\dim M$.

Proof. Suppose that $npn\text{-}\dim M = \alpha$. First, we show that M has Noetherian dimension and $n\text{-}\dim M \leq \alpha$. Since $T(M) \nsubseteq \text{rad}(M)$, there exists a maximal submodule N of M such that $N \nparallel M$ (otherwise, we have $T(M) \subseteq \text{rad}(M)$ which is a contradiction). Hence, $\frac{M}{N}$ is a simple module and clearly it has zero Noetherian dimension. Since $N \nparallel M$, Theorem 4.4 implies that N has Noetherian dimension and $n\text{-}\dim N \leq \alpha$. Therefore, by the dual of Proposition 2.13, M has Noetherian dimension and

$$n\text{-}\dim M = \sup\{n\text{-}\dim N, n\text{-}\dim \tfrac{M}{N}\} \leq \alpha.$$

On the other hand, Proposition 4.2 implies that

$$\alpha \leq nen\text{-}\dim M \leq n\text{-}\dim M.$$

It follows that $n\text{-}\dim M = nen\text{-}\dim M = \alpha$ and we are done. □

The following result is an analogue of Proposition 2.15.

Theorem 4.9. Let M be an R-module. Then M has non-parallel Krull dimension if and only if it has non-parallel Noetherian dimension.

Proof. Suppose that M has non-parallel Krull dimension α. By Theorem 4.4, it suffices to show that every non-parallel submodule of M has Noetherian dimension $\leq \beta$ for some ordinal number β. By our assumption and

Theorem 3.6, N has Krull dimension at most α, so by Proposition 2.15, N has Noetherian dimension for each non-parallel submodule N of M. Put $\beta = \sup\{n\text{-dim}\, N \mid N \in \mathrm{NP}(M)\}$. Now, Theorem 4.4 implies that M has non-parallel Noetherian dimension at most β. □

Corollary 4.10. Every np-Artinian module has non-parallel Noetherian dimension.

Corollary 4.11. Let M be an R-module such that $\frac{M}{N}$ has non-parallel Noetherian dimension for all $0 \neq N \in \mathrm{NP}(M)$. Then M has non-parallel Noetherian dimension.

Proof. It follows by Proposition 3.5 and Theorem 4.9. □

In the following example we show that the converse of the previous corollary and Theorem 3.16 are not true in general.

Example 4.12. Consider $M := \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. Then it is clear that M has Krull (resp., Noetherian) dimension, so it has non-parallel Krull (resp., Noetherian) dimension. Note that $\mathbb{Z} \oplus 0$ (resp., $0 \oplus \mathbb{Z}_{p^\infty}$) is a non-parallel submodule of M which is not Artinian (resp., Noetherian). Now, in view of [13, Theorems 3.4 and 3.12], it is easy to see that M is neither np-Artinian nor np-Noetherian, see also [5, Example 2.7].

Using Theorems 3.11 and 4.9, Corollary 3.12 and Proposition 3.15, the proof of the following result is immediate.

Proposition 4.13. Let M be an R-module with non-parallel Noetherian dimension. The following statements hold.

(1) M is either atomic or λ-f.e. for some ordinal λ.
(2) M is either atomic or it has finite Goldie dimension.
(3) Any non-atomic (non-zero) submodule of M contains a critical submodule.
(4) Any non-zero submodule of M contains an atomic submodule.
(5) The intersection of all maximal type submodules of M is equal to 0.

In the following result we show that in any semiprime ring with at least one non-parallel right ideal, the existence of any discussed dimensions is equivalent.

Theorem 4.14. Let R be a semiprime ring which is not right atomic. Then the following are equivalent.

(1) R has Krull dimension.

(2) R has Noetherian dimension.
(3) R has non-essential Noetherian dimension.
(4) R has non-essential Krull dimension.
(5) R has non-parallel Noetherian dimension.
(6) R has non-parallel Krull dimension.

Proof. ($1 \Rightarrow 2$) It follows from Proposition 2.15.

($2 \Rightarrow 3$) It is Proposition 4.2(1).

($3 \Rightarrow 4$) See the text before [2, Corollary 3.1].

($4 \Rightarrow 5$) By Theorem 3.22, R has Krull dimension (note that, by our assumption, R is not right uniform). Hence, R has non-parallel Krull dimension, by Proposition 3.2(4). Now, Theorem 4.9 implies that R has Noetherian dimension.

($5 \Rightarrow 6$) It follows from Theorem 4.9.

($6 \Rightarrow 1$) It follows from Theorem 3.20. □

Proposition 4.15. Let R be a ring such that either

(1) R does not have Krull dimension or $k\text{-dim}\, R > \alpha$ but $npk\text{-dim}\, R \leq \alpha$, or
(2) R does not have Noetherian dimension or $n\text{-dim}\, R > \alpha$ but

$$npn\text{-dim}\, R \leq \alpha.$$

If $c \in R$, then cR is a parallel right ideal of R or $\mathbf{r}(c)$ is a parallel right ideal of R.

Proof. Suppose that (1) holds (the proof for (2) is similar). Let cR be a non-parallel right ideal of R. By Theorem 3.6, cR has Krull dimension and $k\text{-dim}\, cR \leq \alpha$. Since $cR \cong \frac{R}{\mathbf{r}(c)}$, it is easy to see that $\mathbf{r}(c)$ does not have Krull dimension or $k\text{-dim}\, \mathbf{r}(c) > \alpha$, see Proposition 2.13. Hence, $\mathbf{r}(c)$ is a parallel right ideal of R, by Theorem 3.6. □

We recall that a ring R has many parallel right ideals provided, for every element a in R, aR is a parallel right ideal or $\mathbf{r}(a)$ is a parallel right ideal of R, see [13]. Now, we have the following corollary which is an extension of [13, Corollary 3.21].

Corollary 4.16. Let R be a ring. If $npk\text{-dim}\, R \leq \alpha$ (resp., $nen\text{-dim}\, R \leq \alpha$), then either $k\text{-dim}\, R \leq \alpha$ (resp., $n\text{-dim}\, R \leq \alpha$) or R has many parallel right ideals.

Acknowledgments

The authors would like to thank the referees for reading the article carefully and giving useful comments and and for their efforts towards improving the manuscript. The first author is grateful to the Research Council of Shahid Chamran University of Ahvaz, Iran, for financial support, and Grant No. SCU.MM1400.473.

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Journal of Algebraic Systems

NON-PARALLEL KRULL DIMENSION OF MODULES

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در این مقاله، مفهوم بُعد کرول (به ترتیب، نویتری) غیرموازی یک R-مدول معرفی شده و برخی از خواص مرتبط با آن بررسی شده است. با استفاده از این مفاهیم، ما برخی از نتایج کار قبلی خود را تعمیم داده و سعی می‌کنیم نتایج مناسب جدیدی درباره‌ی مدول‌های np-آرتینی (به ترتیب، np-نویتری) به دست بیاوریم. ما یک دسته بندی برای مدول‌های با بعد کرول (به ترتیب، نویتری) غیرموازی ارائه می‌دهیم و نشان می‌دهیم که این مدول‌ها دارای بعد همسان متناهی هستند. نشان داده شده است که هر R-مدول M با بعد کرول غیرموازی حداکثر α، اتمی یا به ازای یک $\lambda \leq \alpha$، λ-متناهی نشانده شده است. هم‌چنین، ثابت شده است که هر R-مدول np-نویتری دارای بعد کرول غیرموازی است. به ویژه، برای حلقه‌های غیراتمی راست نیم‌اول، نشان می‌دهیم که بعد کرول و بعد کرول غیرموازی، منطبق هستند.

کلمات کلیدی: زیرمدول‌های غیرموازی، مدول‌های اتمی، بعد کرول غیرموازی، بعد نویتری غیرموازی، بعد همسان.