Journal of Algebraic Systems, Vol. 14(No. 3): (2026), pp 439-454.
https://doi.org/10.22044/JAS.2024.14092.1798

SOME ASPECTS OF K-HYPERIDEALS OF TERNARY HYPERSEMIRINGS

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Abstract. This paper introduces the concept of k-hyperideals of ternary hypersemirings and explores some of its fundamental characteristics. Next, k-hyperideals are used to characterize prime ternary hypersemirings and ternary hypersemifields. Moreover, k-hyperideals explain the quotient of ternary hypersemirings. Finally, the new kind of ternary hypersemirings is formed using regular equivalence relations, and the set of all k-hyperideals is generated from them.

1. Introduction

The article "Sur une generalisation de la notion de group" [11], presented by French mathematician F. Marty at the 8th Congress of Scandinavian Mathematicians in 1934, set the pace for the development of the algebraic hyperstructure. The field of hyperstructure theory has experienced rapid growth in the past several years and has numerous applications in computer science, information science, theoretical physics, graphs and hypergraphs, cryptography, geometry, and coding theory, among other fields. might be discovered in the books "Hyperring Theory and Applications" by B. Davvaz and Leoreanu-Fotea [5] and "Applications of Hyperstructures Theory" by P. Corsini and Leoreanu-Fotea [3]. Krasner's hyperring, as it is often known, was first introduced by M. Krasner [7] in 1983. This hypercompositional structure is called Krasner's hyperring $(R, +, \circ)$, where $(R, +)$ is the canonical hypergroup, $(R, \circ)$ is a semigroup, and the zero element 0 (i.e., the identity element of the canonical hypergroup $(R, +)$ is bilaterally absorbing (that is, $x \circ 0 = 0 \circ x = \{0\}$, $\forall x \in R$). The operation $\circ$ is a two-sided distributive one over the hypercomposition '+'. Rota [12] developed the concept of multiplicative hyperring in 1990. In this concept, addition and multiplication are binary operations. Following then, multiplicative hypersemirings were introduced and examined by Dasgupta et al. [4]. The Krasner ternary hyperring was first presented by J. R. Castello and J. S. Paradero-Vilela [2] in 2014.

MSC(2020): Primary: 16Y60; Secondary: 20N20, 16Y99.
Keywords: Ternary hypersemirings; Ternary hypersemifields; k-Hyperideals; Homomorphism; Regular equivalence relation.
Received: 21 January 2024, Accepted: 23 December 2024.

Binary hyperoperations '+' and ternary multiplications '$\cdot$' are found in the Krasner ternary hyperring $(R, +, \cdot)$.

The concept of a ternary ring was initially put out and some important aspects of it were examined by W. G. Lister [8] in 1971. An initial proposal for ternary semiring was made in 2003 by Dutta and Kar [6]. Salim et al. [13] presented the class of multiplicative ternary hyperring in 2015; in this class, ternary hyperoperations are represented by the symbol '$\circ$' and binary operations by the symbol '+'. Ternary hypersemiring, an expanded idea of multiplicative ternary hyperring and ternary semiring, was conceived and investigated by Tamang and Mandal [18].

M. K. Sen et al. examined the characteristics of the maximum and one-sided k-ideals of semirings in [17, 19]. As opposed to ternary hypersemiring, R. Ameri and H. Hedayati investigated the k-hyperideals of semihyperrings in [1]. While in ternary hypersemiring $(R, +, \circ)$ '+' is a binary operation and '$\circ$' is a ternary hyperoperation, in semihyperring $(R, +, \cdot)$, $(R, +)$ is a semihypergroup, $(R, \cdot)$ is a semigroup, and the operation $\cdot$ is a two-sided distributive one over the hypercomposition '+'.

The radical characteristics, prime hyperideals, and primary hyperideals of ternary hypersemirings were investigated in 2021 by Mandal et al. [10]. Salim and Sinha [15] explored the relationships between ternary hypersemirings in 2022 and came up with three isomorphism theorems related to them.

The quasi-hyperideals and bihyperideals in ternary hypersemirings were investigated in 2023 by Kostaq Hilla, Md. Salim, and D. Sinha [16]. The semiprime hyperideals in ternary hypersemirings were once again examined by Md. Salim and B. Dutta in [14]. Different types of hyperideals, such as primary, prime, semiprime, quasi-hyperideal, and bi-hyperideal, have up to this point been explored in ternary hypersemirings. Following Sen et al.'s discovery of the k-ideals for semirings in [19], Ameri et al.[1] presented the notion of k-hyperideals for semihyperrings. The k-hyperideal of ternary hypersemirings, an extension of hyperideals, is a particular kind of hyperideals that was introduced for this reason.

This paper investigates the properties of k-hyperideals of ternary hypersemirings and shows that the set of k-hyperideals is a complete lattice. A ternary hypersemifield is also introduced, with k-hyperideals serving as its defining characteristic. By using a regular equivalency relation $\nu*$ on S, we were able to create a ternary hypersemiring $S_{\nu*}$ in [15]. A one-to-one correspondence between S and $S_{\nu*}$ is obtained by illuminating the set of all k-hyperideals of this ternary hypersemiring.

2. Preliminaries

A ternary hyperoperation '$\circ$' on a nonempty set H, we shall mean a mapping $\circ$: $H \times H \times H \to P^*(H)$ where $P^*(H)$ is the set of all nonempty subsets of H. For $x, y, z \in H$, the image of the element $(x, y, z) \in H \times H \times H$ under the mapping '$\circ$' will be denoted by $x \circ y \circ z$ (which is called the ternary hyperproduct of x, y, z).

Definition 2.1. [18] A ternary hypersemiring $(S, +, \circ)$ is an additive commutative semigroup $(S, +)$ endowed with a ternary hyperoperation '$\circ$' such that the following conditions hold :

(i): $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$
(ii): $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;
(iii): $a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;
(iv): $a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$;

for all $a, b, c, d, e \in S$, where if the inclusions in $(ii) - (iv)$ are replaced by equalities, then the ternary hypersemiring is called a strongly distributive ternary hypersemiring.

Definition 2.2. [18] The additive identity '0' of a ternary hypersemiring $(S, +)$ is said to be a zero (strong zero) of $(S, +\circ)$ if

$$0 \in a \circ b \circ 0 = a \circ 0 \circ b = 0 \circ a \circ b$$

(resp. $\{0\} = a \circ 0 \circ b = 0 \circ a \circ b = a \circ b \circ 0$) for all $a, b \in S$.

Definition 2.3. [18] Let $(S, +, \circ)$ be a ternary hypersemiring. A nonempty finite subset $\epsilon = \{(e_i, f_i) : i = 1, 2, 3 \cdots, n\}$ of $S \times S$ is called a left (resp. lateral, right) identity set of S if for any $a \in S, a \in \sum_{i=1}^{n} e_i \circ f_i \circ a$ (resp. $a \in \sum_{i=1}^{n} e_i \circ a \circ f_i$ and $a \in \sum_{i=1}^{n} a \circ e_i \circ f_i$). A nonempty finite subset $\epsilon = \{(e_i, f_i) : i = 1, 2, 3 \cdots, n\}$ of $S \times S$ where S is a ternary hypersemiring is called an identity set if it is a left and a lateral and a right identity set of S.

An element $e \in S$ is called a hyperidentity or unital element of S if $a \in (e \circ e \circ a) \cap (e \circ a \circ e) \cap (a \circ e \circ e)$ for all $a \in S$.

Definition 2.4. A ternary hypersemiring $(S, +, \circ)$ is called commutative if $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$ for all $a_1, a_2, a_3 \in S$.

Definition 2.5. An element a of a ternary hypersemiring S is said to be invertible in S if there exists an element b in S (called the ternary semiring-inverse of a) such that $x \in a \circ b \circ x = b \circ a \circ x = x \circ a \circ b = x \circ b \circ a$ for all $x \in S$.

Definition 2.6. [18] An additive subsemigroup T of a ternary hypersemiring $(S, +, \circ)$ is called a ternary subhypersemiring if $t_1 \circ t_2 \circ t_3 \subseteq T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.7. [18] Let $(S, +, \circ)$ be ternary hypersemiring. An additive subsemigroup I of S is called

- **(i):** a left hyperideal of S if $s_1 \circ s_2 \circ x \subseteq I$, for all $x \in I$ and for all $s_1, s_2 \in S$;
- **(ii):** a right hyperideal of S if $x \circ s_1 \circ s_2 \subseteq I$, for all $x \in I$ and for all $s_1, s_2 \in S$;
- **(iii):** a lateral hyperideal of S if $s_1 \circ x \circ s_2 \subseteq I$, for all $x \in I$ and for all $s_1, s_2 \in S$;
- **(iv):** a two sided hyperideal of S if I is both a left and a right hyperideal of S;
- **(v):** a hyperideal of S if I is a left, a right, and a lateral hyperideal of S.

Let S be a ternary hypersemiring. If I, J and K are three nonempty subsets of S, then

$$I \circ J \circ K = \cup\{\sum_{finite} a_i \circ b_i \circ c_i : a_i \in I, b_i \in J, c_i \in K\}.$$

Proposition 2.8. *Let $(S, +, \circ)$ be a ternary hypersemiring and A be a nonempty subset of S. Then $S \circ S \circ A$, $A \circ S \circ S$ and $S \circ S \circ A \circ S \circ S$ are respectively a left hyperideal, a right hyperideal and a hyperideal of S. If the ternary hypersemiring S has a hyperidentity e, then $\langle A\rangle_l = S \circ S \circ A, \langle A\rangle_r = A \circ S \circ S$ and $\langle A\rangle = S \circ S \circ A \circ S \circ S$.*

3. k-hyperideals of ternary hypersemirngs

Definition 3.1. [18] A (right, lateral, left) hyperideal I of a ternary hypersemiring $(S, +, \circ)$ is called a (right, lateral, left) k-hyperideal if $x \in S$, $a \in I$ and $a + x \in I \Rightarrow x \in I$.

A hyperideal I of S is said to be k-hyperideal of S if it is a right, a letaral and a left hyperideal of S.

The family of all right (lateral, left) k-hyperideals of a ternary hypersemiring S is denoted by $R_k(S)($ resp. $M_k(S), L_k(S))$. The family of k-hyperideals of a ternary hypersemiring S is denoted by $I_k(S)$.

Example 3.2. Let $S = \mathbb{Z}^-$ denote the set of all negative integers. Then $(S, +)$ is a commutative semigroup, where $+$ is the usual addition of integers. Define a ternary hyperoperation '$\circ$' on S by $x \circ y \circ z = \{x \cdot a \cdot y \cdot b \cdot z : a, b \in \mathbb{Z}^-\}$, for all $x, y, z \in S$. Then the system $(S, +, \circ)$ is a ternary hypersemiring. Consider the subset $I = 2\mathbb{Z}^-$ of negative integers. Then I is a k-hyperideal of S, because if $a \in I$, $x \in S$ and $a + x \in I$, then $a + x$ must also be even. Therefore $x \in I$.

Remark 3.3. Every k-hyperideal is a hyperideal but the converse is not true.

Example 3.4. Consider the following ternary hypersemirings $(S, +, \circ)$, where $S = \{0, 1\}$ and "+" and "$\circ$" are defined as follows: $a + b = \min(a, b)$ and $a \circ b \circ c = \{a, b, c\}$ for all $a, b, c \in S$. Then $I = \{0\}$ is a hyperideal of S. But it is not a k-hyperideal of S, since $0 + 1 \in I$ and $0 \in I$ does not imply $1 \in I$.

Example 3.5. Consider the following ternary hypersemirings $(\mathbb{Z}^-, +, \circ)$, where $\mathbb{Z}^-$ is the set of all negative integers and '+' and '$\circ$' are defined as follows: $a + b = \max(a, b)$ and $a \circ b \circ c = \{a, b, c\}$ for all $a, b, c \in S$. Then $I = 3\mathbb{Z}^-$ is a hyperideal of $(\mathbb{Z}^-, +, \circ)$. Since $(-3) + (-4) \in I$ and $-3 \in I$ but $-4 \notin I$. Hence it is not a k-hyperideal.

Definition 3.6. Let I be an hyperideal of ternary hypersemirng S. Then the k-closure of I, denoted by $\bar{I}$, defined by $\bar{I} = \{a \in S : a+b \in I \text{ for some } b \in I\}$.

Remark 3.7. The k-closure of I is the smallest k-hyperideal of S containing I.

We can show that $(i) A \subseteq \bar{A}$, $(ii) A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$, $(iii) \bar{\bar{A}} = \bar{A}$ and (iv) The k-closure of A is a k-hyperideal of S.

Proposition 3.8. *Let A be a k-hyperideal of strongly distributive ternary hypersemirings S, then the k-closure of A is a k-hyperideal of S.*

Proof. Proof is similar to the Lemma 2.2[17]. □

Definition 3.9. A mapping f from a ternary hypersemiring $(S, +, \circ)$ to a ternary hypersemiring $(S', +, \circ)$ is called a

(i): homomorphism if $f(a + b) = f(a) + f(b)$ and
$$f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c).$$
(ii): good homomorphism if $f(a + b) = f(a) + f(b)$ and
$$f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)).$$

Remark 3.10. Every good homomorphism is a homomorphism but converse is not true.

Example 3.11. Let $S = \{0, 1\}$ be a two points set. Let us define addition '+' and ternary hyperoperation '$\circ$', by $a + b = \min\{a, b\}$ and $a \circ b \circ c = \{a, b, c\}$ for all $a, b, c \in S$. Then $(S, +, \circ)$ is a ternary hypersemiring. Let

$$X = \mathbb{Z}^- \cup \{0\}$$

where $\mathbb{Z}^-$ denotes the set of all negative integers. Let $A \in P(X)^*$ with $|A| \geq 2$ and $0 \notin A$. Then under the usual addition of integers and multiplication of integers, X forms a commutative ternary semiring. Clearly, the mapping f from the ternary hypersemiring X_A over the ternary semiring X induced by A to the ternary hypersemiring S defined by $f(a) = 0$, for all $a \in \mathbb{Z}^-$ and $f(a) = 1$ is a homomorphism but not a good homomorphism since $f(-1 \circ -1 \circ 0) = f(0) \subset \{0, 1\} = f(1) \circ f(1) \circ f(0)$.

Proposition 3.12. *The homomorphic image of hyperideal of a ternary hypersemiring is again a hyperideal.*

Proposition 3.13. *(i) Every hyperideal of a multiplicative ternary hyperring*[13] $(R, +, \circ)$ *is a k-hyperideal of* R.

(ii) Let S *be a ternary hypersemiring with zero, then the k-hyperideal* K *of* S *contains* 0.

Proof. (i) Let I be a hyperideal of a multiplicative ternary hyperring R. Then I is an additive commutative subgroup of $(R, +)$. Let $x \in R$, $a \in I$ and $x + a \in I \Rightarrow (x + a) - a \in I$ implies that $x \in I$. Hence I is a k-hyperideal of R.

(ii) It follows from $0 \in S$, $x \in K$ and $x + 0 \in K \Rightarrow 0 \in K$. □

Proposition 3.14. *Arbitrary intersection of a family of k-hyperideals of ternary hypersemiring is again a k-hyperideal of* S.

Proof. Let $\{I_j\}_{j \in J}$ be a family of k-hyperideals of S and J be the nonempty index set. Let $s \in S$, $a \in \bigcap_{j \in J} I_j$ and $s + a \in \bigcap_{j \in J} I_j$. Then $s + a \in I_j$ for all $j \in J$. Therefore $s \in I_j$ for all $j \in J$, since I_j is a k-hyperideal of S. This shows that $s \in \bigcap_{j \in J} I_j$. □

Proposition 3.15. *The family of k-hyperideals* $I_k(S)$ *of a ternary hypersemiring* $(S, +, \circ)$ *with zero forms a partial order set with respect to set inclusion. If we define for any two k-hyperideals* $I, J \in I_k(S)$ *such that* $I \vee J = I \cap J$ *and* $I \wedge J = I + J$ *then* $(I_k(S), \subseteq, \wedge, \vee)$ *is a complete latice.*

Proof. Let $\{K_k : k \in \Omega\} \subseteq I_k(S)$. Then $K = \sum_{k\in\Omega} K_k = \bigcup_{\Lambda\subseteq\Omega} \sum_{k\in\Omega} K_k$ (where Ω is an index set and Λ is finite) is in $I_k(S)$. It is obvious that l.u.b $\{K_k : k \in \Omega\} = \sum_{k\in\Omega} K_k$ and g.l.b $\{K_k : k \in \Omega\} = \bigcap_{k\in\Omega} K_k$. Thus $(I_k(S), \subseteq, \wedge, \vee)$ is a complete lattice. □

Proposition 3.16. *Let $(S, +, \circ)$ be a ternary hypersemiring with zero. If every hyperideal of S is k-hyperideal then the family of hyperideals forms a modular lattice.*

Proposition 3.17. *Let S and $S^{'}$ be two ternary hypersemiring. Let $\Phi : S \to S^{'}$ be an epimorphism. Then the statements hold true:*
(i) *If K is a k-hyperideal of S, then $\Phi(K)$ is a k-hyperideal of $S^{'}$.*
(ii) *If $K^{'}$ is a k-hyperideal of $S^{'}$, then $\Phi^{-1}(K^{'})$ is a k-hyperideal of S.*

Proof. (i) By Proposition 3.12, we can say that $\Phi(K)$ is a hyperideal of $S^{'}$. Let $s' \in S'$ and $k' \in \Phi(K)$ and $s' + k' \in \Phi(K)$. Then there exists $s, k \in S$ such that $s' = \Phi(s)$ and $k' = \Phi(k)$ since Φ is an epimorphism. Now

$$s' + k' = \Phi(s) + \Phi(k) = \Phi(s + k) = \Phi(t)$$

for some $t = s + k \in K$. This implies that $s \in K$, since K is a k-hyperideal of S. Thus $s' = \Phi(s) \in \Phi(K)$. Hence $\Phi(K)$ is a k-hyperideal of S'.

(ii) For $k_1, k_2 \in \Phi^{-1}(K')$ we have $\Phi(k_1), \Phi(k_2) \in K'$. Therefore

$$\Phi(k_1 + k_2) = \Phi(k_1) + \Phi(k_2) \in K' \Rightarrow k_1' + k_2' \in \Phi^{-1}(K').$$

Again let $s_1, s_2 \in S$ and $k \in \Phi^{-1}(K') \Rightarrow \Phi(k) \in K'$. Now

$$\Phi(s_1 \circ s_2 \circ k) \subseteq \Phi(s_1) \circ \Phi(s_2) \circ \Phi(k) \subseteq K',$$

since K' is a hyperideal of S'. This implies that $s_1 \circ s_2 \circ k \subseteq \Phi^{-1}(K')$. Hence $\Phi^{-1}(K')$ is a left hyperideal of S. Similarly, we can show that $\Phi^{-1}(K')$ is a lateral hyperideal and a right hyperideal of S. It remains to show that $\Phi^{-1}(K')$ is a k-hyperideal of S. For that, let $s \in S$ and

$$k \in \Phi^{-1}(K') \Rightarrow \Phi(k) \in K'$$

and $s + k \in \Phi^{-1}(K')$. Now $\Phi(s) + \Phi(k) = \Phi(s + k) \in K'$. This implies that $\Phi(s) \in K'$(since K' is a hyperideal of S') $\Rightarrow s \in \Phi^{-1}(K')$. Hence $\Phi^{-1}(K')$ is a k-hyperideal of S. □

Definition 3.18. Let f be a homomorphism from a ternary hypersemiring S with zero to another ternary hypersemiring S' with zero. Then the kernel of f is denoted by $\ker(f)$ and defined by $\ker(f) = \{s \in S : f(s) = 0_{S'}\}$.

Proposition 3.19. *Let f be a homomorphism from a ternary hypersemiring S with zero to a ternary hypersemiring S' with zero. Then ker(f) is a k-hyperideal of S.*

Proof. It is obvious that $\ker(f)$ is an hyperideal of S and $0 \in \ker(f)$. We now show that $\ker(f)$ is a k-hyperideal. Let $s \in S$, $k \in \ker(f)$ and $s+k \in \ker(f)$. Then $0_{S'} = f(s+k) = f(s) + f(k) = f(s)$. This implies that $s \in \ker(f)$. Hence $\ker(f)$ is a k-hyperideal of S. □

Proposition 3.20. *Let $(S, +, \circ)$ be a ternary hypersemiring with zero and I be a k-hyperideal of S. Let $S/I = \{a+I : a \in S\}$. Then we define the addition and ternary hyperoperation on S/I by $(a+I)+(b+I) = (a+b)+I$ and $(a+I)\circ(b+I)\circ(c+I) = \{p+I : p \in a\circ b\circ c\}$. Then with respect to the above addition and ternary hyperoperation S/I forms a ternary hypersemiring.*

Proof. Let I be a k-hyperideal of $(S, +, \circ)$. Then we have the quotient semigroup $(S/I, +)$. Obviously $(S/I, +)$ is an additive commutative semigroup. Now we shall show that above defined ternary hyperoperation is well defined. Let $a+I = a'+I$, $b+I = b'+I$ and $c+I = c'+I$. Let $p+I \in (a+I)\circ(b+I)\circ(c+I)$. Then $p \in a\circ b\circ c$. Again

$$a+I = a'+I \Rightarrow a+i_1 = a'+i_2(\text{ for some } i_1, i_2 \in I) \Rightarrow a \in a'+I,$$

since I is a k-hyperideal. Similarly $b \in b'+I$ and $c \in c'+I$, since I is a k-hyperideal of S. Now $a\circ b\circ c \subseteq (a'+i_1)\circ(b'+i_2)\circ(c'+i_3) \subseteq a'\circ b'\circ c'+I$. This implies that $p = q+i$ where $q \in a'\circ b'\circ c'$. Therefore

$$p+I = q+I \in (a'+I)\circ(b'+I)\circ(c'+I).$$

So, $(a+I)\circ(b+I)\circ(c+I) \subseteq (a'+I)\circ(b'+I)\circ(c'+I)$. Similarly $(a'+I)\circ(b'+I)\circ(c'+I) \subseteq (a+I)\circ(b+I)\circ(c+I)$. Consequently $(a+I)\circ(b+I)\circ(c+I) = (a'+I)\circ(b'+I)\circ(c'+I)$ and ternary hyperoperation '$\circ$' is well defined.

Let $x+I \in ((a+I)\circ(b+I)\circ(c+I))\circ(d+I)\circ(e+I)$ where $a, b, c, d, e \in S$. Then $x \in p\circ d\circ e$ where $p+I \in (a+I)\circ(b+I)\circ(c+I) \Rightarrow p \in a\circ b\circ c$. Then $x \in (a\circ b\circ c)\circ d\circ e = a\circ b\circ(c\circ d\circ e)$, so $x+I \in (a+I)\circ(b+I)\circ(y+I)$, where $y \in c\circ d\circ e$. So, $y+I \in (c+I)\circ(d+I)\circ(e+I)$. Thus

$$x+I \in (a+I)\circ(b+I)\circ((c+I)\circ(d+I)\circ(e+I)).$$

Thus $x+I \in (a+I)\circ(b+I)\circ((c+I)\circ(d+I)\circ(e+I))$. Thus

$$\begin{aligned}&((a+I)\circ(b+I)\circ(c+I))\circ(d+I)\circ(e+I)\\ &\subseteq (a+I)\circ(b+I)\circ((c+I)\circ(d+I)\circ(e+I)).\end{aligned}$$

Similarly we can prove the converse. Hence

$$(a+I)\circ(b+I)\circ((c+I)\circ(d+I)\circ(e+I))$$
$$=((a+I)\circ(b+I)\circ(c+I))\circ(d+I)\circ(e+I).$$

Similarly we can prove that

$$(a+I)\circ((b+I)\circ(c+I)\circ(d+I))\circ(e+I)$$
$$=(a+I)\circ(b+I)\circ((c+I)\circ(d+I)\circ(e+I)).$$

So, '$\circ$' is associative.

Again let

$$\begin{aligned} x+I &\in (a+I)\circ(b+I)\circ((c+I)+(d+I)) \\ &= (a+I)\circ(b+I)\circ((c+d)+I), \end{aligned}$$

where $a,b,c,d\in S$. This implies that $x\in a\circ b\circ(c+d)\subseteq a\circ b\circ c+a\circ b\circ d$. So, $x=y+z$ where $y\in a\circ b\circ c$ and $z\in a\circ b\circ d$. Hence

$$(x+I)=(y+z)+I=(y+I)+(z+I)$$

where $y+I\in(a+I)\circ(b+I)\circ(c+I)$ and $z+I\in(a+I)\circ(b+I)\circ(d+I)$. Thus

$$(a+I)\circ(b+I)\circ((c+I)+(d+I))$$
$$\subseteq(a+I)\circ(b+I)\circ(c+I)+(a+I)\circ(b+I)\circ(d+I).$$

Similarly we can prove that

$$((a+I)+(b+I))\circ(c+I)\circ(d+I)$$
$$\subseteq(a+I)\circ(c+I)\circ(d+I)+(b+I)\circ(c+I)\circ(d+I)$$

and

$$(a+I)\circ((b+I)+(c+I))\circ(d+I)$$
$$\subseteq(a+I)\circ(b+I)\circ(d+I)+(a+I)\circ(c+I)\circ(d+I)$$

for all $a,b,c,d\in R$. Now,

$$\begin{aligned} (a+I)\circ(b+I)\circ(0+I) &= \{x+I : x\in a\circ b\circ 0=\{0_S\}\} \\ &= \{0+I\} \end{aligned}$$

Similarly we can prove that

$$(a+I)\circ(0+I)\circ(b+I)=(0+I)\circ(a+I)\circ(b+I)=\{0+I\}.$$

Therefore $(R/I,+,\circ)$ is a ternary hypersemiring. $\square$

Theorem 3.21. *Let S be a ternary hypersemiring with zero and I be a hyperideal of S, then T is a k-hyperideal of S/I if and only if $T = J/I$ such that J is a k-hyperideal of S and $I \subseteq J$.*

Proof. Let T is a k-hyperideal of S/I. Consider $J = \{x \in S : x + I \in T\}$. Obviously $T = J/I$. Suppose $i \in I$ then $i + I = I \in T = J/I \Rightarrow i \in J$. Therefore $I \subseteq J$. We shall show that J is a hyperideal of S. Let $x, y \in J$, then $(x + I) + (y + I) = (x + y) + I \in T = J/I \Rightarrow x + y \in J$. Also

$$(s_1 + I) \circ (s_2 + I) \circ (x + I) = \{p + I : p \in s_1 \circ s_2 \circ x\} \subseteq T = J/I.$$

Thus $p \in s_1 \circ s_2 \circ x \Rightarrow p + I \in T = J/I \Rightarrow p \in J$. Hence $s_1 \circ s_2 \circ x \subseteq J$. Similarly we can show that $s_1 \circ x \circ s_2 \subseteq J$ and $x \circ s_1 \circ s_2 \subseteq J$. Hence J is a hyperideal of S. Now we prove that J is a k-hyperideal of S. Let $x \in J$, $s \in S$ and $s + x \in J$. Then $(r + I) + (x + I) = (r + x) + I \in T = J/I \Rightarrow (r + I) \in T = J/I$(since T is a k-hyperideal of S/I) $\Rightarrow r \in J$. Hence J is a k-hyperideal of S and $I \subseteq J$.

Conversely, let J is a k-hyperideal of S, $I \subseteq J$ and $T = J/I$ is a hyperideal of S/I, by Lemma 3.13[10]. We now show that J/I is a k-hyperideal of S/I. Let $s + I \in S/I$, $x + I \in J/I$ and $(s + I) + (x + I) \in J/I$. Then

$$(s + I) + (x + I) = (s + x) + I \in J/I \Rightarrow s + x \in J \Rightarrow s \in J$$

(since J is a k-hyperideal of S) $\Rightarrow s + I \in T = J/I$. This implies that $T = J/I$ is a k-hyperideal of S/I. □

Theorem 3.22. *Let S be a ternary hypersemiring and K be a k-hyperideal of S. Then the quotient ternary hypersemiring S/K is a prime ternary hypersemiring if and only if K is a prime k-hyperideal of S.*

Proof. Let K be a k-hyperideal of S. Suppose that $a+K, b+K, c+K \in S/K$ and $(a + K) \circ (S/K) \circ (b + K) \circ (S/K) \circ (c + K) \subseteq \{0 + K\}$. This implies that $\{t + K : t \in a \circ S \circ b \circ S \circ c\} \subseteq \{0 + K\} \Rightarrow a \circ S \circ b \circ S \circ c \subseteq K$(as K is a k-hyperideal of S). Similarly, we can show that

$$(a + K) \circ (S/K) \circ (S/K) \circ (b + K) \circ (S/K) \circ (S/K) \circ (c + K) \subseteq \{0 + K\}$$
$$\Rightarrow a \circ S \circ S \circ b \circ S \circ S \circ c \subseteq K,$$

$$(a + K) \circ (S/K) \circ (S/K) \circ (b + K) \circ (S/K) \circ (c + K) \circ (S/K) \subseteq \{0 + K\}$$
$$\Rightarrow a \circ S \circ S \circ b \circ S \circ c \circ S \subseteq K$$

and

$$(S/K) \circ (a + K) \circ (S/K) \circ (b + K) \circ (S/K) \circ (S/K) \circ (c + K) \subseteq \{0 + K\}$$
$$\Rightarrow S \circ a \circ S \circ b \circ S \circ S \circ c \subseteq K.$$

Since K is a prime hyperideal of S, by Theorem 3.4[10], we obtain $a \in K$ or $b \in K$ or $c \in K$. This shows that $a + K \in \{0 + K\}$ or $b + K \in \{0 + K\}$ or $c + K \in \{0 + K\}$. Thus $\{0 + K\}$ is a prime hyperideal of S/K and S/K is a prime ternary hypersemiring.

Converse follows by reversing the above arguments. □

Definition 3.23. A ternary hypersemiring $(T, +, \circ)$ is said to be semiprime ternary hypersemiring if (0) is a semiprime hyperideal of T.

Theorem 3.24. *A k-hyperideal K of a ternary hypersemiring $(T, +, \circ)$ is semiprime k-hyperideal if and only if T/K is a semiprime ternary hypersemiring.*

Proof. The proof is comparable to the application of Theorem 3.6 [14] and Theorem 3.22. □

Proposition 3.25. *Let K be a k-hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$ with an identity set, then the quotient ternary hypersemiring S/K is also a commutative ternary hypersemiring with an identity set.*

Definition 3.26. A commutative ternary hypersemiring $(S, +, \circ)$ with an identity set ϵ is said to be ternary hypersemifield if every element of $(R\backslash\{0\}, \circ)$ is invertible.

Example 3.27. Let $\mathbb{R}^-$ denote the set of all negative real numbers and $\mathbb{R}_0^- = \mathbb{R}^- \cup \{0\}$. Suppose $n \in \mathbb{Z}$ is arbitrarily choosen but fixed integer. Define a ternary hyperoperation '$\circ$' on $\mathbb{R}_0^-$ by

$$a \circ b \circ c = \{abc + nk : k \in \mathbb{Z} \text{ and for all } a, b, c \in \mathbb{R}^-\}.$$

Then with respect to the usual addition of non-positive real numbers and defined ternary hyperoperation, the system $(\mathbb{R}_0^-, +, \circ)$ forms a ternary hypersemifield.

Theorem 3.28. *Let S be a commutative ternary hypersemiring with an identity set and K be a proper k-hyperideal of S. Then K is maximal if and only if S/K is a ternary hypersemifield.*

Proof. Let S be a commutative ternary hypersemiring with an identity set $\{(e_i, f_i) : (i = 1, 2, \cdots m)\}$ and K be a k-hyperideal of S. By Proposition 3.25, S/K is a commutative ternary hypersemiring with identity set. Let $a/K \neq 0/K$. Then $a \notin K$. Now $K + \langle a \rangle$ is an hyperideal properly containing

K. Since K is maximal hyperideal, by Theorem 3.14[10], we get $K+\langle a\rangle = S$. This implies there exists $k_i \in K$ and $s_{ij}, t_{ij} \in S$ such that

$$e_1 \in \{k_1\} + \sum_{j=1}^{n} s_{1j} \circ t_{1j} \circ a$$
$$e_2 \in \{k_2\} + \sum_{j=1}^{n} s_{2j} \circ t_{2j} \circ a$$
$$\vdots$$
$$e_i \in \{k_i\} + \sum_{j=1}^{n} s_{ij} \circ t_{ij} \circ a$$

From above we get

$$e_1 \circ f_1 \circ x \subseteq k_1 \circ f_1 \circ x + \sum_{j=1}^{n} (s_{1j} \circ t_{1j} \circ f_1) \circ a \circ x$$
$$e_2 \circ f_2 \circ x \subseteq k_2 \circ f_2 \circ x + \sum_{j=1}^{n} (s_{2j} \circ t_{2j} \circ f_2) \circ a \circ x$$
$$\vdots$$
$$e_i \circ f_i \circ x \subseteq k_i \circ f_i \circ x + \sum_{j=1}^{n} (s_{ij} \circ t_{ij} \circ f_i) \circ a \circ x$$

This implies that

$$x \in \sum_{i=1}^{m} e_i \circ f_i \circ x \subseteq \sum_{i=1}^{m} k_i \circ f_i \circ x + (\sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} \circ t_{ij} \circ f_i) \circ a \circ x$$
$$\Rightarrow x \in p + K,$$

where $\sum_{i=1}^{m} k_i \circ f_i \circ x \subseteq K$, $p \in b \circ a \circ x$ and $b = (\sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} \circ t_{ij} \circ f_i) \in S$. Hence $x + K \in \{p + K : p \in b \circ a \circ x\} = (b + K) \circ (a + K) \circ (x + K)$ for all $x + K \in S/K$. Hence S/K is a ternary hypersemifield.

Conversely, suppose that for a proper k-hyperideal K of S, the quotient ternary hypersemiring S/K is ternary hypersemifield. Let M be another k-hyperideal of S such that $K \subset M \subseteq S$. Then there exists an element $m \in M$ such that $m \notin K$. So, $m + K \neq 0 + K$ and hence there exists $k + K \in S/K$ such that $x + K \in (m + K) \circ (k + K) \circ (x + k)$ for all

$$x + K \in S/K \Rightarrow x + K \in m \circ k \circ x + K$$

for all $x+K \in S/K$. This implies that $x+k_1 = y+k_2$ for some $k_1, k_2 \in K \subset M$ and $y \in m \circ k \circ x \subseteq M$(since $m \in M$ and M is a hyperideal of S). Hence $y+k_2 = x+k_1 \in M$ for all $x \in S$. Since M is a k-hyperideal, $x+k_1 \in M$ and $k_1 \in M$ implies that $x \in M$. Consequntly $M = S$. Hence K is a maximal k-hyperideal of S. □

Let $\nu*$ be an equivalence relation on a non-empty set S and $P(S)$ denote the power set of S. Let $P^*(S) = P(S) - \{\phi\}$. Then, we define two relations $\overline{\nu*}$ and $\overline{\overline{\nu*}}$ on $P^*(S)$ as follows:

- **(i):** For any $A, B \in P^*(S), A\overline{\nu*}B$ holds if and only if for each $a \in A$ there exists $b \in B$ such that $a\nu * b$ holds and also for each $b' \in B$ there exists $a' \in A$ such that $a'\nu * b'$ holds.
- **(ii):** $A\overline{\overline{\nu*}}B$ holds if and only if $a\nu * b$ holds for all $a \in A$ and $b \in B$.

Definition 3.29. [15] An equivalence relation $\nu*$ defined on a ternary hypersemiring $(S, +, \circ)$ is called

- **(i):** regular if $\nu*$ is a congruence relation on the commutative semigroup $(S, +)$ i.e. $a\nu * b \Rightarrow (a + c)\nu * (b + c)$ for $a, b, c \in S$ and
$$a\nu * b, c\nu * d, e\nu * f \Rightarrow (a \circ c \circ e)\overline{\nu*}(b \circ d \circ f)$$
for $a, b, c, d, e, f \in S$;
- **(ii):** strongly regular if $\nu*$ is a congruence relation on the commutative semigroup $(S, +)$ i.e. $a\nu * b \Rightarrow (a + c)\nu * (b + c)$ for $a, b, c \in S$ and $a\nu * b, c\nu * d, e\nu * f \Rightarrow (a \circ c \circ e)\overline{\overline{\nu*}}(b \circ d \circ f)$ for $a, b, c, d, e, f \in S$.

Remark 3.30. The second condition stated in (i) and (ii) of the Definition 3.29. are equivalent to the following conditions respectively:
$a\nu * b \Rightarrow (a \circ c \circ d)\overline{\nu*}(b \circ c \circ d), (c \circ a \circ d)\overline{\nu*}(c \circ b \circ d)$ and $(c \circ d \circ a)\overline{\nu*}(c \circ d \circ b)$ for all $a, b, c, d \in S$ and $a\nu * b \Rightarrow (a \circ c \circ d)\overline{\overline{\nu*}}(b \circ c \circ d), (c \circ a \circ d)\overline{\overline{\nu*}}(c \circ b \circ d)$ and $(c \circ d \circ a)\overline{\overline{\nu*}}(c \circ d \circ b)$ for all $a, b, c, d \in S$.
It is clear that the strongly regular equivalence relation is a regular equivalence relation on a ternary hypersemiring.

Proposition 3.31. [15] *An equivalence relation $\nu*$ on a ternary hypersemiring $(S, +, \circ)$ is regular if and only if $(S_{\nu*}, +, \circ)$ is a ternary hypersemiring, where $S_{\nu*} = \{a_{\nu*} : a \in S\}$ and $a_{\nu*}$ is the equivalence class containing $a, a_{\nu*} + b_{\nu*} = (a+b)_{\nu*}$ and $a_{\nu*} \circ b_{\nu*} \circ c_{\nu*} = \{x_{\nu*} : x \in a \circ b \circ c\}$ for any $a, b, c \in S$.*

Proposition 3.32. *Let $(S, +, \circ)$ be a ternary hypersemiring and $\nu*$ be a regular equivalence relation on S. If K is a k-hyperideal of S then $K_{\nu*}$ is a k-hyperideal of $(S_{\nu*}, +, \circ)$*

Proof. Let $x_{\nu*}, y_{\nu*} \in K_{\nu*}$. Then $x_{\nu*} = k_{\nu*}$ and $y_{\nu*} = k'_{\nu*}$ for some $k, k' \in K$. Now $x_{\nu*} + y_{\nu*} = k_{\nu*} + k'_{\nu*} = (k+k')_{\nu*} \in K_{\nu*}$, since K is a hyperideal of S. Again let $s_{\nu*}, s'_{\nu*} \in S_{\nu*}$ and $x_{\nu*} \in K_{\nu*}$, then $x_{\nu*} = k_{\nu*}$ for some $k \in K$. Now $s_{\nu*} \circ s'_{\nu*} \circ x_{\nu*} = s_{\nu*} \circ s'_{\nu*} \circ k_{\nu*} = \{p_{\nu*} : p \in s \circ s' \circ p \subseteq K\}$. This implies that $p_{\nu*} \in K_{\nu*}$. Consequently $s_{\nu*} \circ s'_{\nu*} \circ x_{\nu*} \subseteq K_{\nu*}$. Similarly we can show that $s_{\nu*} \circ x_{\nu*} \circ s'_{\nu*} \subseteq K_{\nu*}$ and $x_{\nu*} \circ s_{\nu*} \circ s'_{\nu*} \subseteq K_{\nu*}$. Hence $K_{\nu*}$ is a hyperideal of $S_{\nu*}$. Finally, we show that $K_{\nu*}$ is a k-hyperideal of $S_{\nu*}$. Let $s_{\nu*} \in S_{\nu*}$, $x_{\nu*} \in K_{\nu*} \Rightarrow x_{\nu*} = k_{\nu*}$ for some $k \in K$ and

$$x_{\nu*} + s_{\nu*} \in K_{\nu*} \Rightarrow k_{\nu*} + s_{\nu*} = (k+s)_{\nu*} \in K_{\nu*}.$$

Then $k + s \in K$. Now $s \in S$ and $k + s \in K$ imples that $s \in K$, since K is a k-hyperideal of S. Therefore $s_{\nu*} \in K_{\nu*}$. Hence $K_{\nu*}$ is a k-hyperideal of $S_{\nu*}$. □

Proposition 3.33. *Let $(K, +, \circ)$ be a k-hyperideal of a ternary hypersemiring $(S, +, \circ)$ and μ^* be a regular equivalence relation on $(S, +, \circ)$. Then $K_{\mu^*} = \{a \in S :$ there exists $b \in K$ such that $a\mu^*b\}$ is a k-hyperideal of $(S, +, \circ)$*

Proof. Obviously $K \subseteq K^*_\mu$ and hence $K_{\mu^*} \neq \phi$. Let $a_1, a_2 \in K^*_\mu$. Then there exist $b_1, b_2 \in K$ such that $a_1\mu^*b_1$ and $a_2\mu^*b_2$. So $(a_1 + a_2)\mu^*(b_1 + b_2)$. Hence $a_1 + a_2 \in K^*_\mu$. Let $a_1, a_2, a_3 \in K^*_\mu$. Then $a_i\mu^*b_i$ for $i = 1, 2, 3$ for some $b_1, b_2, b_3 \in K$. So $(a_1 \circ a_2 \circ a_3)\overline{\mu^*}(b_1 \circ b_2 \circ b_3)$(since μ^* is a regular equivalence relation on S). Let $x \in a_1 \circ a_1 \circ a_3$. By (1) there exists an element $y \in b_1 \circ b_2 \circ b_3$ such that $x\mu^*y$, where $y \in K$. So $x \in K^*_\mu$. Thus $a_1 \circ a_2 \circ a_3 \in P^*(K_{\mu^*})$. Hence K_{μ^*} is a ternary subhypersemiring of $(S, +, \circ)$.

Now let us consider $s_1, s_2 \in S$ and $t \in K_{\mu^*}$. Then there exists $b \in K$ such that $t\mu^*b$ and holds. Hence $(s_1 \circ s_2 \circ t)\mu^*(s_1 \circ s_2 \circ b)$, since μ^* is regular. Thus $s_1 \circ s_2 \circ t \in K_{\mu^*}$. Similarly we can show that $s_1 \circ t \circ s_2 \in K_{\mu^*}$ and $t \circ s_1 \circ s_2 \in K_{\mu^*}$. Therefore K_{μ^*} is a hyperideal of S. It remains to show that K_{μ^*} is a k-hyperideal. Let $s \in S$, $t \in K_{\mu^*}$ and $s + t \in K_{\mu^*}$. Then $t_1\mu^*b$

for $t_1 = s + t$ and there exists $b \in K$. This implies that $t_{1^*_\mu} \in K_{\mu^*}$. i.e. $t_{1_{\mu^*}} = s_{\mu^*} + t_{\mu^*} \in K_{\mu^*} \Rightarrow s_{\mu^*} \in K_{\mu^*}$, since K_{μ^*} is a k-hyperideal of S_{μ^*}. Thus $s \in K \subseteq K_{\mu^*}$. Consequently K_{μ^*} is a k-hyperideal of S. □

Acknowledgments

The author would like to thank the referee for his/her valuable suggestions.

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Journal of Algebraic Systems

SOME ASPECTS OF k-HYPERIDEALS OF TERNARY HYPERSEMIRINGS

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برخی جنبه‌های k-ابرایده‌آل‌ها در ابرنیم‌حلقه‌های سه‌تایی

اِم دی. سلیم

گروه ریاضی، کالج دولتی باراسات، باراسات، هند

در این مقاله، مفهوم k-ابرایده‌آل‌ها در ابرنیم‌حلقه‌های سه‌تایی معرفی شده و برخی از ویژگی‌های بنیادی آن بررسی می‌شود. سپس از k-ابرایده‌آل‌ها برای مشخصه‌سازی ابرنیم‌حلقه‌های سه‌تایی اول و ابرنیم‌میدان‌های سه‌تایی استفاده می‌شود. علاوه بر این، k-ابرایده‌آل‌ها در تبیین خارج‌قسمت ابرنیم‌حلقه‌های سه‌تایی به‌کار گرفته می‌شوند. در پایان، نوع جدیدی از ابرنیم‌حلقه‌های سه‌تایی با استفاده از روابط هم‌ارزی منظم ساخته می‌شود و مجموعه‌ی تمامی k-ابرایده‌آل‌ها از آن‌ها تولید می‌گردد.

کلمات کلیدی: ابرنیم‌حلقه‌های سه‌تایی، ابرنیم‌میدان‌های سه‌تایی، k-ابرایده‌آل‌ها، همریختی، رابطه‌ی هم‌ارزی منظم.