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SOME RESULTS ON SUBRINGS OF $C(X)$ AND $C(X,\mathbb{C})$

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Abstract. Let X be a Tychonoff space and $\mathcal{R}[X]$ be the collection of all subrings of $C(X)$ that separate points and contain the identity element 1. In this paper, we establish a correspondence between the ideals in $A(X) \in \mathcal{R}[X]$ and the z_A^γ-filters on the completion of X with respect to a uniform structure arising from the functions in $A(X)$. We also explore some properties of z-ideals, z_A-ideals and maximal ideals in these types of subrings of $C(X)$. For each subset $A(X)$ of $C(X)$, let $[A(X)]_c = \{f+ig : f, g \in A(X)\}$. We demonstrate that $[A(X)]_c$ is a c-type subring of $C(X,\mathbb{C})$ when $A(X) \in \mathcal{R}[X]$ is a c-type subring of $C(X)$. Finally, for an intermediate subring $A(X)$ of $C(X)$, we show that the completion of X with respect to a suitable uniform structure derived from $[A(X)]_c$ is equal to $\upsilon_A X$, the A-compactification of X.

1. Introduction and Preliminaries

Let X be a Tychonoff space and $C(X)$, $C^*(X)$ denote the rings of all continuous real-valued functions on X and bounded continuous real-valued functions on X respectively. In this paper, by subrings of $C(X)$, we mean those subrings which contain the constant functions and we consider all ideals as proper ideals. The correspondence between the ideals in $C(X)$ and the z-filters on X has been described by Gillman and Jerison in [6], where they have also derived a similar correspondence in the case of $C^*(X)$. A subring $A(X)$ of $C(X)$ is an intermediate subring of $C(X)$ if $C^*(X) \subseteq A(X) \subseteq C(X)$. To capture the algebraic properties of an intermediate subring $A(X)$ of $C(X)$, Redlin and Watson in [12] have introduced the concept of local invertibility of a function $f \in A(X)$, through which they have provided a correspondence between ideals in $A(X)$ and z-filters on X. This correspondence has been studied further by other mathematicians in [4, 9]. In [3], further exploration of similar correspondence between other types of ideals called z_A^β-ideals and filters called z_A^β-filters has been done for an intermediate subring of $C(X)$.

A subring $A(X)$ of $C(X)$ is called a c-type subring if there exists a Tychonoff space Y such that $A(X)$ is isomorphic to $C(Y)$. A subring $A(X)$ of $C(X)$ is said to separate points if for any two distinct points x, $y \in X$,

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there exists a function $f \in A(X)$ such that $f(x) \neq f(y)$. De and Acharyya in [5] have provided the characterization of intermediate c-type subrings of $C(X)$ and this problem has been further studied for subrings of $C(X)$ that separate points and contain the identity element 1 by Mitra and Chowdhury in [8] and for the subrings of the ring of complex-valued continuous functions by Acharyya et al. in [1].

A realcompactification of X is a realcompact space in which X is dense. The concept of realcompactness of a space was introduced by Hewitt in [7] and a generalization called A-compactness was given by Redlin and Watson in [12], where $A(X)$ is an intermediate subring of $C(X)$. For any f in an intermediate subring $A(X)$ of $C(X)$, let $\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}$ and $\upsilon_A X = \bigcap_{f\in A(X)} \upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty \ \forall\ f \in A(X)\}$. For each intermediate subring $A(X)$ of $C(X)$, $\upsilon_A X$ is a realcompactification of X called the A-compactification of X and X is A-compact if and only if $X = \upsilon_A X$ [2].

Let $\sum(X)$ denote the collection of all intermediate subrings of $C(X)$ and $\mathcal{R}[X]$ denote the collection of subrings of $C(X)$ that separate points and contain the identity element 1. It is worth noting that there exists a member of $\mathcal{R}[X]$ which is not in $\sum(X)$, see [8, Example 1]. For each f in a subring $A(X)$ of $C(X)$, Plank [11] defined $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0\ \forall\ g \in A(X)\}$, where $f^* : \beta X \to \mathbb{R}^*$ is the unique continuous extension of f [6] and $\mathbb{R}^*$ is the one-point compactification of $\mathbb{R}$. An ideal I in a commutative ring R is called a z-ideal if $f \in I$ implies that $M_f(R) \subseteq I$, where $M_f(R)$ represents the intersection of all the maximal ideals in R containing f. In particular, an ideal I in $C(X)$ is a z-ideal if and only if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C(X)$ imply that $g \in I$. Further, an ideal I in an intermediate subring $A(X)$ of $C(X)$ is a z-ideal if and only if $S_A(f) \subseteq S_A(g)$, $f \in I$ and $g \in A(X)$ imply that $g \in I$. An ideal I in a subring $A(X)$ of $C(X)$ is said to be a z_A-ideal, if whenever $Z(f) \subseteq Z(g)$, where $f \in I$ and $g \in A(X)$, then $g \in I$ [10].

The study of uniform structure generated by pseudometrics in the theory of $C(X)$ has enhanced the understanding of the completion of a space as each function $f \in C(X)$ gives rise to a pseudometric $\psi_f : X \times X \to \mathbb{R}$ defined by $\psi_f(x, y) = |f(x) - f(y)|$. The family $\{\psi_f : f \in A(X)\}$, for a given subring $A(X)$ of $C(X)$, serves as a subbase for a specific uniform structure on X. A uniform space X together with a uniform structure $\mathcal{D}$ is denoted by $(X, \mathcal{D})$. A z-filter $\mathcal{F}$ on a uniform space $(X, \mathcal{D})$ is said to be Cauchy if whenever $\psi \in \mathcal{D}$ and $\epsilon > 0$, there exists $Z \in \mathcal{F}$ such that $\psi(x, y) < \epsilon$ for all $x, y \in Z$. A uniform structure $\mathcal{D}$ on X is said to be Hausdorff if for any two distinct points $x, y \in X$, there exists $\psi \in \mathcal{D}$ such that $\psi(x, y) \neq 0$. For each

$A(X) \in \mathcal{R}[X]$, let u_A denote the uniformity generated by $\{\psi_f : f \in A(X)\}$ and let $(\gamma_A X, u_A)$ be the unique completion of X with respect to u_A. Here we use the same symbol u_A for the uniform structure on $\gamma_A X$. Gillman and Jerison [6] proved that $(\gamma_{C^*} X, u_{C^*}) = \upsilon X$, where υX is the Hewitt realcompactification of X and $(\gamma_C X, u_C) = \beta X$. Moreover, Mitra and Chowdhury [8] proved that $(\gamma_A X, u_A) = \upsilon_A X$ when $A(X) \in \sum(X)$. We refer the reader for undefined terms mentioned in this paper to [6].

Let $C(X, \mathbb{C})$ and $C^*(X, \mathbb{C})$ be the rings of all complex-valued continuous functions on X and bounded complex-valued continuous functions on X respectively. An intermediate subring of $C(X, \mathbb{C})$ is a subring of $C(X, \mathbb{C})$ which contains $C^*(X, \mathbb{C})$. For each subset $A(X)$ of $C(X)$, let

$$[A(X)]_c = \{f + ig : f, g \in A(X)\}.$$

If $A(X)$ is a subring of $C(X)$, then $[A(X)]_c$ is a subring of $C(X, \mathbb{C})$. In fact, if $A(X) \in \sum(X)$, then $[A(X)]_c$ is the smallest intermediate subring of $C(X, \mathbb{C})$ containing $A(X)$ [1]. Let $\mathcal{R}[X, \mathbb{C}]$ be the collection of all subrings of $C(X, \mathbb{C})$ that separate points and contain 1. Then it is easy to see that for each $A(X) \in \mathcal{R}[X]$, $[A(X)]_c \in \mathcal{R}[X, \mathbb{C}]$. A subring $A(X, \mathbb{C})$ of $C(X, \mathbb{C})$ is called a c-type subring if there exists a Tychonoff space Y such that $A(X, \mathbb{C})$ is isomorphic to $C(Y, \mathbb{C})$.

In Section 2, we provide a correspondence between ideals in $A(X) \in \mathcal{R}[X]$ and a special category of filters called z_A^γ-filters on the completion $\gamma_A X$. We also introduce the concept of z_A^γ-ideals, extending the notion of z_A^υ-ideals initially presented in [10]. We prove some properties of z-ideals, z_A-ideals and maximal ideals in subrings $A(X) \in \mathcal{R}[X]$ and derive the inter-relationships that exist between them. In Section 3, we prove that $[A(X)]_c$ is a c-type subring of $C(X, \mathbb{C})$ when $A(X) \in \mathcal{R}[X]$ is a c-type subring of $C(X)$. Finally, in Section 4, we show that the completion of X with respect to a uniformity generated by a suitable collection of pseudometrics arising from the functions in $[A(X)]_c$, where $A(X) \in \sum(X)$, is equal to $\upsilon_A X$.

2. z_A^γ-FILTERS AND z_A^γ-IDEALS

Let $A(X)$ be a subring of $C(X)$ and $f \in A(X)$. Then by [8, Theorem 4], f can be extended to a uniformly continuous function $f^{\gamma_A} : \gamma_A X \to \mathbb{R}$. For $I \subseteq A(X)$, let $Z_A^\gamma[I] = \{Z(f^{\gamma_A}) : f \in I\}$ and $Z_A^\gamma[X] = \{Z(f^{\gamma_A}) : f \in A(X)\}$.

Since X is dense in $\gamma_A X$, the following lemma is immediate.

Lemma 2.1. *Let f, $g \in A(X)$. Then*

(1) $(f + g)^{\gamma_A} = f^{\gamma_A} + g^{\gamma_A}$ *and*

(2) $(fg)^{\gamma_A} = f^{\gamma_A} g^{\gamma_A}$.

The following lemma holds as a subring $A(X) \in \mathcal{R}[X]$ is a c-type subring of $C(X)$ if and only if it is isomorphic to $C(\gamma_A X)$ [8].

Lemma 2.2. *Let $A(X) \in \mathcal{R}[X]$ be a c-type subring. Then $f \in A(X)$ is invertible in $A(X)$ if and only if $Z(f^{\gamma_A}) = \phi$.*

Definition 2.3. A non-empty subset $\mathcal{F}$ of $Z_A^\gamma[X]$ is called a z_A^γ-filter on $\gamma_A X$ if the following hold:

(1) $\phi \notin \mathcal{F}$,
(2) if Z_1, $Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$ and
(3) if $Z \in \mathcal{F}$ and $Z' \in Z_A^\gamma[X]$ with $Z' \supseteq Z$, then $Z' \in \mathcal{F}$.

Theorem 2.4. *Let $A(X) \in \mathcal{R}[X]$ be a c-type subring.*

(1) *If I is an ideal in $A(X)$, then $Z_A^\gamma[I]$ is a z_A^γ-filter on $\gamma_A X$.*
(2) *If $\mathcal{F}$ is a z_A^γ-filter on $\gamma_A X$, then $Z_A^{\gamma^{\leftarrow}}[\mathcal{F}] = \{f \in A(X) : Z(f^{\gamma_A}) \in \mathcal{F}\}$ is an ideal in $A(X)$.*

Proof. (1) Clearly $\phi \notin Z_A^\gamma[I]$, for if $\phi = Z(f^{\gamma_A})$ for some $f \in I$, then f would be invertible in $A(X)$ by Lemma 2.2, contradicting that I is a proper ideal. Also, for $Z(f^{\gamma_A})$, $Z(g^{\gamma_A}) \in Z_A^\gamma[I]$, by Lemma 2.1,

$$\begin{aligned} Z(f^{\gamma_A}) \cap Z(g^{\gamma_A}) &= Z((f^{\gamma_A})^2 + (g^{\gamma_A})^2) \\ &= Z((f^2)^{\gamma_A} + (g^2)^{\gamma_A}) \\ &= Z((f^2 + g^2)^{\gamma_A}) \end{aligned}$$

and thus $Z(f^{\gamma_A}) \cap Z(g^{\gamma_A}) \in Z_A^\gamma[I]$. Finally, for $Z(h^{\gamma_A}) \in Z_A^\gamma[I]$ and $Z(k^{\gamma_A}) \in Z_A^\gamma[X]$ such that $Z(k^{\gamma_A}) \supseteq Z(h^{\gamma_A})$, we have

$$Z(k^{\gamma_A}) = Z(h^{\gamma_A}) \cup Z(k^{\gamma_A}) = Z(h^{\gamma_A} k^{\gamma_A}) = Z((hk)^{\gamma_A}).$$

So $Z(k^{\gamma_A}) \in Z_A^\gamma[I]$. Thus $Z_A^\gamma[I]$ is a z_A^γ-filter on $\gamma_A X$.

(2) Clearly by Lemma 2.2, $Z_A^{\gamma^{\leftarrow}}[\mathcal{F}]$ contains no unit of $A(X)$. Let $f, g \in Z_A^{\gamma^{\leftarrow}}[\mathcal{F}]$ and $h \in C(X)$. Then by Lemma 2.1,

$$Z((f-g)^{\gamma_A}) = Z(f^{\gamma_A} - g^{\gamma_A}) \supseteq Z(f^{\gamma_A}) \cap Z(g^{\gamma_A}) \in \mathcal{F}.$$

Also, $Z((hf)^{\gamma_A}) = Z(h^{\gamma_A} f^{\gamma_A}) \supseteq Z(f^{\gamma_A}) \in \mathcal{F}$. Hence $Z_A^{\gamma^{\leftarrow}}[\mathcal{F}]$ is an ideal in $A(X)$. □

Definition 2.5. A z_A^γ-filter on $\gamma_A X$ which is not contained in any other z_A^γ-filter is called a z_A^γ-ultrafilter.

Theorem 2.6. *Let $A(X) \in \mathcal{R}[X]$ be a c-type subring.*

(1) *If M is a maximal ideal in $A(X)$, then $Z_A^\gamma[M]$ is a z_A^γ-ultrafilter on $\gamma_A X$.*
(2) *If $\mathcal{F}$ is a z_A^γ-ultrafilter on $\gamma_A X$, then $Z_A^{\gamma^\leftarrow}[\mathcal{F}]$ is a maximal ideal in $A(X)$.*

Proof. It is easy to see that Z_A^γ and $Z_A^{\gamma^\leftarrow}$ preserve inclusion. Hence the result follows from Theorem 2.4. □

Parsinia [10] defined an ideal I of an intermediate subring $A(X)$ of $C(X)$ to be a z_A^υ-ideal if $Z(f^{\upsilon_A}) \subseteq Z(g^{\upsilon_A})$, $f \in I$ and $g \in A(X)$ imply that $g \in I$. We now give a more general notion of z_A^υ-ideal.

Definition 2.7. Let $A(X \in \mathcal{R}[X]$. An ideal I in $A(X)$ is called a z_A^γ-ideal if whenever $Z(f^{\gamma_A}) \subseteq Z(g^{\gamma_A})$ with $f \in I$ and $g \in A(X)$, then $g \in I$.

Remark 2.8. By [8, Theorem 10], we note that when $A(X) \in \sum(X)$, z_A^γ-ideal reduces to z_A^υ-ideal.

In [10], it is shown that the notion of z_C^υ-ideals and the notion of z-ideals of $C(X)$ are equivalent to each other. Also the notion of $z_{C^*}^\upsilon$-ideals coincides with the notion of z-ideals of $C^*(X)$. We now give the interrelationships that exist between z_A^γ-ideals, z_A-ideals and maximal ideals in a subring $A(X) \in \mathcal{R}[X]$.

Theorem 2.9. *Let $A(X) \in \mathcal{R}[X]$ be a c-type subring of $C(X)$. Then every maximal ideal in $A(X)$ is a z_A^γ-ideal.*

Proof. Let M be a maximal ideal in $A(X)$. Then $Z_A^\gamma[M]$ is a z_A^γ-ultrafilter on $\gamma_A X$. But since $Z_A^{\gamma^\leftarrow}[Z_A^\gamma[M]]$ is a maximal ideal in $A(X)$ and $M \subseteq Z_A^{\gamma^\leftarrow}[Z_A^\gamma[M]]$, we get $M = Z_A^{\gamma^\leftarrow}[Z_A^\gamma[M]]$. Let $Z(f^{\gamma_A}) \subseteq Z(g^{\gamma_A})$ with $f \in M$ and $g \in A(X)$. Since $Z_A^\gamma[M]$ is a z_A^γ-filter, $Z(g^{\gamma_A}) \in Z_A^\gamma[M]$, i.e., $g \in Z_A^{\gamma^\leftarrow}[Z_A^\gamma[M]] = M$. Therefore M is a z_A^γ-ideal. □

Theorem 2.10. *Let $A(X) \in \mathcal{R}[X]$. Then every z_A-ideal in $A(X)$ is a z_A^γ-ideal in $A(X)$.*

Proof. Let I be a z_A-ideal in $A(X)$ and let $Z(f^{\gamma_A}) \subseteq Z(g^{\gamma_A})$, where $f \in I$ and $g \in A(X)$. Then $Z(f) \subseteq Z(g)$. Hence $g \in I$. □

Remark 2.11. The converse of the above theorem is false as provided in [10, Example 2.5].

3. c-TYPE SUBRINGS OF $C(X,\mathbb{C})$

For each member $A(X,\mathbb{C})$ of $\mathcal{R}[X,\mathbb{C}]$, we construct a unique completion of X denoted by $\breve{\gamma}_{A(X,\mathbb{C})}X$. We will show that, when $A(X) \in \mathcal{R}[X]$ is a c-type subring of $C(X)$, $[A(X)]_c$ is a c-type subring of $C(X,\mathbb{C})$.

Let $A(X,\mathbb{C}) \in \mathcal{R}[X,\mathbb{C}]$. For each $f \in A(X,\mathbb{C})$, the function

$$\breve{\psi}_f : X \times X \to \mathbb{R}$$

defined by $\breve{\psi}_f(x,y) = |f(x)-f(y)|$ is clearly a pseudometric on X. Therefore the collection $\{\breve{\psi}_f : f \in A(X,\mathbb{C})\}$ is a subbase for some uniform structure on X. Let us denote this uniform structure by $\breve{u}_{A(X,\mathbb{C})}$. Since $A(X,\mathbb{C})$ separates points, $\breve{u}_{A(X,\mathbb{C})}$ is Hausdorff. Thus by [6, Theorem 15.9], there exists a unique completion of $(X, \breve{u}_{A(X,\mathbb{C})})$, say $(\breve{\gamma}_{A(X,\mathbb{C})}X, \breve{u}_{A(X,\mathbb{C})})$. Here we use the same symbol $\breve{u}_{A(X,\mathbb{C})}$ for the uniform structure on $\breve{\gamma}_{A(X,\mathbb{C})}X$.

Theorem 3.1. *Let $A(X) \in \mathcal{R}[X]$ be a c-type subring of $C(X)$. Then $[A(X)]_c$ is a c-type subring of $C(X,\mathbb{C})$.*

Proof. Since $A(X) \in \mathcal{R}[X]$ is a c-type subring of $C(X)$, by [8, Theorem 9], there exists an isomorphism $H : A(X) \to C(\gamma_A X)$. Let

$$\breve{H} : [A(X)]_c \to C(\gamma_A X, \mathbb{C})$$

be defined by $\breve{H}(f+ig) = H(f) + iH(g)$. Then clearly $\breve{H}$ is an isomorphism. □

Remark 3.2. Let $A(X) \in \mathcal{R}[X]$ and

$$\breve{c}_{A(X,\mathbb{C})}X = \{p \in \beta X : A^p \text{ is a Cauchy } z\text{-filter on } X \text{ with respect to the uniformity } \breve{u}_{A(X,\mathbb{C})}X\}$$

Then $\breve{c}_{A(X,\mathbb{C})}X \subseteq \upsilon_A X$. Here A^p denotes the z-ultrafilter on X that converges to the point p in βX.

Proof. Suppose $p \notin \upsilon_A X$. Then there exists f in $A(X)$ such that $f^*(p) = \infty$, i.e., f is unbounded on every member of A^p. Since $A(X) \subseteq [A(X)]_c$, $f \in [A(X)]_c$ and so $\breve{\psi}_f \in \breve{u}_{A(X,\mathbb{C})}$. Since f is unbounded on every member of A^p, for any $\epsilon > 0$ and for any $Z \in A^p$, there exist x, y in Z such that $\breve{\psi}_f(x,y) \not< \epsilon$. So A^p is not Cauchy, i.e., $p \notin \breve{c}_{A(X,\mathbb{C})}X$. Hence $\breve{c}_{A(X,\mathbb{C})}X \subseteq \upsilon_A X$. □

Remark 3.3. When X is A-compact, $\breve{c}_{A(X,\mathbb{C})}X = \upsilon_A X$. For an intermediate subring $A(X)$ of $C(X)$, Mitra and Chowdhury [8] have shown that

$\gamma_A X = c_A X = \upsilon_A X$, where

$$c_A X = \{p \in \beta X : A^p \text{ is a Cauchy } z\text{-filter on } X \text{ with respect to the uniformity } u_A X\}.$$

4. Intermediate subrings of $C(X,\mathbb{C})$

In this section, we discuss on the completion of X with respect to a suitable collection of pseudometrics arising from each function in $[A(X)]_c$, where $A(X) \in \sum(X)$ and show that such completion is equal to $\upsilon_A X$.

Let $A(X)$ be a subring of $C(X)$. For each function $f \in [A(X)]_c$, we define $\tilde{\psi}_f : X \times X \to \mathbb{R}$ by $\tilde{\psi}_f(x,y) = ||f|(x) - |f|(y)|$ for all x, $y \in X$. Clearly for each x, y, $z \in X$, we have $\tilde{\psi}_f(x,y) \geq 0$, $\tilde{\psi}_f(x,x) = 0$, $\tilde{\psi}_f(x,y) = \tilde{\psi}_f(y,x)$ and

$$\begin{aligned}\tilde{\psi}_f(x,z) &= ||f|(x) - |f|(z)| \\ &= |(|f|(x) - |f|(y)) + (|f|(y) - |f|(z))| \\ &\leq ||f|(x) - |f|(y)| + ||f|(y) - |f|(z))| \\ &= \tilde{\psi}_f(x,y) + \tilde{\psi}_f(y,z),\end{aligned}$$

showing that $\tilde{\psi}_f$ is a pseudometric on X for each $f \in [A(X)]_c$. Therefore the family $\{\tilde{\psi}_f : f \in [A(X)]_c\}$ is a subbase for some uniform structure on X.

Remark 4.1. Since $C^*(X)$ separates points, for distinct points p, p' in X, there exists $f \in C^*(X)$ such that $f(p) \neq f(p')$. If $|f|(p) \neq |f|(p')$, then $\tilde{\psi}_f(p,p') \neq 0$. If $|f|(p) = |f|(p')$, we can choose some constant function k such that $f + k \in C^*(X)$ and $|f+k|(p) \neq |f+k|(p')$ so that $\tilde{\psi}_{f+k}(p,p') \neq 0$.

By Remark 4.1, for each $A(X) \in \sum(X)$, the uniformity generated by the family $\{\tilde{\psi}_f : f \in [A(X)]_c\}$ is Hausdorff. Let this uniform structure be denoted by $\tilde{u}_{[A(X)]_c}$. Then there exists a unique completion of X with respect to $\tilde{u}_{[A(X)]_c}$ and let us denote this complete uniform structure by $(\tilde{\gamma}_{[A(X)]_c}X, \tilde{u}_{[A(X)]_c})$. Here we use the same symbol $\tilde{u}_{[A(X)]_c}$ for the uniform structure on $\tilde{\gamma}_{[A(X)]_c}X$.

For each $A(X) \in \sum(X)$, Mitra and Chowdhury [8] have proved that $\gamma_A X = \upsilon_A X$. Now we aim to prove that for each $A(X) \in \sum(X)$,

$$\tilde{\gamma}_{[A(X)]_c}X = \upsilon_A X.$$

We define $\upsilon_{[A(X)]_c}X = \{p \in \beta X : |f|^*(p) \neq \infty \text{ for all } f \in [A(X)]_c\}$.

Lemma 4.2. *For each $A(X) \in \sum(X)$, $\upsilon_A X = \upsilon_{[A(X)]_c}X$.*

Proof. Let $p \in \upsilon_A X$ and $f \in [A(X)]_c$. By [1, Theorem 2.3], $|f| \in A(X)$. Therefore since $p \in \upsilon_A X$, $|f|^*(p) \neq \infty$. Thus $p \in \upsilon_{[A(X)]_c}X$. Conversely, let

$p \in \upsilon_{[A(X)]_c}X$ and $f \in A(X)$. Since f also belongs to $[A(X)]_c$, $|f|^*(p) \neq \infty$ as $p \in \upsilon_{[A(X)]_c}X$. As $f \in A(X)$, $|f|(x) = f(x)$ or $-f(x)$ for any $x \in X$. Hence $|f|^*(p) = f^*(p)$ or $-f^*(p)$. Hence $f^*(p) \neq \infty$. Therefore $p \in \upsilon_A X$. $\square$

Theorem 4.3. *Let $A(X) \in \sum(X)$ and*

$$\tilde{c}_{[A(X)]_c}X = \{p \in \beta X : A^p \text{ is a Cauchy z-filter on } X \text{ with respect to the uniformity } \tilde{u}_{[A(X)]_c}\}.$$

Then $\tilde{c}_{[A(X)]_c}X = \upsilon_{[A(X)]_c}X$.

Proof. Let $p \in \beta X$ be such that $p \notin \upsilon_{[A(X)]_c}X$. Then there exists $f \in [A(X)]_c$ such that $|f|^*(p) = \infty$. So for any $\epsilon > 0$ and any $Z \in A^p$, $||f|(x) - |f|(y)| > \epsilon$, for some $x, y \in Z$. That is, $\tilde{\psi}_f$-diameter of Z is greater than ϵ. Thus $p \notin \tilde{c}_{[A(X)]_c}X$.

For the reverse inclusion, let p be a point in $\upsilon_{[A(X)]_c}X$. Let $f \in [A(X)]_c$. Since $|f|^*(p) \neq \infty$, for any $\epsilon > 0$, there exists a zero-set neighborhood of p in βX, say Z', such that

$$|f|^*(Z') \subseteq (|f|^*(p) - \tfrac{\epsilon}{2}, |f|^*(p) + \tfrac{\epsilon}{2}).$$

Let $Z = Z' \cap X$. Then $p \in cl_{\beta X} Z$ as X is dense in βX and $p \in Z'$. Hence by [6, Theorem 6.5 (c)], $Z \in A^p$. Since $Z \subseteq X$, for all x, $y \in Z$, we have

$$\tilde{\psi}_f(x, y) = ||f(x)| - |f(y)|| = ||f|^*(x) - |f|^*(y)| < \epsilon.$$

So $\tilde{\psi}_f$-diameter of Z is less than ϵ. Let $d \in \tilde{u}_{[A(X)]_c}$. Since $\{\tilde{\psi}_f : f \in [A(X)]_c\}$ is a subbase for the uniform structure $\tilde{u}_{[A(X)]_c}$, so for any $\epsilon > 0$, there exist $f_1, f_2, \ldots, f_n \in [A(X)]_c$ and $\delta > 0$ such that $\bigvee_{i=1}^n \tilde{\psi}_{f_i}(x, y) \leq \delta$ implies that $d(x, y) \leq \epsilon$ for all $x, y \in X$. Again for each f_i, there exists $Z_i \in A^p$ such that for all x, $y \in Z_i$, $\tilde{\psi}_{f_i}(x, y) < \delta$. Let $G = \bigcap_{i=1}^n Z_i$. Then $G \in A^p$ and for all x, $y \in G$, $\bigvee_{i=1}^n \tilde{\psi}_{f_i}(x, y) \leq \delta$ and hence $d(x, y) \leq \epsilon$. This shows that A^p is Cauchy with respect to the uniformity $\tilde{u}_{[A(X)]_c}$. Therefore $p \in \tilde{c}_{[A(X)]_c}X$. Thus $\upsilon_{[A(X)]_c}X \subseteq \tilde{c}_{[A(X)]_c}X$. $\square$

Theorem 4.4. *For any $A(X) \in \sum(X)$, $\tilde{\gamma}_{[A(X)]_c}X = \upsilon_A X$.*

Proof. By Remark 4.1, for distinct points p, p' in X, there exists $f \in [A(X)]_c$ such that $\tilde{\psi}_f(p, p') \neq 0$. Therefore by [6, Remark 15.9], $\tilde{c}_{[A(X)]_c}X = \tilde{\gamma}_{[A(X)]_c}X$. But by Theorem 4.3, $\tilde{c}_{[A(X)]_c}X = \upsilon_{[A(X)]_c}X$ and by Lemma 4.2,

$$\upsilon_A X = \upsilon_{[A(X)]_c}X.$$

Therefore $\tilde{\gamma}_{[A(X)]_c}X = \upsilon_A X$. $\square$

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SOME RESULTS ON SUBRINGS OF $C(X)$ AND $C(X,\mathbb{C})$

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برخی نتایج درباره‌ی زیرحلقه‌های $C(X)$ و $C(X,\mathbb{C})$

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فرض کنید X یک فضای تیخونوف باشد و $\mathcal{R}[X]$ مجموعه‌ی تمامی زیرحلقه‌های $C(X)$ باشد که نقاط X را از یکدیگر متمایز می‌کنند و تابع همانی ۱ را در بر دارند. در این مقاله، تناظری میان ایده‌آل‌های $A(X)\in\mathcal{R}[X]$ و z_A^γ-فیلترهای روی کامل‌سازی X نسبت به ساختار یکنواختی که از توابع موجود در $A(X)$ حاصل می‌شود، برقرار می‌کنیم. همچنین برخی ویژگی‌های z-ایده‌آل‌ها، z_A-ایده‌آل‌ها و ایده‌آل‌های ماکسیمال را در این‌گونه زیرحلقه‌های $C(X)$ بررسی می‌کنیم. برای هر زیرمجموعه‌ی $A(X)$ از $C(X)$، تعریف می‌کنیم: $[A(X)]_c=\{f+ig : f,g\in A(X)\}$. نشان می‌دهیم که هرگاه $A(X)\in\mathcal{R}[X]$ یک زیرحلقه از c-تایپ در $C(X)$ باشد، آن‌گاه $[A(X)]_c$ نیز یک زیرحلقه c-تایپ از $C(X,\mathbb{C})$ خواهد بود. در پایان، برای یک زیرحلقه‌ی میانی $A(X)$ از $C(X)$، ثابت می‌کنیم که کامل‌سازی X نسبت به ساختار یکنواخت مناسبی که از $[A(X)]_c$ به دست می‌آید، با $\upsilon_A X$، A-فشرده‌سازی فضای X، برابر است.

کلمات کلیدی: زیرحلقه‌های c-تایپ، فشرده‌سازی استون-چک، ساختار یکنواخت، z_A^γ-فیلتر، z_A^γ-ایده‌آل.