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GENERALIZED SHERMAN-MORRISON-WOODBURY FORMULA FOR THE G-DRAZIN INVERSE

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Abstract. In this paper, we present generaized Sherman-Morrison-Woodbury formula for the g-Drazin inverse in a Banach algebra. New results for the g-Drazin invertibility of a modified element $a - cdb$ with the generalized Schur complement $d^d - ba^dc$ are given. The g-Drazin invertibility of an operator matrix with nonsingular generalized Schur complement under weaker restrictions is thereby established.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. Let

$$\mathcal{A}^{qnil} := \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}.$$

As is well known, we have

$$a \in \mathcal{A}^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} \to 0.$$

An element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $b \in \mathcal{A}$ such that $ab = ba, b = bab, a - a^2b \in \mathcal{A}^{qnil}$. Such b is unique, if it exists, and denote it by a^d. The Drazin inverse a^D of a is defined by replacing the preceding $\mathcal{A}^{qnil}$ into the set of all nilpotents in $\mathcal{A}$. The Drazin and g-Drazin inverses are useful in matrix and operator theory. It has been applied in many fields such as ordinary differential equations, statistics and probability, Marcov chain, etc (see [2, 10, 14, 21]).

If A and D are invertible matrices and B and C are matrices such that $D-BA^{-1}C$ and $A-CD^{-1}B$ are invertible, the Sherman-Morrision-Woodbury formula is expressed as

$$(A - CD^{-1}B)^{-1} = A^{-1} + A^{-1}C(D - BA^{-1}C)^{-1}BA^{-1}.$$

This formula has lots of applications in statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations. Recently, Sherman-Morrison-Woodbury formula was extended to Drazin (g-Drazin) inverse, e.g., [4, 5, 13, 15, 16, 19]. In [20], Zhang and Du derived

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formulas for the Drazin inverse of $A - CD^DB$ in terms of the Drazin inverse of $D - BA^DC$ for complex matrices. This inspires us to further study generalized Sherman-Morrison-Woodbury formula for the g-Drazin inverse in a Banach algebra. Applications to an operator matrix with nonsingular generalized Schur complement are thereby obtained.

Let $a, d, z := d^d + ba^dc \in \mathcal{A}^d$. In Section 2, we investigate wider conditions under which $a + cdb \in \mathcal{A}^d$ with the generalized Schur complement $d^d - ba^dc$. New Sherman-Morrision-Woodbury formula for the g-Drazin inverse is established in a Banach algebra. These also give generalizations of [5, Theorem 2.1], [20, Theorem 2.5] and [7, Theorem 2.3].

The generalized Schur complement of a in $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ is stated as $d - ca^db$. Many authors studied the Drazin inverse of a complex matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$ when the generalized Schur complement $D - CA^DB$ is nonsingular (see [6, 9]). Finally, in Section 3, as applications, we generalized these results to various conditions for the g-Drazin invertibility of the operator matrix M over a Banach algebra. These also extend [6, Theorem 3.1] to the wider case by a new route.

Throughout the paper, all Banach algebras are complex with an identity. We use $\mathcal{A}^{-1}, \mathcal{A}^D$ and $\mathcal{A}^d$ to denote the set of all invertible, Drazin and g-Drazin invertible elements in $\mathcal{A}$, respectively. Let $a^\pi = 1 - aa^d$ for any $a \in \mathcal{A}^d$. We denote by $\mathbb{C}^{n\times n}$ the Banach algebra of all $n \times n$ complex matrices.

2. Main results

Let a, d and z have g-Drazin inverse. Set $s_a = aa^dsaa^d$ and $z_d = aa^dzaa^d$. We begin with

Lemma 2.1. *Let a, d and s_a have g-Drazin inverses. If $a^\pi cdb = 0$, then $s_a = saa^d$ and*

$$s^d = s_a^d + \sum_{i=0}^{\infty} (s_a^d)^{i+2} sa^i a^\pi.$$

Proof. Since $a^\pi cdb = 0$, we have $s_a = saa^d$, and so $s = sa^\pi + saa^d = sa^\pi + s_a$. Since $a^\pi s = aa^\pi \in \mathcal{A}^{qnil}$, it follows by [2, Lemma 6.4.10] that $sa^\pi \in \mathcal{A}^{qnil}$. As

$(sa^\pi)s_a = 0$, it follows by [11, Lemma 1.2] that

$$\begin{array}{rcl} s^d & = & s_a^d + \sum_{i=1}^{\infty}(s_a)^{i+1}(sa^\pi)^i \\ & = & s_a^d + \sum_{i=0}^{\infty}(s_a)^{i+2}sa^ia^\pi, \end{array}$$

as asserted. □

Let $h = ba^d, k = a^dc$. For future use, we now record the following.

Lemma 2.2. *Let $s_a = aa^dsaa^d$ and $m = a^d - kz^dh$. Then the following are equivalent:*

(1) $s_am = aa^d$.
(2) $kd^\pi z^dh = kdz^\pi h$.

Proof. We compute that

$$\begin{aligned} aa^d - akz^dh + akdh - akd(z - d^d)z^dh &= am + aa^dcdh - aa^dcd(z - d^d)z^dh \\ &= am + aa^dcd[1 - (z - d^d)z^d]h \\ &= am + aa^dcd[1 - ba^dcz^d]ba^d \\ &= am + aa^dcdba^d[1 - cz^dba^d] \\ &= a(aa^d)m + aa^dcdb[a^d - kz^dh] \\ &= (aa^d)(a + cdb)(aa^d)m \\ &= s_am. \end{aligned}$$

Hence, $s_am = aa^d$ if and only if

$$\begin{aligned} ak(d^\pi z^d - dz^\pi)h &= ak[z^d - d + d(z - d^d)z^d]h \\ &= akz^dh - akdh + akd(z - d^d)z^dh \\ &= 0, \end{aligned}$$

i.e., $kd^\pi z^dh = kdz^\pi h$. □

Likewise, we can derive

Lemma 2.3. *Let $s_a = aa^dsaa^d$ and $m = a^d - kz^dh$. Then the following are equivalent:*

(1) $ms_a = aa^d$.
(2) $kz^\pi dh = kz^dd^\pi h$.

It is convenient at this state to prove the main result of this section.

Theorem 2.4. *Let a, d and z have g-Drazin inverses. If $a^\pi cdb = 0$, $kd^\pi z^d h = kdz^\pi h$ and $kz^\pi dh = kz^d d^\pi h$, then s has g-Drazin inverse. In this case,*

$$s^d = a^d - kz^d h + \sum_{i=0}^{\infty} (a^d - kz^d h)^{i+2} s a^i a^\pi.$$

Proof. In view of Lemma 2.2 and Lemma 2.3, $s_a m = aa^d = m s_a$. Hence, s_a has group inverse and $s^\# = m$. Since $a^\pi cdb = 0$, we have

$$s^d = s_a^d + \sum_{i=0}^{\infty} (s_a^d)^{i+2} s a^i a^\pi.$$

Therefore $s^d = a^d - kz^d h + \sum_{i=0}^{\infty} (a^d - kz^d h)^{i+2} s a^i a^\pi$, as asserted. □

Corollary 2.5. *Let a, d and z have g-Drazin inverses. If $a^\pi c = 0$, $cd^\pi = 0, d^\pi b = 0, cz^\pi = 0, z^\pi b = 0$, then s has g-Drazin inverse. In this case,*

$$s^d = a^d - kz^d h + \sum_{i=0}^{\infty} (a^d - kz^d h)^{i+2} s a^i a^\pi.$$

Let $A, B, C, D \in \mathbb{C}^{n\times n}$. Let $S = A + CDB, Z = D^D + BA^D C$. Set $S_A = AA^D SAA^D$ and $H = BA^D, K = A^D C$.

Corollary 2.6. *If $A^\pi CDB = 0, KD^\pi Z^D H = KDZ^\pi H$, then*

$$S^D = A^D - KZ^D H + \sum_{i=0}^{m-1} (A^D - KZ^D H)^{i+2} S A^i A^\pi,$$

where $m = i(A)$.

Proof. For a complex matrix, the Drazin and g-Drazin inverses coincide with each other. Therefore we complete the proof by Theorem 2.4. □

Example 2.7. Let

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C}).$$

By a simple computing, we deduce that, $z = d = I_2, k = c, h = b$ *and* $s = a$. Since s is invertible, so $s^d = s^{-1} = I_2$. Also we have, $a^\pi cdb = 0$, $kd^\pi z^d h = kdz^\pi h$ and $kz^\pi dh = kz^d d^\pi h$. Then by Theorem 2.4,

$$s^d = a^d - kz^d h + \sum_{i=0}^{\infty} (a^d - kz^d h)^{i+2} s a^i a^\pi.$$

As $a^\pi = 0$ and $kz^dh = 0$, we conclude that, $s^d = I_2$. This confirms the correctness of the previous theorem.

Lemma 2.8. *Let a, d and s_a have g-Drazin inverses. If $cdba^\pi = 0$, then $s_a = aa^ds$ and*

$$s^d = s_a^d + \sum_{i=0}^{\infty} a^\pi a^i s(s_a^d)^{i+2}.$$

Proof. Since $cdba^\pi = 0$, we have $s_a = aa^ds$; hence, $s = a^\pi s + aa^ds = s_a + a^\pi s$. Since $sa^\pi = aa^\pi \in \mathcal{A}^{qnil}$, it follows by the Cline's formula that $a^\pi s \in \mathcal{A}^{qnil}$. Since $s_a(a^\pi s) = 0$, by using by [11, Lemma 1.2] that

$$s^d = s_a^d + \sum_{i=1}^{\infty}(a^\pi s)^i(s_a)^{i+1} = s_a^d + \sum_{i=0}^{\infty} a^\pi a^i s(s_a^d)^{i+2},$$

as required. □

Theorem 2.9. *Let a, d and z have g-Drazin inverses. If $cdba^\pi = 0$, $kd^\pi z^dh = kdz^\pi h$ and $kz^\pi dh = kz^dd^\pi h$, then s has g-Drazin inverse. In this case,*

$$s^d = a^d - kz^dh + \sum_{i=0}^{\infty} a^\pi a^i s(a^d - kz^dh)^{i+2}.$$

Proof. According to Lemma 2.2 and Lemma 2.3, we have $s_am = aa^d = ms_a$, and then $s^d = m$. Since $cdba^\pi = 0$, by Lemma 2.7, we have

$$s^d = s_a^d + \sum_{i=0}^{\infty} a^\pi a^i s(s_a^d)^{i+2}.$$

Consequently, $s^d = a^d - kz^dh + \sum_{i=0}^{\infty} a^\pi a^i s(a^d - kz^dh)^{i+2}$, as asserted. □

Corollary 2.10. *Let a, d and z have g-Drazin inverses. If $ba^\pi = 0$, $cd^\pi = 0, d^\pi b = 0, cz^\pi = 0, z^\pi b = 0$, then s has g-Drazin inverse. In this case,*

$$s^d = a^d - kz^dh + \sum_{i=0}^{\infty} a^\pi a^i s(a^d - kz^dh)^{i+2}.$$

Proof. Since $cd^\pi = 0, d^\pi b = 0, cz^\pi = 0$ and $z^\pi b = 0$, we check that

$$kd^\pi z^dh = 0 = kdz^\pi h, kz^\pi dh = 0 = kz^dd^\pi h.$$

According to Theorem 2.9, we complete the proof. □

As an immediate consequence of Theorem 2.9, we now derive

Corollary 2.11. *Let* $A, D, Z \in \mathbb{C}^{n\times n}$ *have Drazin inverses. If*
$$CDBA^{\pi} = 0, KD^{\pi}Z^{D}H = KDZ^{\pi}H,$$
then
$$S^{D} = A^{D} - KZ^{D}H + \sum_{i=0}^{m-1} A^{\pi}A^{i}S(A^{D} - KZ^{D}H)^{i+2},$$
where $m = i(A)$

We are ready to prove:

Theorem 2.12. *Let* $a, d, z \in \mathcal{A}^{d}$*. If*
$$a^{\pi}cdb = 0, d^{\pi}ba^{d}c = 0$$
and $dba^{d}a = d^{d}dba^{d}$*, then* $s \in \mathcal{A}^{d}$*.*

Proof. Clearly, we check that
$$\begin{pmatrix} a^2a^d & -aa^dc \\ dd^dbaa^d & d^d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -dbaa^d & 1 \end{pmatrix} = \begin{pmatrix} s_a & -aa^dc \\ 0 & -d^d \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ -dbaa^d & 1 \end{pmatrix}\begin{pmatrix} a^2a^d & -aa^dc \\ dd^dbaa^d & d^d \end{pmatrix} = \begin{pmatrix} a^2a^d & -aa^dc \\ -dba^2a^d + dd^dbaa^d & dbaa^dc + d^d \end{pmatrix}.$$
By hypothesis, we have
$$dbaa^dc + d^d = d^ddba^dc + d^ddd^d = dd^dz,$$
$$-dba^2a^d + dd^dbaa^d = -dba^2a^d + dba^da^2 = 0.$$
Since $d^{\pi}ba^dc = 0$, we see that $d^{\pi}z = d^{\pi}(d^d - ba^dc) = 0$. Since $z \in \mathcal{A}^d$, it follows by [8, Theorem 2.1] that $zdd^d \in \mathcal{A}$ and $(zdd^d)^d = z^ddd^d$. By using Cline's formula, we have $(dd^dz)^d = dd^d(z^ddd^d)^2z = dd^dz^d$. Thus, $\begin{pmatrix} a^2a^d & -aa^dc \\ 0 & dd^dz \end{pmatrix}$ has g-Drazin inverse. Then we have a $y \in \mathcal{A}$ such that
$$\begin{pmatrix} a^2a^d & -aa^dc \\ 0 & dd^dz \end{pmatrix}^d = \begin{pmatrix} a^d & y \\ 0 & dd^dz^d \end{pmatrix}.$$
Hence, we have
$$\begin{aligned} &\begin{pmatrix} s_a & -aa^dc \\ 0 & -d^d \end{pmatrix}^d \\ &= \begin{pmatrix} a^2a^d & -aa^dc \\ dd^dbaa^d & d^d \end{pmatrix}\begin{pmatrix} a^d & y \\ 0 & dd^dz^d \end{pmatrix}^2\begin{pmatrix} 1 & 0 \\ -dbaa^d & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^2a^d & -aa^dc \\ dd^dbaa^d & d^d \end{pmatrix}\begin{pmatrix} (a^d)^2 & a^dy + ydd^dz^d \\ 0 & dd^d(z^d)^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -dbaa^d & 1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a^2a^d & -aa^dc \\ dd^dbaa^d & d^d \end{pmatrix} \begin{pmatrix} (a^d)^2 - [a^dy + ydd^dz^d]dbaa^d & a^dy + ydd^dz^d \\ -dd^d(z^d)^2dbaa^d & dd^d(z^d)^2 \end{pmatrix}.$$

Therefore

$$s_a^d = a^d + [aa^dcdd^d(z^d)^2d - aa^dy - a^2a^dydd^dz^d]dbaa^d.$$

Since $a^\pi cdb = 0$. By virtue of Lemma 2.1, we conclude that $s \in \mathcal{A}^d$, as asserted. □

Corollary 2.13. *Let $a, d \in \mathcal{A}^d$. If $a^\pi c = 0$ and $ba^d = 0$, then $s \in \mathcal{A}^d$.*

Proof. This is obvious by Theorem 2.12. □

3. APPLICATIONS

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$. We come now to apply the preceding results to the matrix M with nonsingular generalized Schur complement $d - ca^db$. We leave the detailed formula of the g-Drazin inverse of M to the interested readers as they can be derived by some straightforward computations according to our proof.

Lemma 3.1. *Let $a \in \mathcal{A}^d$. If $a^\pi b = 0$ and $d - ca^db$ is invertible, then M has g-Drazin inverse.*

Proof. We easily check that

$$M = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -b & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} z &:= \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}^d - \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^d \begin{pmatrix} -b & 0 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -d & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -b & 0 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} ca^db - d & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since $d - ca^db \in \mathcal{A}^{-1}$, we see that $z \in M_2(\mathcal{A})$ is invertible. Then $z^\pi = 0$. Since $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$ is invertible, we have $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}^\pi = 0$. We easily check that $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^\pi \begin{pmatrix} -b & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -a^\pi b & 0 \\ 0 & 0 \end{pmatrix} = 0$. In light of Corollary 2.5, M has g-Drazin inverse. □

Theorem 3.2. *Let* $a \in \mathcal{A}^d$*. If*

$$aa^\pi bc = 0, ca^\pi bc = 0, aa^\pi bd = 0, ca^\pi bd = 0$$

and $d - ca^d b$ *is invertible, then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} a & aa^d b \\ c & d \end{pmatrix}, Q = \begin{pmatrix} 0 & a^\pi b \\ 0 & 0 \end{pmatrix}.$$

Obviously, $a^\pi(aa^d b) = 0$ and $d - ca^d(aa^d)b = d - ca^d b \in \mathcal{A}^{-1}$. Applying Lemma 3.1 to P, P has g-Drazin inverse. Clearly, $Q^2 = 0$. Since $aa^\pi bc = 0, ca^\pi bc = 0, aa^\pi bd = 0$ and $ca^\pi bd = 0$, we have

$$\begin{aligned} PQP &= \begin{pmatrix} a & aa^d b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & a^\pi b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & aa^d b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & aa^\pi b \\ 0 & ca^\pi b \end{pmatrix} \begin{pmatrix} a & aa^d b \\ c & d \end{pmatrix} \\ &= 0. \end{aligned}$$

According to [2, Corollary 15.2.4], M has g-Drazin inverse, as asserted. □

Example 3.3. Let

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C}).$$

It is clear that

$$aa^\pi bc = 0, ca^\pi bc = 0, aa^\pi bd = 0, ca^\pi bd = 0$$

and $d - ca^d b$ is invertible, so $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has g-Drazin inverse. Also as $abc = 0, abd = 0, cbc = 0$ and $cbd = 0$, [18, Theorem 3.2] confirms the g-Drazin invertibility of M.

Corollary 3.4. *Let* $d \in \mathcal{A}^d$*. If*

$$dd^\pi cb = 0, bd^\pi cb = 0, dd^\pi ca = 0, bd^\pi ca = 0$$

and $a - bd^d c$ *is invertible, then* M *has g-Drazin inverse.*

Proof. By using Theorem 3.2, $N := \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ has g-Drazin inverse. Obviously, we have

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and therefore

$$M^d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N^d \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

hence the result. □

For a complex, the Drazin and g-Drazin inverses coincide with each other. The following is a generalization of [6, Theorem 3.1].

Corollary 3.5. *Let $a \in \mathcal{A}^d$. If $aa^\pi b = 0, ca^\pi b = 0$ and $d - ca^d b$ is invertible, then M has g-Drazin inverse.*

Proof. This is obvious by Theorem 3.2. □

Lemma 3.6. *Let $a \in \mathcal{A}^d$. If $ca^\pi = 0$ and $d - ca^d b$ is invertible, then M has g-Dra inverse.*

Proof. As in the proof of Lemma 3.1, we have

$$\begin{aligned} M &= \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -b & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \\ z &:= \begin{pmatrix} ca^d b - d & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

By hypothesis, we check that

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^\pi = \begin{pmatrix} ca^\pi & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

According to Corollary 2.9, M has g-Drazin inverse. □

Theorem 3.7. *Let $a \in \mathcal{A}^d$. If $bca^\pi a = 0, bca^\pi b = 0, dca^\pi a = 0, dca^\pi b = 0$ and $d - ca^d b$ is invertible, then M has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} a & b \\ caa^d & d \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ ca^\pi & 0 \end{pmatrix}.$$

Clearly, $Q^2 = 0$. Since $bca^\pi a = 0, bca^\pi b = 0, dca^\pi a = 0$ and $dca^\pi b = 0$, we have

$$\begin{aligned} PQP &= \begin{pmatrix} a & b \\ caa^d & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ ca^\pi & 0 \end{pmatrix} \begin{pmatrix} a & b \\ caa^d & d \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ caa^d & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ ca^\pi a & ca^\pi b \end{pmatrix} \\ &= 0. \end{aligned}$$

Clearly, $(caa^d)a^\pi = 0$ and $d - (caa^d)a^d b = d - ca^d b \in \mathcal{A}^d$. Applying Lemma 3.4 to P, P has g-Drazin inverse. In light of [2, Corollary 15.2.4], M has g-Drazin inverse, as desired. □

Corollary 3.8. *$abd^\pi d = 0, abd^\pi c = 0, cbd^\pi d = 0, cbd^\pi c = 0$ and $a - bd^d c$ is invertible, then M has g-Drazin inverse.*

Proof. Applying Theorem 3.6 to the matrix $N := \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Then the proof is true as in the proof of Corollary 3.3. □

4. OPEN PROBLEMS

We end our work with the following questions of some intrest and importance.

(i) Does generalized Sherman-Morrison-Woodbury formula work for the generalized Drazin inverses of elements over rings?

(ii) How can we apply generalized Sherman-Morrison-Woodbury formula for other generalized inverses, speccially for pseudo n-strong Drazin inverses, which has been extensively studied in [3]?

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References

1. H. Chen and M. S. Abdolyousefi, Generalized Jacobson's lemma in a Banach algebra, *Comm. Algebra*, **49** (2021), 3263–3272.
2. H. Chen and M. S. Abdolyousefi, *Theory of clean rings and matrices*, World Scientific, Hackensack, NJ, 2022.
3. J. Cui, P. Danchev and Y. Zeng, Some new results on pseudo n-strong Drazin inverses in rings, (2023), https://doi.org/10.48550/arXiv.2312.02347.
4. C. Y. Deng, A generalization of the Sherman-Morrison-Woodbury formula, *Appl. Math. Lett.*, **24** (2011), 1561–1564.
5. E. Dopazo and M. F. Martinez-Serrano, On deriving the Drazin inverse of a modified matrix, *Linear Algebra Appl.*, **438** (2013), 1678–1687.
6. R. Hartwig, X. Li and Y. Wei, Representations for the Drazin inverse of a 2×2 block matrix, *SIAM J. Matrix Anal. Appl.*, **27** (2006), 757–771.
7. M. Z. Kolundzija, Generalized Sherman-Morrison-Woodbury formula for the generalized Drazin inverse in Banach algebra, *Filomat*, **31** (2017), 5159–5167.

8. X. Liu, X. Qin and J. Benitez, New additive results for the generalized Drazin inverse in a Banach algebra, *Filomat*, **30** (2016), 2289–2294.
9. M. F. Martinez-Serano and N. Castro-Gonzalez, On the Drazin inverse of block matrices and generalized Schur complement, *Appl. Math. Comput.*, **215** (2009), 2733–2740.
10. D. Mosić, Additive results for the generalized Drazin inverse in a Banach algebra, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), 1465–1478.
11. D. Mosić, Expressions for the generalized Drazin inverse of a block matrix in a Banach algebra, *Int. J. Comput. Math.*, **91** (2014), 514–526.
12. D. Mosić, Extensions of Jacobson's lemma for Drazin inverses, *Aequationes Math.*, **91** (2017), 419–428.
13. D. Mosić, Some results on the Drazin inverse of a modified matrix, *Calcolo*, **50** (2013), 305–311.
14. D. Mosić and D. S. Djordjevic, Block representations of generalized Drazin inverse, *Appl. Math. Comput.*, **331** (2018), 200–209.
15. A. Shakoor, H. Yang and I. Ali, Some presentations for the Drazin inverse of a modified matrix, *Calcolo*, **51** (2014), 505–514.
16. L. Stojković and D. Mosić, Expressions for the Drazin inverse of a modified matrix, *Filomat*, **32** (2018), 6387–6394.
17. K. Yan, Q. Zeng and Y. Zhu, Generalized Jacobson's lemma for Drazin inverses and its applications, *Linear Multilinear Algebra*, **68** (2020), 81–93.
18. H. Yang and X. Liu, The Drazin inverse of the sum of two matrices and its applications, *J. Comput. Appl. Math.*, **235** (2011), 1412–1417.
19. D. Zhang, Representations for the Drazin inverse of a modified matrix, *Filomat*, **29** (2015), 853–863.
20. D. Zhang and X. Du, Representations for the Drazin inverse of a modified matrix, *Filomat*, **29** (2015), 853–863.
21. D. Zhang and D. Mosić, Explicit formulae for the generalized Drazin inverse of block matrices over a Banach algebra, *Filomat*, **32** (2018), 5907–5917.

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GENERALIZED SHERMAN–MORRISON–WOODBURY FORMULA FOR THE g-DRAZIN INVERSE

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تعمیم فرمول وودباری-موریسون-شرمن برای معکوس g-درازین

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ما در این مقاله به تعمیم فرمول وودباری-موریسون-شرمن برای معکوس g-درازین در یک جبر باناخ می‌پردازیم. نتایج جدیدی برای معکوس‌پذیری g-درازین عنصر $a - cdb$ با مکمل شور تعمیم یافته $d^d - ba^dc$ ارائه شده است. تحت شرایط ضعیف‌تر، معکوس‌پذیری g-درازین از یک ماتریس عملگر با مکمل شور تعمیم یافته غیر منفرد بدست آمده است.

کلمات کلیدی: فرمول وودباری-موریسون-شرمن، معکوس g-درازین، معکوس درازین، جبر باناخ.