

TORSION MAXIMAL SUBGROUPS OF $GL_n(D)$

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ABSTRACT. Let D be a division ring over its center F . Let $GL_n(D)$ be the general linear group over D . In this article we prove that if $GL_n(D)$ contains a torsion maximal subgroup M , then $D = F$, $\text{char}F = p > 0$ and F is algebraic over its prime subfield \mathbb{F}_p whenever one of the following conditions occurs: (1) D is algebraic over F ; (2) there exists an element $a \in M$ such that $C_{M_n(D)}(F[a])$ is algebraic over F .

1. INTRODUCTION

The structure of linear groups is now well-understood, but the structure of skew linear group is completely different. The most important results concerning such groups can be found in [12], [23] and [24]. In some papers the structure of maximal subgroups of general skew linear groups is investigated. For some recently related results, see [1], [2], [6], [9] and [20]. By Snider's Theorem ([23, p. 207]), a soluble absolutely irreducible skew linear group is abelian by-locally finite. Also, by Theorem 3.7.21 of [23, p. 215], a nilpotent absolutely irreducible skew linear group is center by-locally finite. In [1], the following conjecture appears:

Conjecture. Let D be a division ring and M a torsion maximal subgroup of $GL_n(D)$; then $D = F$, $\text{char}F = p > 0$ and F is algebraic over \mathbb{F}_p .

The following result is obtained in [9]:

Let D be a division ring with center F and M be a maximal subgroup of $GL_n(D)$. If $D \neq F$ or $n > 1$, then $M/M \cap F^*$ cannot be locally finite unless $\text{char}F = p > 0$ and either:

- (1) $[D : F] = p^2$, $n = 1$, and $M \cup \{0\}$ is a maximal subfield of D , or
- (2) $D = F$, $n = p$, and $M \cup \{0\}$ is a maximal subfield of $M_p(F)$, or
- (3) $D = F$ and F is locally finite.

In this paper we continue the study of the algebraic properties of torsion (or locally finite) maximal subgroups of $GL_n(D)$ and $SL_n(D)$, where D is a division ring.

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The question on the existence of non-cyclic free subgroups in linear groups over a field was studied by Tits in [25]. The main theorem of Tits asserts that in the characteristic 0, every subgroup of the general linear group $GL_n(F)$ over a field F either contains a non-cyclic free subgroup or is soluble-by-finite, and the same conclusion for finitely generated subgroups in the case of prime characteristic. Lichtman in [18] showed that there exists a finitely generated group which is not soluble-by-finite and does not contain a non-cyclic free subgroup. See also [8], [11] and [10].

The question of the existence of maximal subgroups in a division ring is difficult and it has not been settled completely. Despite this, various aspects of maximal subgroups in division rings have been studied substantially by many authors (see [1], [3], [13] and [17]). For instance, it was shown in [13] that there is a close connection between the question of the existence of maximal subgroups in the multiplicative group of a finite dimensional division algebra and the Albert's conjecture concerning the cyclicity of division algebras of prime degree. Moreover, it can also be shown that the monomial subgroup of $GL_n(D)$ is maximal (see [17]). It was proved that $\mathbb{C}^* \cup \mathbb{C}^*j$ is a soluble maximal subgroup of the multiplicative group of the real quaternion division algebra (see [19]). In [1] it was shown that for any division ring D ,

$$H := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(D), B \in M_{r,n-r}(D), C \in GL_{n-r}(D) \right\},$$

is a maximal subgroup of $GL_n(D)$. Also, in [17] the following result was obtained:

Let $n \geq 2$, D be a division ring and N be the monomial subgroup of $G = GL_n(D)$. Then N is maximal in G except when $G = GL_2(\mathbb{F}_3)$ or $G = GL_2(\mathbb{F}_5)$.

We obtained in this way the following result:

Let n be a natural number and F be an n -divisible field. Then, we have the following results:

- (1) $GL_n(F) = F^*SL_n(F)$.
- (2) $GL_n(F)/F^* \cong SL_n(F)/Z(SL_n(F)) = PSL_n(F)$. In particular, when $F \neq \mathbb{F}_2, \mathbb{F}_3$, $GL_n(F)/F^*$ is a simple group and there exists a one-to-one correspondence between maximal subgroups of $GL_n(F)$ containing F^* and maximal subgroups of $SL_n(F)$ containing $Z(SL_n(F))$.
- (3) $(F^* \wr S_n) \cap SL_n(F)$ is a maximal subgroup of $SL_n(F)$.

(4) When

$$H := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(F), B \in M_{r,n-r}(F), C \in GL_{n-r}(F) \right\},$$

we have $H \cap SL_n(F)$ is a maximal subgroup of $SL_n(F)$.

2. NOTATIONS AND CONVENTIONS

We recall here some of the notations that we will need throughout this article. Given a subset S and a subring K of a ring R , the subring generated by K and S is denoted by $K[S]$. The unit group of R is written by R^* . For a group G and a subset $S \subset G$, $Z(G)$ and $C_G(S)$ are the center and the centralizer of S in G , and the same notations are applied for R . $N_G(S)$ is used for the normalizer of S in G and G' for the derived subgroup. A field is locally finite, if every its finitely generated subfield is finite. Given a division ring D with center F and a subgroup G of $GL_n(D)$, the space of column n -vectors $V = D^n$ over D is a G - D bimodule. G is called irreducible (resp. completely reducible, reducible) if V is irreducible (resp. completely reducible, reducible) as G - D bimodule. Furthermore, G is absolutely irreducible if $F[G] = M_n(D)$.

An irreducible group G is said to be imprimitive if for some integer $m \geq 2$, there exist subspaces V_1, \dots, V_m of V such that $V = \bigoplus_{i=1}^m V_i$ and for any $g \in G$ the mapping $V_i \rightarrow gV_i$ is a permutation of the set $\{V_1, \dots, V_m\}$; otherwise G is called primitive.

Here, we define the wreath product of a skew linear group and a permutation group (see [24, p. 106-109]).

Let U be a linear space over a division ring D , G_1 a subgroup of $GL(U)$, and Γ a subgroup of the symmetric group S_k on $\{1, \dots, k\}$, $k > 1$. The cartesian product $U^k = V_1$ can be regarded as a linear space over D , and we write any vector $v \in V_1$ in the form $v = (u_1, \dots, u_k)$, $u_j \in U$. For any $f_1, \dots, f_k \in G_1$ and $s \in \Gamma$, we define a mapping $f : V_1 \rightarrow V_1$, $f = \langle f_1, \dots, f_k, s \rangle$, by setting

$$f(v) = f(u_1, \dots, u_k) = \bar{v} \in V_1,$$

where the $s(v)$ th component of \bar{v} is $f_v(u_v)$, $v = 1, \dots, k$. Obviously f is an automorphism of V_1 . The group of all such automorphisms is called the wreath product of the skew linear group G_1 and the permutation group Γ , and is denoted by $G_1 \wr \Gamma$. The group $G_1 \wr \Gamma$ is imprimitive. In addition, by Lemma 5 of [24, p.108], any imprimitive subgroup P of $GL_n(D)$ is conjugate to a subgroup of $(GL_r(D) \wr S_k)$, when r and k are such that $n = rk$ with $k > 1$.

For example, recall that a monomial matrix is a square matrix with exactly one non-zero entry in each row and column. It is not hard to see that the set of all $n \times n$ monomial matrices over D is conjugate to $D^* \wr S_n$, when $n > 1$.

Let $n = rk$, for some natural numbers r and k with $k > 1$. Obviously, $(GL_r(D) \wr S_k) \subseteq GL_n(D)$. Let \mathcal{A} be the set of all $k \times k$ monomial matrices with entries in D and choose $A \in \mathcal{A}$. We construct a new matrix in $GL_n(D)$ as follows. We replace each nonzero entry in A with a matrix from $GL_r(D)$. Also, we replace each zero entry in A with the zero matrix from $M_r(D)$. Denote the set of these new matrices by \mathcal{B} . Hence $\mathcal{B} \subseteq GL_n(D)$. It is not hard to see that $\mathcal{B} = (GL_r(D) \wr S_k)$.

3. PRELIMINARY RESULTS

This section contains some preliminary results that we use throughout this article.

Theorem A. ([14]) Let D be a division ring and N be a subnormal subgroup of the multiplicative group of D . If N is periodic, then $N \subseteq Z(D)$.

Theorem B. ([6]) Let N be a normal subgroup in a primitive subgroup M of $GL_n(D)$. Then, we have:

- (1) $F[N]$ is a prime ring.
- (2) $C_{M_n(D)}(N)$ is a simple Artinian ring.
- (3) If $C_{M_n(D)}(N)$ is a division ring, then N is irreducible.

Theorem C. ([21]) If A is a simple ring with identity and $n \geq 2$, then the subrings of $M_n(A)$ which are invariant under the all inner automorphisms of $M_n(A)$ are subrings of the center or $M_n(A)$ itself.

Theorem D. ([2]) Let D be a division ring with center F and put $A = M_n(D)$. Let either $n = 2$ but D contains at least four elements or $n \geq 3$. If N is a subnormal subgroup of A^* , then either $N \subseteq F$ or $A' \subseteq N$, where A' is the derived group of A^* .

Theorem E. ([23, p. 154]) Let D be a division ring that is not a locally finite field, and let $n > 1$ be an integer. If N is any non-central normal subgroup of $GL_n(D)$, then N contains a non-cyclic free subgroup.

First, we prove the following lemma, which will play an important role in proving our main theorems.

Lemma 3.1. *Given an F -central division algebra D and N a subnormal subgroup of $GL_n(D)$, if M is a maximal subgroup of N , then, one of the following cases occurs:*

- (1) M is reducible and there exists an invertible matrix P and a natural number $0 < r < n$ such that $PMP^{-1} \subseteq H$,

$$H = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(D), B \in M_{r,n-r}(D), C \in GL_{n-r}(D) \right\}.$$

In addition, M contains a copy of D' .

- (2) M is imprimitive and there exists an invertible matrix $P \in GL_n(D)$ such that $PMP^{-1} \subseteq H = GL_r(D) \wr S_k$, where $rk = n$. Also, M contains a copy of D' .
- (3) M is primitive.

Proof. The conclusion is clear for $n = 1$, so assume that $n > 1$. By [4], $SL_n(D) \subseteq N$ and N is a normal subgroup of $GL_n(D)$. If M is reducible, by 1.1.1 of [23], there exists an invertible matrix P and a natural number $0 < r < n$ such that

$$PMP^{-1} \subseteq H,$$

$$H = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(D), B \in M_{r,n-r}(D), C \in GL_{n-r}(D) \right\}.$$

Notice that PMP^{-1} is a maximal subgroup of N , so two cases may occur:

Case 1. $H \cap N = PMP^{-1}$. So,

$$SL_n(D) \cap \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(D), B \in M_{r,n-r}(D), C \in GL_{n-r}(D) \right\}$$

is a subset of PMP^{-1} . We conclude that both PMP^{-1} and M contain a copy of

$$D_1 = \{diag(a, b) : a, b \in D^*\} \cap SL_2(D) = \{diag(a, b) : ab \in D'\}.$$

This is a group which fits into the short exact sequence

$$1 \rightarrow D' \rightarrow D_1 \rightarrow D^* \rightarrow 1,$$

so a copy of D' is immediate.

Case 2. $H \cap N = N$. We have $SL_n(D) \subseteq N$. But obviously $SL_n(D) \not\subseteq H$, which is a contradiction.

If M is imprimitive, by Lemma 5 of [24, p. 108], we conclude that it is conjugate to a subgroup of $GL_r(D) \wr S_k$, where $rk = n$. Assume that there exists an invertible matrix $P \in GL_n(D)$ such that

$$PMP^{-1} \subseteq H = GL_r(D) \wr S_k.$$

So two cases may occur:

Case 1. $H \cap N = PMP^{-1}$.

$$D_1 = \{\text{diag}(a, b) : a, b \in D^*\} \cap SL_2(D) = \{\text{diag}(a, b) : ab \in D'\}$$

is a subgroup of $SL_n(D) \cap GL_r(D) \wr S_k$. Thus, in this case again M contains an isomorphic copy of D' .

Case 2. $H \cap N = N$. We have $SL_n(D) \subseteq N$. But obviously

$$SL_n(D) \not\subseteq GL_r(D) \wr S_k,$$

which is a contradiction. \square

In mathematics, specifically in the field of group theory, a divisible group is an abelian group in which every element can, in some sense, be divided by positive integers, or more accurately, every element is an n -th multiple for each positive integer n . A famous example of a divisible abelian group is the additive group \mathbb{Q} of rational numbers, a torsion free divisible abelian group. Let F is a field, we say that F is divisible, when F^* is a divisible group. Furthermore, for a natural number n , F is n -divisible when $F^{*n} = F^*$. For example, for any odd natural number n , \mathbb{R} is n -divisible.

Let D be a division algebra and $Z(D) = F$. For any subgroup $G \subseteq F^*$, $\{aI_n \mid a \in G\}$ is a subgroup of $GL_n(D)$. For convenience, we use G instead of $\{aI_n \mid a \in G\}$.

By stating the following results, we intend to find out the relation between the maximal subgroups of $GL_n(D)$ and the maximal subgroups of $SL_n(D)$.

Lemma 3.2. *Let n be a natural number and F be an n -divisible field. We have the following results:*

- (1) $GL_n(F) = F^*SL_n(F)$.
- (2)

$$GL_n(F)/F^* \cong SL_n(F)/Z(SL_n(F)) = PSL_n(F).$$

Then, when $F \neq \mathbb{F}_2, \mathbb{F}_3$, $GL_n(F)/F^$ is a simple group and there exists a one-to-one correspondence between maximal subgroups of $GL_n(F)$ containing F^* and maximal subgroups of $SL_n(F)$ containing $Z(SL_n(F))$.*

- (3) $(F^* \wr S_n) \cap SL_n(F)$ is a maximal subgroup of $SL_n(F)$.
- (4) If $H := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(F), B \in M_{r, n-r}(F), C \in GL_{n-r}(F) \right\}$, then $H \cap SL_n(F)$ is a maximal subgroup of $SL_n(F)$.

Proof. (1) F is n -divisible, so $F^{*n} = F^*$. For any $A \in GL_n(F)$, we write $A = a(\frac{1}{a}A)$, when $a^n = \det(A)$. Thus, $GL_n(F) \subseteq F^*SL_n(F) \subseteq GL_n(F)$. Hence, we conclude that $GL_n(F) = F^*SL_n(F)$.

(2) By Theorem 21.2 of [7], $SL_n(F)/Z(SL_n(F)) = PSL_n(F)$ is a simple group.

- (3) By [17], the monomial subgroup is a maximal subgroup of $SL_n(F)$.
- (4) By Theorem 7 of [1],

$$H := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(F), B \in M_{r,n-r}(F), C \in GL_{n-r}(F) \right\}$$

is a maximal subgroup of $GL_n(F)$. □

Corollary 3.3. *Let n be an odd natural number. Then, we have the following results:*

- (1) $GL_n(\mathbb{R}) = \mathbb{R}^*SL_n(\mathbb{R})$.
- (2) $GL_n(\mathbb{R})/\mathbb{R}^* \cong SL_n(\mathbb{R})$. $GL_n(\mathbb{R})/\mathbb{R}^*$ is a simple group and there exists a one-to-one correspondence between maximal subgroups of $GL_n(\mathbb{R})$ containing F^* and maximal subgroups of $SL_n(\mathbb{R})$.
- (3) $(\mathbb{R}^* \wr S_n) \cap SL_n(\mathbb{R})$ is a maximal subgroup of $SL_n(\mathbb{R})$.
- (4) If $H := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in GL_r(\mathbb{R}), B \in M_{r,n-r}(\mathbb{R}), C \in GL_{n-r}(\mathbb{R}) \right\}$, then $H \cap SL_n(\mathbb{R})$ is a maximal subgroup of $SL_n(\mathbb{R})$.

In group theory, a group G is called metacyclic if it has a cyclic normal subgroup L such that G/L is cyclic. The following lemma is a generalization of Herstein’s lemma (Theorem 13.8 of [16]). Also, this lemma is a nice result of Skolem-Nother Theorem.

Lemma 3.4. *Let D be a noncommutative division ring over its center F . Suppose a is a noncentral torsion element of $GL_n(D)$. Assume that $a^m = 1$, for some natural number $m > 1$. Then, we have the following results:*

- (1) For any natural number k such that $\gcd(k, m) = 1$, there exists a unique $y_k \in GL_n(D)$ such that $y_k a y_k^{-1} = a^k \neq a$.
- (2) $\langle a, y_k \rangle$ is a metacyclic group.
- (3) Let $\varphi(m)$ be Euler’s totient function, then $a y_k^{\varphi(m)} = y_k^{\varphi(m)} a$. Thus, $F[\langle a, y_k^{\varphi(m)} \rangle]$ is a commutative ring. In case $n = 1$, $F[\langle a, y_k^{\varphi(m)} \rangle]$ is a field.
- (4) For any natural number k such that $\gcd(k, m) = 1$, there exists an additive commutator $x_k \in M_n(D)$ such that $x_k a = a^k x_k$. In fact, $x_k = a y_k - y_k a$.

Proof. (1) Since $\gcd(k, m) = 1$, we have the homomorphisms

$$f, g_k : F[a] \rightarrow F[a]$$

such that $f(a) = a$ and $g_k(a) = a^k$. On the other hand, $F[a]$ is a field with $[F[a] : F] < \infty$. Now, using Skolem-Noether Theorem, we conclude that there exists a unique element $y_k \in GL_n(D)$ such that $y_k a y_k^{-1} = a^k \neq a$.

(2) $\langle a \rangle \triangleleft \langle a, y_k \rangle$ and $\langle a, y_k \rangle / \langle y_k \rangle \cong \langle a \rangle / (\langle a \rangle \cap \langle y_k \rangle)$. So, $\langle a, y_k \rangle$ is a metacyclic group.

(3) It is easily checked that for any natural number t , $y_k^t a y_k^{-t} = a^{k^t}$. Since $\gcd(k, m) = 1$, so $k^{\varphi(m)} \equiv 1 \pmod{m}$. Thus, $a y_k^{\varphi(m)} = y_k^{\varphi(m)} a$ and so $F[\langle a, y_k^{\varphi(m)} \rangle]$ is a commutative ring. In case $n = 1$, $F[\langle a, y_k^{\varphi(m)} \rangle]$ is a field, as we desired.

(4) Let $x_k = a y_k - y_k a$. Therefore, $x_k a = a^k x_k$. □

Example 3.5. Let $a = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in GL_3(\mathbb{C})$. Then, $a^3 = I_3$. Let

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \in GL_3(\mathbb{C}).$$

Then, $y a y^{-1} = a^2$. Also, $x = a y - y a = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin GL_3(\mathbb{C})$. We have

$$x a = a^2 x$$

Example 3.6. Let $a = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in GL_3(\mathbb{C})$. set

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in GL_3(\mathbb{C}).$$

We have $y a y^{-1} = a^2$, $y^2 = a^3 = I_3$. Thus, $\langle a, y \rangle \cong S_3 \subseteq GL_3(\mathbb{C})$. Let

$$y_n = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{n}} \end{bmatrix} \in GL_3(\mathbb{C}).$$

When n is a natural number. $y_n a y_n^{-1} = a^2$, $y_n^{lcm(2,n)} = a^3 = I_3$.

Lemma 3.7. Let D be a division ring over its center F . Suppose that M is a maximal subgroup of $GL_n(D)$. If $a \in Z(M)$ is a torsion element then $a \in F^*$.

Proof. Suppose on the contrary that $a \in Z(M) \setminus F^*$. By Lemma 3.4, there exists an element $y \in GL_n(D)$ such that $yay^{-1} = a^i \neq a$, for some natural number i . So, $\langle M, y \rangle \subseteq N_{GL_n(D)}(\langle a \rangle)$. By maximality of M , we conclude that $\langle M, y \rangle = GL_n(D) = N_{GL_n(D)}(\langle a \rangle)$. Thus, $F[\langle a \rangle]^*$ is a normal subgroup of $GL_n(D)$. Using Cartan–Brauer–Hua theorem in case $n = 1$ and Theorem C in case $n > 1$, we conclude that either a is central or $F[a] = M_n(D)$. Both cases lead us to a contradiction, so our assumption must be false. Consequently $a \in F^*$, as we claimed. \square

4. TORSION MAXIMAL SUBGROUPS OF $GL_n(D)$

The following results give us useful information about the torsion maximal subgroups of $GL_n(D)$.

Theorem 4.1. *Let D be a division ring which is algebraic over its center F . Let M be a torsion maximal subgroup of $GL_n(D)$. Then $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p .*

Proof. Assume that $F^* \not\subseteq M$. So by maximality of M , we have

$$F^*M = GL_n(D).$$

Therefore, we obtain that in case $n > 1$, $SL_n(D) = (GL_n(D))' = M' \subseteq M$ and in case $n = 1$, $D' \subseteq M$. But M is a torsion group. Using Theorem E in case $n > 1$, we conclude that $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . Now, assume that $n = 1$. By Theorem A, we conclude that D^* is a soluble group. Using Hua’s Theorem ([15]), we obtain that $D = F$. Since M is a torsion maximal subgroup of F^* , F^*/M is a simple abelian group. Therefore, $[F^* : M] < \infty$. Thus we conclude that F^* is a torsion group. Consequently, F is algebraic over \mathbb{F}_p .

Now, assume that $F^* \subseteq M$. Thus, F^* is a torsion group. Consequently, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . On the other hand, D is algebraic over F and F is algebraic over \mathbb{F}_p . Hence, D is algebraic over \mathbb{F}_p . Using Theorem 13.11 of [16], we conclude that $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p , as we claimed. \square

Theorem 4.2. *Let D be a division ring with center F . Also, consider that M be a torsion maximal subgroup of $GL_n(D)$, then one of the following cases occurs:*

- (1) $D = F$, $\text{char} F = p > 0$, and F is algebraic over \mathbb{F}_p .
- (2) D is not algebraic over F , $\text{char} F = p > 0$, and F is algebraic over \mathbb{F}_p . Also, $Z(M) = F^*$, $SL_n(D) \not\subseteq M$ and $F[M]$ is a prime ring. In

addition, for any element $a \in M$, $C_{M_n(D)}(F[a]) \cong M_m(D_1)$, when D_1 is a division ring with center $F[a]$ and $m \leq n$ is a natural number. Also, D_1 is not algebraic over F and $M_n(D) \otimes_F (F[a]) \cong M_{tm}(D_1)$, when $[F[a] : F] = t$.

Proof. By a similar argument as Theorem 4.1, we may assume that $F^* \subseteq M$, $SL_n(D) \not\subseteq M$ and D is not algebraic over F . By lemma 3.7, $Z(M) = F^*$. By lemma 3.1, either M is primitive or M contains an isomorphic copy of D' . When M contains an isomorphic copy of D' , by a similar argument as previous theorem, we conclude that $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . So case (1) occurs.

Now, Let M be a primitive subgroup of $GL_n(D)$. By Theorem B, $F[M]$ is a prime ring. Let a be an element of M . a is torsion, so $F[a]$ is a field and $[F[a] : F] < \infty$. Using Centralizer Theorem ([7]), we obtain that

$$C_{M_n(D)}(F[a]) \cong M_m(D_1),$$

when D_1 is a division ring with center $F[a]$. By Theorem 1.1.9 of [23], we conclude that $m \leq n$. Let $[F[a] : F] = t$. By Centralizer Theorem, we obtain:

$$M_n(D) \otimes_F (F[a])^{op} \cong C_{M_n(D)}(F[a]) \otimes_F M_t(F) \cong M_t(C_{M_n(D)}(F[a])).$$

So, $M_n(D) \otimes_F (F[a]) \cong M_t(M_m(D_1)) \cong M_{tm}(D_1)$. If D_1 is algebraic over F , then $C_{M_n(D)}(F[a])$ algebraic over F . Therefore, we conclude that D is algebraic over F . Using Theorem 4.1, we conclude that $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . \square

Corollary 4.3. *Let D be a division ring with center F . Also, consider that M be a torsion maximal subgroup of $GL_n(D)$. In addition, assume that there exists an element $a \in M$ such that $C_{M_n(D)}(F[a])$ is algebraic over F . Then $D = F$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p .*

In continuation of this article, we will investigate some group-theoretic properties of maximal subgroups of subnormal subgroups of $GL_n(D)$.

Theorem 4.4. *Let D be a noncommutative division ring with center F . Assume that N be a subnormal subgroup of $GL_n(D)$, if M is a locally finite maximal subgroup of N , then one of the following cases occurs:*

- (1) $n > 1$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . In addition, for some natural number $s \leq n$, $F[M] \cong M_s(F)$, $M \trianglelefteq F[M]^*$ and $SL_n(D) \cap F[M]^* \subseteq M$.
- (2) $M \subseteq N \subseteq F^*$.
- (3) $F[M] = M_n(D)$ and D is locally finite dimensional over F .

Proof. Assume that M is reducible. By Lemma 3.1, M contains an isomorphic copy of D' . By Theorem A, we conclude that D^* is a soluble group. Using Hua's Theorem, we obtain $D = F$, which is a contradiction. Hence, we may assume that M is irreducible. Using Theorem 1.1.12 of [23], we obtain that $F[M]$ is a simple Artinian ring. Thus, by maximality of M , either $N \cap F[M]^* = M$ or $N \subseteq F[M]^*$.

If $N \cap F[M]^* = M$, then M is normal in $F[M]^*$. In case $n > 1$, by Theorem 4.5.1 of [23], F is a locally finite field and $F[M]^*$ is a locally finite group. If $N \not\subseteq F^*$, then by Theorem 4.5.1 of [23], we conclude that $F[M]^* \cong GL_s(F)$, for some natural number s . By theorem 1.1.9 of [23], we conclude that $s \leq n$. By Theorem D, $SL_n(D) \subseteq N$. Therefore, $SL_n(D) \cap F[M]^* \subseteq M$. In case $n = 1$, by Theorem A, we obtain that $M \subseteq (Z(F[M]) = F$. So $F[M] = F$. Thus $N \cap F^*$ is a maximal subgroup of N . This means, $N/N \cap F^*$ has no proper subgroup. Therefore N is a torsion group. Now, by Theorem A, we conclude that N is central.

Now we may assume $N \subseteq F[M]^*$. Then, by Theorem 14.3.8 of [22] and Corollary 1 of [21], we have $F[M] = M_n(D)$. Since M is locally finite, we conclude that $F[M]$ is locally finite dimensional over F . Therefore D is algebraic over F . \square

Corollary 4.5. *Let D be a noncommutative division ring with center F . Assume that M is a locally finite maximal subgroup of $SL_n(D)$, then one of the following cases occurs:*

- (1) $n > 1$, $\text{char} F = p > 0$ and F is algebraic over \mathbb{F}_p . In addition, for some natural number $s \leq n$, $F[M] \cong M_s(F)$, $M \trianglelefteq F[M]^*$ and $SL_n(D) \cap F[M]^* = M$.
- (2) $M \subseteq N \subseteq F^*$.
- (3) $F[M] = M_n(D)$ and D is locally finite dimensional and algebraic over F .

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