Journal of Algebraic Systems, Vol. 14(No. 3): (2026), pp 489-508.
https://doi.org/10.22044/JAS.2025.14919.1880

ON THE SPECTRA OF TENSOR JOIN OF HYPERGRAPHS

R. Vishnupriya and R. Rajkumar*

Abstract. In this paper, we consider certain classes of hypergraphs constructed from the tensor join of hypergraphs, specifically the tensor join of hypergraphs constrained by vertex subsets and the $(H, \mathcal{T}_{\mathcal{S}})$-join of hypergraphs constrained by $\mathcal{S}$. We determine some eigenvalues of the adjacency tensor of these classes of hypergraphs by establishing corresponding eigenvectors. We demonstrate that the eigenvalues of the adjacency tensor of the constituting hypergraphs are also eigenvalues of the adjacency tensor of the join of a set of hypergraphs. Furthermore, as a special case of our results, we provide some eigenvalues and eigenvectors of the adjacency tensor for the join of non-uniform hypergraphs on a backbone hypergraph H (and, similarly, for the join of m-uniform hypergraphs on a backbone hypergraph H). Additionally, we establish a relationship between the eigenvalues of the adjacency tensor of a hypergraph H and certain eigenvalues of the adjacency tensor of the $(H, \mathcal{T}_{\mathcal{S}})$-join of hypergraphs constrained by $\mathcal{S}$. Using this relationship, we determine some eigenvalues and their corresponding eigenvectors for the adjacency tensor of the lexicographic product of two hypergraphs.

1. Introduction

Spectral hypergraph theory involves associating various tensors and connectivity matrices with a hypergraph and studying the properties of the hypergraph through the spectra of these tensors and matrices. The adjacency tensor for an m-uniform hypergraph was introduced by Bulò and Pelillo [12]. Later, Cooper and Dutle defined a degree-normalized mm-adjacency tensor for m-uniform hypergraphs [3]. Hu and Qi introduced the Laplacian tensor for a uniform hypergraph [5], followed by an alternative definition of the Laplacian tensor by Li et al. [7]. Xie and Chang introduced the signless Laplacian tensor for a uniform hypergraph [15]. Subsequently, in [10], Qi proposed simple and natural definitions for both the Laplacian tensor and the signless Laplacian tensor for a uniform hypergraph. For additional details, we refer the reader to [11].

Recently, Anirban Banerjee et al. [2] introduced the adjacency tensor, Laplacian tensor, and signless Laplacian tensor for general hypergraphs. In their work, they studied several properties of general hypergraphs using the

MSC(2020): Primary: 05C50; Secondary: 05C65, 05C76, 15A18.
Keywords: Hypergraphs; Tensor join; Adjacency tensor; Eigenvalues.
Received: 9 August 2024, Accepted: 27 January 2025.
*Corresponding author.

spectra of these tensors. Further properties of these tensors on general hypergraphs have been explored by various researchers; see, for instance, [8, 6, 16]. In [1], Anirban Banerjee introduced the adjacency matrix, Laplacian matrix, and normalized Laplacian matrix for hypergraphs, investigating several structural properties of hypergraphs through the spectral properties of these matrices.

On the other hand, over the past decades, several hypergraph products have been defined, and their structural properties extensively studied. For a comprehensive overview, we refer the reader to the survey [4]. A natural question that arises in spectral hypergraph theory is: "*how far the spectrum of a hypergraph be related to the spectrum of some hypergraphs or its subhypergraphs?*" In this context, the spectra of hypergraphs constructed through various hypergraph products have been investigated by numerous authors. Cooper and Dutle [3] studied the spectra of a degree-normalized m-adjacency tensor on hypergraphs formed through certain operations, where the constituent graphs are m-uniform. Pearson and Zhang analyzed the spectra of the adjacency tensor for the Cartesian and tensor products of two hypergraphs [9]. Amitesh Sarkar and Anirban Banerjee defined several types of hypergraph joins and studied the spectra of the adjacency matrix of weighted hypergraphs formed by these operations [13]. More recently, in [14], the authors introduced three families of operations on hypergraphs, showed their equivalence, and collectively referred to them as the tensor join. Furthermore, it was noted that any hypergraph can be viewed as a tensor join of some of its subhypergraphs. In that work, the spectra of the adjacency matrix, Laplacian matrix, signless Laplacian matrix, and normalized Laplacian matrix (as defined in [1]) were studied for weighted hypergraphs obtained via various tensor join operations.

In this paper, we focus on specific operations on hypergraphs as particular cases of the tensor join. We derive certain eigenvalues and their corresponding eigenvectors of the adjacency tensor, as defined in [2], for the hypergraphs constructed through these operations, expressed in terms of the eigenvalues and eigenvectors of the constituting hypergraphs.

The rest of the paper is organized as follows. Section 2 provides basic definitions and preliminary notations. In Section 3, we introduce the tensor join of hypergraphs constrained by vertex subsets as a specific case of the tensor join of hypergraphs, and compute the eigenvalues and eigenvectors of the adjacency tensor for hypergraphs constructed through these operations. Additionally, we derive eigenvalues and eigenvectors of the adjacency tensor for hypergraphs formed by the join of a set of hypergraphs or uniform hypergraphs, as defined in [13]. Section 4 introduces the $(H, \mathcal{T}_{\mathcal{S}})$-join of hypergraphs as a particular case of the $(H, \mathcal{T})$-join and presents the couple join of hypergraphs in $\mathcal{G}$, constrained by $\mathcal{S}$, as a specific instance of the $(H, \mathcal{T}_{\mathcal{S}})$-join. We compute certain eigenvalues and corresponding eigenvectors of the adjacency tensor for hypergraphs constructed by these operations and determine eigenvalues and eigenvectors for the adjacency tensor of the lexicographic product of two hypergraphs.

2. Preliminaries

A hypergraph $H = (V(H), E(H))$ consists of a non-empty set $V(H)$ and a multiset $E(H)$ of subsets of V. The elements of $V(H)$ are called vertices, and the elements of $E(H)$ are called hyperedges, or simply edges of H. The rank of H, denoted by $rank(H)$, is defined as $\max_{e \in E(H)} \{|e|\}$, if $E(H) \neq \Phi$; 0, otherwise. H is said to be uniform if all of it's edges have the same cardinality. A uniform hypergraph in which all of its edges have cardinality m is said to be m-uniform. The degree of a vertex v in a hypergraph H, denoted by $deg_H(v)$, is the number of edges in H that contains v. H is said to be r-regular if each vertex of H is of degree r. Throughout this paper, we consider only the hypergraphs having finite number of vertices.

An r order n dimensional hypermatrix $A := (a_{i_1 i_2 \ldots i_r})$ is an r dimensional array of n^r entries from the field $\mathbb{C}$, where $a_{i_1 i_2 \ldots i_r} \in \mathbb{C}$; $i_j \in [n] := \{1, 2, \ldots, n\}$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and let x^r be an r order n dimensional hypermatrix with $(i_1, i_2, \ldots, i_r)$-th entry as $x_{i_1} x_{i_2} \ldots x_{i_r}$. Then Ax^{r-1} denotes an n tuple whose j-th component is

$$\sum_{i_2, i_3, \ldots, i_r = 1}^{n} a_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}.$$

A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n - \{0\})$ is said to be an eigenvalue and eigenvector (or simply eigenpair) of the hypermatrix A if it satisfies the following equation

$$Ax^{r-1} = \lambda x^{[r-1]}.$$

Here $x^{[r]}$ is a vector in $\mathbb{C}^n$ whose i-th entry is x_i^r. We call (λ, x) an H-eigenpair (i.e., λ and x are called H-eigenvalue and H-eigenvector, respectively) if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n - \{0\}$.

A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n - \{0\})$ is said to be an E-eigenpair (i.e., λ and x are called E-eigenvalue and E-eigenvector, respectively) of the hypermatrix

A if it satisfies the following equations

$$\begin{aligned} Ax^{r-1} &= \lambda x, \\ \sum_{i=1}^{n} x_i^2 &= 1. \end{aligned}$$

We call (λ, x) an Z-eigenpair (i.e., λ and x are called Z-eigenvalue and Z-eigenvector, respectively) if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n - \{0\}$.

Definition 2.1. ([2]) Let H be the hypergraph with vertex set

$$V(H) = \{v_1, v_2, \ldots, v_n\}$$

and edge set $E(H) = \{e_1, e_2, \ldots, e_k\}$. Let $rank(H) = r$. Then the adjacency hypermatrix of H, denoted by $A(H)$, is defined as

$$A(H) = (a_{i_1 i_2 \ldots i_r}),\ 1 \leq i_1, i_2, \ldots, i_r \leq n.$$

For each edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\} \in E(H)$ of cardinality $s \leq r$, $a_{p_1 p_2 \ldots p_r} = \frac{s}{\alpha}$, where

$$\alpha = \sum_{\substack{k_1, k_2, \ldots, k_s \geq 1, \\ \sum k_i = r}} \frac{r!}{k_1! k_2! \ldots k_s!},$$

and $p_1, p_2, \ldots, p_r$ are chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set. The other positions of the hypermatrix are zero. The eigenvalue and eigenvector of the hypermatrix $A(H)$ are called the eigenvalue and eigenvector of the hypergraph H, respectively. Also, we call the eigenpair (resp. H-eigenpair, E-eigenpair and Z-eigenpair) of $A(H)$ as the eigenpair (resp. H-eigenpair, E-eigenpair and Z-eigenpair) of H.

Notation 2.2. Let $\mathcal{R}(a_1, a_2, \ldots, a_m)$ denote the range set of the sequence $(a_i)_{i=1}^{m}$. Let

$$\mathcal{R}^{\blacktriangledown}(a_1, a_2, \ldots, a_m) := \begin{cases} \mathcal{R}(a_1, a_2, \ldots, a_m) \setminus \{\blacktriangledown\}, & \text{if } \blacktriangledown \in \{a_1, a_2, \ldots, a_m\}; \\ \mathcal{R}(a_1, a_2, \ldots, a_m), & \text{otherwise.} \end{cases}$$

Definition 2.3. ([14]) For $i \in [k]$, let A_i be mutually disjoint sets of n_i elements. Let $\mathcal{A}$ be the sequence $(A_i)_{i=1}^{k}$. *An indicating tensor corresponding to* $\mathcal{A}$, denoted by $T[\mathcal{A}] := (T[\mathcal{A}]_{p_1 p_2 \ldots p_N})$, is a $0-1$ tensor of order $N := \sum_{i=1}^{k} n_i$ and dimension

$$(\underbrace{n_1+1, \ldots, n_1+1}_{n_1\ times}, \underbrace{n_2+1, \ldots, n_2+1}_{n_2\ times}, \ldots, \underbrace{n_k+1, \ldots, n_k+1}_{n_k\ times}),$$

where $p_1, p_2, \ldots, p_{n_1} \in A_1 \cup \{\blacktriangledown\}$, $p_{n_1+n_2+\cdots+n_i+1}, \ldots, p_{n_1+n_2+\cdots+n_{i+1}} \in A_{i+1} \cup \{\blacktriangledown\}$ for $i \in [k-1]$; $\blacktriangledown$ is an arbitrary element that is not an element of any A_i, $i \in [k-1]$; and is satisfying the following:

(i) If there exists $p_1, p_2, \ldots, p_N$ such that $\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N) \subseteq A_i$ for some $i \in [k]$, then $T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 0$.

(ii) If there exists $p_1, p_2, \ldots, p_N$ such that $T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 1$, then
$$T[\mathcal{A}]_{p'_1 p'_2 \ldots p'_N} = 1$$
whenever $\mathcal{R}^{\blacktriangledown}(p'_1, p'_2, \ldots, p'_N) = \mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N)$.

Notice that if $p_1 = p_2 = \cdots = p_N = \blacktriangledown$, then $\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N) = \Phi \subseteq A_i$ and so $T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 0$.

In the rest of the paper, whenever we consider a sequence of hypergraphs $(G_i)_{i=1}^k$, without loss of generality, we assume that the vertex sets of G_is are mutually disjoint for $i \in [k]$.

Notation 2.4. Let $E(T[\mathcal{A}]) = \{(p_1, p_2, \ldots, p_N) \colon T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 1\}$.

Definition 2.5. ([14]) Let $\mathcal{G} = (G_i(V_i, E_i))_{i=1}^k$ be a sequence of k hypergraphs. Let $\mathcal{V} = (V_i)_{i=1}^k$. Consider an indicating tensor $T[\mathcal{V}]$. Then the $T[\mathcal{V}]$-*join of hypergraphs in* $\mathcal{G}$, denoted by $\bigvee_{T[\mathcal{V}]} \mathcal{G}$, is the hypergraph constructed as follows:

- Take one copy of G_i, $i \in [k]$;
- For each $D \subseteq \bigcup_{i=1}^k V_i$, join the vertices in D as an edge in $\bigvee_{T[\mathcal{V}]} \mathcal{G}$ if and only if $D \in E(T[\mathcal{V}])$.

Definition 2.6. ([14]) Let H be a hypergraph with $V(H) = [k]$. Let $\mathcal{G} = (G_i(V_i, E_i))_{i=1}^k$ be a sequence of hypergraphs with $|V_i| = n_i$ for all $i \in [k]$. For each $e \in E(H)$, let $\mathcal{V}_e = (V_i)_{i \in e}$, $N_e := \sum_{i \in e} n_i$ and $\mathcal{G}_e = \{G_i \colon i \in e\}$. Let $\mathcal{T} = \{T[\mathcal{V}_e] \colon e \in E(H)\}$, where for each $e \in E(H)$, $T[\mathcal{V}_e]$ is a non-zero indicating tensor with
$$T[\mathcal{V}_e]_{p_1 p_2 \ldots p_{N_e}} = 0 \text{ if } \mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_{N_e}) \cap V_i = \Phi \text{ for some } i \in e.$$
Then we construct the hypergraph by taking a copy of each G_i and applying the $T[\mathcal{V}_e]$-join of hypergraphs in $\mathcal{G}_e$ for each edge $e \in E(H)$. This hypergraph is denoted by $\mathcal{G}(H, \mathcal{T})$ and is called the $(H, \mathcal{T})$-*join of hypergraphs in* $\mathcal{G}$.

Notation 2.7. Let $\mathcal{A} = \{A_i\}_{i=1}^k$, where each A_i is a set whose elements are sets. We define $\beta_{\mathcal{A}} := \sum_{i=1}^k \{\min_{S \in A_i} |S|\}$ and $\gamma_{\mathcal{A}} := \sum_{i=1}^k \{\max_{S \in A_i} |S|\}$. Clearly $\beta_{\mathcal{A}} \le \gamma_{\mathcal{A}}$.

Notation 2.8. Let H be a hypergraph and let $i_1, i_2, \ldots, i_n \in V(H)$. Whenever we write $(i_1, i_2, i_3, \ldots, i_n) \in E(H)$, we mean the ordered set

$$\{i_1, i_2, i_3, \ldots, i_n\} \in E(H).$$

3. Tensor join of hypergraphs constrained by vertex subsets

In this section, we consider a particular case of $T[\mathcal{V}]$-join of hypergraphs in $\mathcal{G}$ as given in Definition 2.5. Let $\mathcal{P}(S)$ denote the power set of the set S. Let $\mathcal{P}^*(S) := \mathcal{P}(S) - \{\Phi\}$.

Definition 3.1. For $i \in [k]$, let H_i be a hypergraph with n_i vertices. Let $S_i \subseteq \mathcal{P}(V(H_i))$ for all $i \in [k]$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ and

$$B \subseteq \{\beta_{\mathcal{S}}, \beta_{\mathcal{S}} + 1, \ldots, \gamma_{\mathcal{S}}\}.$$

Let $\mathcal{H} = (H_i)_{i=1}^k$ and $\mathcal{V} = (V(H_i))_{i=1}^k$. Then we consider $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$, the ${}_BT_{\mathcal{S}}[\mathcal{V}]$-join of hypergraphs in $\mathcal{H}$, where ${}_BT_{\mathcal{S}}[\mathcal{V}]$ is an indicating tensor of order $N = \sum\limits_{i=1}^{k} n_i$ with

$$ {}_BT_{\mathcal{S}}[\mathcal{V}]_{p_1 p_2 \ldots p_N} = \begin{cases} 1, & \mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N) \cap V(H_i) \in S_i \text{ for all } i \in [k] \\ & \text{with } |\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N)| \in B; \\ 0, & \text{otherwise.} \end{cases} $$

We call this hypergraph as the ${}_BT_{\mathcal{S}}[\mathcal{V}]$*-join of the hypergraphs in* $\mathcal{H}$ *constrained by* $\mathcal{S}$.

The hypergraph defined above can be described as follows: $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$ is the hypergraph with $V\left(\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}\right) = \bigcup\limits_{i=1}^{k} (V(H_i))$ and $E\left(\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}\right) = \bigcup\limits_{i=0}^{k} (E(H_i))$, where

$$ E(H_0) = \left\{ e \subseteq V\left(\bigvee_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}\right) : |e| \in B,\ e \cap V(H_i) \in S_i \text{ and } e \notin S_i \ \forall i \right\}. $$

Note 3.2. For each $i \in [k]$, if we take $S_i = \mathcal{P}^*(V(H_i))$ in Definition 3.1, then we get $B \subseteq \{k, k+1, \ldots, N\}$. In this case, the operation defined in Definition 3.1 becomes the join of a set $\mathcal{H} = \{H_i\}_{i=1}^k$ of hypergraphs as defined in [13, p. 8].

Example 3.3. Let $\mathcal{H} = (H_1, H_2, H_3)$, where H_1, H_2 and H_3 are shown in Figure 3.3(a), (b) and (c), respectively. Let $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{\{1,2\},\{1,2,3\}\}$, $S_2 = \{\{7\},\{5,6,7\}\}$ and $S_3 = \{\{10\},\Phi\}$. Observe that $\beta_{\mathcal{S}} = 3$ and $\gamma_{\mathcal{S}} = 7$. Let $B = \{3,4\} \subset \{3,4,5,6,7\}$. Take the indicating tensor

$$_BT_{\mathcal{S}}[\mathcal{V}]_{p_1p_2\ldots p_{10}} = \begin{cases} 1, & \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_{10}) = \{1,2,7,10\} \text{ or } \{1,2,7\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then the hypergraph $\bigvee\limits_{_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$ is as shown in Figure 3.3(d).

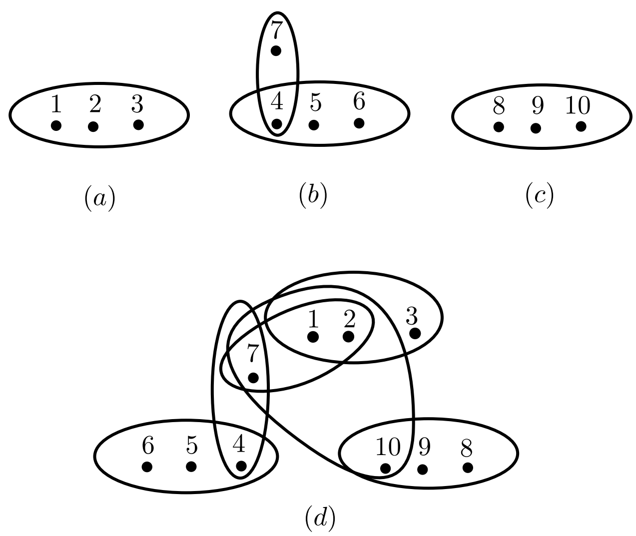

FIGURE 1. (a) The hypergraph H_1, (b) The hypergraph H_2, (c) The hypergraph H_3, (d) $_BT_{\mathcal{S}}[\mathcal{V}]$-join of the hypergraphs in $\mathcal{H}$ constrained by $\mathcal{S}$.

Definition 3.4. Taking H_i as a m-uniform hypergraph for each $i \in [k]$ and $B = \{m\}$ in Definition 3.1 we get the $_{\{m\}}T_{\mathcal{S}}[\mathcal{V}]$-join of the hypergraphs in $\mathcal{H}$. We simply denote it by $\bigvee\limits_{_mT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$ and refer to this hypergraph as the $_mT_{\mathcal{S}}[\mathcal{V}]$-*uniform join of the hypergraphs in* $\mathcal{H}$ *constrained by* $\mathcal{S}$.

Note 3.5. For each $i \in [k]$, if we take $S_i = \mathcal{P}^*(V(H_i))$ in Definition 3.4, we get the join of a set $\mathcal{H} = \{H_i\}_{i=1}^k$ of m-uniform hypergraphs as defined in [13, p. 4].

Theorem 3.6. *For each* $i \in [k]$, *let* H_i *be a hypergraph on* n_i *vertices with* $rank(H_i) = r$. *Let* $\mathcal{H} = (H_i)_{i=1}^k$, $\mathcal{V} = (V(H_i))_{i=1}^k$ *and* $S_i \subseteq \mathcal{P}(V(H_i))$ *be such that* $|W| > 1$ *for any* $W \in S_i$, *for all* $i \in [k]$. *Let* $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ *and*

$B \subseteq \{\beta_{\mathcal{S}}, \beta_{\mathcal{S}}+1, \dots, \gamma_{\mathcal{S}}\}$ be such that $\max B \leq r$. Then any eigenvalue of H_i is an eigenvalue of $\bigvee\limits_{{}_B T_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$.

Furthermore, if (λ, x) is an H-eigenpair (resp. E-eigenpair, Z-eigenpair) of H_i for $i \in [k]$, then (λ, X) is an H-eigenpair (resp. E-eigenpair and Z-eigenpair) of $\bigvee\limits_{{}_B T_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$, where $X = (\mathbf{0}_{n_1}, \dots, \mathbf{0}_{n_{i-1}}, x, \mathbf{0}_{n_{i+1}}, \dots, \mathbf{0}_{n_k})$ and $\mathbf{0}_{n_j}$ is an n_j-tuple whose entries are zero for all $j \in [k]$.

Proof. For simplicity, in this proof, we denote the hypergraph $\bigvee\limits_{{}_B T_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$ by $\mathcal{H}^*$. Since $rank(H_i) = r$ and $\max B \leq r$, by Definition 3.1, we have $rank(\mathcal{H}^*) = r$. For each $i \in [k]$, let $A(H_i) := (a^{(i)}_{t_1 t_2 \dots t_r})$, where $t_1, t_2, \dots, t_r \in V(H_i)$. Let $A(\mathcal{H}^*) := (a_{i_1 i_2 \dots i_r})$. For a hypergraph H_t, let $(\lambda, x) \in (\mathbb{C}, \mathbb{C}^{n_t} \backslash \{\mathbf{0}\})$ be it's eigenpair.

Let $X = (\mathbf{0}_{n_1}, \dots, \mathbf{0}_{n_{t-1}}, x, \mathbf{0}_{n_{t+1}}, \dots, \mathbf{0}_{n_k})$. Clearly, X is an $\sum\limits_{i=1}^{k} n_i$-tuple in $\mathbb{C}^{n_1+n_2+\dots+n_k}$.

For each $j \in \{1, 2, \dots, \sum\limits_{i=1}^{k} n_i\}$, we consider

$$(A(\mathcal{H}^*)X^{r-1})_j = \sum_{(j,i_2,i_3,\dots,i_r) \in E(\mathcal{H}^*)} a_{j i_2 \dots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}. \tag{3.1}$$

If $j \in V(H_t)$, then by Definition 3.1, the above equation (3.1) becomes

$$\begin{aligned}\left(A(\mathcal{H}^*)X^{r-1}\right)_j = &\sum_{(j,i_2,i_3,\dots,i_r) \in E(H_t)} a_{j i_2 \dots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} \\ &+ \sum_{(j,i_2,i_3,\dots,i_r) \in E({}_B T_{\mathcal{S}}[\mathcal{V}])} a_{j i_2 \dots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}. \end{aligned} \tag{3.2}$$

If $(j, i_2, i_3, \dots, i_r) \in E({}_B T_{\mathcal{S}}[\mathcal{V}])$, then $\mathcal{R}^{\blacktriangledown}(j, i_2, \dots, i_r) \notin S_i$ for all $i \in [k]$, since by the definition of an indicating tensor that $\mathcal{R}^{\blacktriangledown}(j, i_2, \dots, i_r) \notin V(H_i)$ for all $i \in [k]$. Therefore, any one of $i_2, i_3, \dots, i_r$ is in $V(H_q)$ for some $q \neq t$. Notice that the corresponding component of X is zero. So, we have

$$\sum_{(j,i_2,i_3,\dots,i_r) \in E({}_B T_{\mathcal{S}}[\mathcal{V}])} a_{j i_2 \dots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} = 0. \tag{3.3}$$

Also since $rank(H_t) = rank(\mathcal{H}^*)$, we have $a_{j i_2 \dots i_r} = a^{(t)}_{j i_2 \dots i_r}$ for all

$$(j, i_2, i_3, \dots, i_r) \in E(H_t).$$

Thus from (3.2) and (3.3), we have

$$\begin{aligned}\left(A(\mathcal{H}^*)X^{r-1}\right)_j &= \sum_{(j,i_2,i_3,\ldots,i_r)\in E(H_t)} a^{(t)}_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} + 0 \\ &= \lambda x_j^{r-1}.\end{aligned} \tag{3.4}$$

If $j \in V(H_l)$ for $l \neq t$, then we have

$$\begin{aligned}\left(A(\mathcal{H}^*)X^{r-1}\right)_j &= \sum_{(j,i_2,i_3,\ldots,i_r)\in E(H_l)} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} \\ &+ \sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r}.\end{aligned} \tag{3.5}$$

Since $l \neq t$, all the components of X correspond to $V(H_l)$ are zero, and so we have

$$\sum_{(j,i_2,i_3,\ldots,i_r)\in E(H_l)} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} = 0. \tag{3.6}$$

Also, being $|W| > 1$ for all $W \in S_i$, any one of $i_2, i_3, \ldots, i_r$ is in $V(H_l)$, and the corresponding component of X is zero. Therefore,

$$\sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} = 0. \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5), we have

$$\left(A(\mathcal{H}^*)X^{r-1}\right)_j = 0. \tag{3.8}$$

Also note that since $x_j = 0$, we have $\lambda x_j^{r-1} = 0$. Thus, from (3.4) and (3.8), we have $A(\mathcal{H}^*)X^{r-1} = \lambda X^{[r-1]}$. Since (λ, x) is an arbitrary eigenpair of H_t for each $t \in [k]$, we conclude that any eigenvalue of H_i is an eigenvalue of $\bigvee_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$.

Furthermore, if we proceed the above arguments with the H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x) of H_i for $i \in [k]$, then we have (λ, X) as an H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x_i) of $\mathcal{H}^*$, since x is a real unit vector. □

The next theorem (resp. its corollary) asserts some eigenvalues and eigenvectors of the join of a set of hypergraphs (resp. join of a set of uniform hypergraphs) defined in [13], by using the notations introduced in Notes 3.2 and 3.5.

Theorem 3.7. *Let $\mathcal{H} = (H_i)_{i=1}^{k}$, where H_i is a hypergraph on n_i vertices with $rank(H_i) = r$. Let $3 \leq k \leq r$ and $S_i = \mathcal{P}^*(V(H_i))$ for all $i \in [k]$. Let $\mathcal{V} = (V(H_i))_{i=1}^{k}$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ and $B \subseteq \{k, k+1, \ldots, r\}$. Then any eigenvalue of H_i is an eigenvalue of $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$.*

Furthermore, if (λ, x) is an H-eigenpair (resp. E-eigenpair, Z-eigenpair) of H_i for $i \in [k]$, then (λ, X) is an H-eigenpair (resp. E-eigenpair and Z-eigenpair) of $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$, where $X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{i-1}}, x, \mathbf{0}_{n_{i+1}}, \ldots, \mathbf{0}_{n_k})$.

Proof. For simplicity, in this proof, we denote the hypergraph $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$ by $\mathcal{H}^*$. Since $B \subseteq \{k, k+1, \ldots, r\}$ and $rank(H_i) = r$ as per Definition 3.1, we conclude that $rank(\mathcal{H}^*) = r$. Let

$$A(H_i) := (a^{(i)}_{t_1 t_2 \ldots t_r}),$$

where $t_1, t_2, \ldots, t_r \in V(H_i)$ for all $i \in [k]$ and $A(\mathcal{H}^*) := (a_{i_1 i_2 \ldots i_r})$. For any eigenpair $(\lambda, x) \in (\mathbb{C}, \mathbb{C}^{n_t} - \{\mathbf{0}\})$ of H_t, we consider

$$X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{t-1}}, x, \mathbf{0}_{n_{t+1}}, \ldots, \mathbf{0}_{n_k}).$$

For every $j \in \{1, 2, \ldots, \sum\limits_{i=1}^{k} n_i\}$, we consider the j-th component of the vector $A(\mathcal{H}^*)X^{r-1}$. Suppose that $j \in V(H_t)$. Then we have

$$\begin{aligned}(A(\mathcal{H}^*)X^{r-1})_j &= \sum_{(j,i_2,i_3,\ldots,i_r) \in E(\mathcal{H}^*)} a_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} \\ &= \sum_{(j,i_2,i_3,\ldots,i_r) \in E(H_t)} a_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} \\ &\quad + \sum_{(j,i_2,i_3,\ldots,i_r) \in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}. \end{aligned} \tag{3.9}$$

Since $j \in V(H_t)$ and $rank(H_t) = rank(\mathcal{H}^*)$, we have

$$\begin{aligned}\sum_{(j,i_2,i_3,\ldots,i_r) \in E(H_t)} a_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} &= \sum_{(j,i_2,i_3,\ldots,i_r) \in E(H_t)} a^{(t)}_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} \\ &= \lambda x_j^{r-1}. \end{aligned} \tag{3.10}$$

Furthermore, since $\mathcal{R}^{\blacktriangledown}(j, i_2, \ldots, i_r) \cap V(H_i) \in S_i$ for each i, any one of $i_2, i_3, \ldots, i_r$ is in $V(H_q)$ for some $q \neq t$. Notice that the corresponding component of X is zero. Therefore, we have

$$\sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{ji_2\ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} = 0. \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.9), we have

$$(A(\mathcal{H}^*)X^{r-1})_j = \lambda x_j^{r-1}. \tag{3.12}$$

Suppose that $j \in V(H_l)$ for $l \neq t$. Then,

$$\begin{aligned}\left(A(\mathcal{H}^*)X^{r-1}\right)_j = &\sum_{(j,i_2,i_3,\ldots,i_r)\in E(H_l)} a_{ji_2\ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} \\ &+ \sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{ji_2\ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}.\end{aligned} \tag{3.13}$$

Since $l \neq t$, all the components of X correspond to $V(H_l)$ are zero and so we have

$$\sum_{(j,i_2,i_3,\ldots,i_r)\in E(H_l)} a_{ji_2\ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} = 0. \tag{3.14}$$

Since $3 \leq k \leq r$, $\mathcal{R}^{\blacktriangledown}(j, i_2, \ldots, i_r) \cap V(H_i) \in S_i$ for each i, the cardinality of a new edge must be atleast three. So, it belongs to a hypergraph other than H_i and H_l. Thus any one of $i_2, i_3, \ldots, i_r$ is in $V(H_s)$ for $s \neq i, l$. So the corresponding component of X is zero. So, from (3.13) and (3.14), we have

$$(A(\mathcal{H}^*)X^{r-1})_j = \sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_BT_{\mathcal{S}}[\mathcal{V}])} a_{ji_2\ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r} = 0. \tag{3.15}$$

Since $x_j = 0$ for all $j \in V(H_l)$ for $l \neq t$, we have $\lambda x_j^{r-1} = 0$. Thus from (3.12) and (3.15), we have $A(\mathcal{H}^*)X^{r-1} = \lambda X^{[r-1]}$. Since (λ, x) is an arbitrary eigenpair of H_t for each $t \in [k]$, it follows that any eigenvalue of H_i is an eigenvalue of $\bigvee\limits_{{}_BT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$.

Furthermore, if we proceed the above arguments with the H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x) of H_i, then we have (λ, X) as an H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x) of $\mathcal{H}^*$, since x is a real unit vector. □

Corollary 3.8. *Let $\mathcal{H} = (H_i)_{i=1}^k$, where H_i is an m-uniform hypergraph with $3 \leq k \leq m$. $S_i = \mathcal{P}^*(V(H_i))$ for all $i \in [k]$. Let $\mathcal{V} = (V(H_i))_{i=1}^k$.*

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ *and* $B = \{m\}$*. Then any eigenvalue of* H_i *is an eigenvalue of* $\bigvee\limits_{{}_mT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$*.*

Furthermore, if (λ, x) *is an* H*-eigenpair (resp.* E*-eigenpair,* Z*-eigenpair) of* H_i *for* $i \in [k]$*, then* (λ, X) *is an* H*-eigenpair (resp.* E*-eigenpair and* Z*-eigenpair) of* $\bigvee\limits_{{}_mT_{\mathcal{S}}[\mathcal{V}]} \mathcal{H}$*, where* $X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{i-1}}, x, \mathbf{0}_{n_{i+1}}, \ldots, \mathbf{0}_{n_k})$*.*

Proof. Since H_i is m-uniform, it follows that $rank(H_i) = m$ for each $i \in [k]$. Thus by taking $r = m$ and $B = \{m\}$ in the above theorem, we get the result. □

4. $(H, \mathcal{T}_{\mathcal{S}})$-JOIN OF THE HYPERGRAPHS CONSTRAINED BY $\mathcal{S}$

In this section, we consider a particular case of $(H, \mathcal{T})$*-join of hypergraphs in* $\mathcal{G}$ given in Definition 2.6.

Definition 4.1. Let H be a hypergraph with vertex set $V(H) = [k]$. Let $\mathcal{G} = \{G_i\}_{i=1}^k$ be family of k hypergraphs. For each $e \in E(H)$, we take $\mathcal{S}_e = \{S_i\}_{i \in e}$, where S_i is a nonempty subset of $\mathcal{P}^*(V(G_i))$,

$$B_e \subseteq \{\beta_{\mathcal{S}_e}, \beta_{\mathcal{S}_e} + 1, \ldots, \gamma_{\mathcal{S}_e}\},$$

$\mathcal{V}_e = (V_i)_{i \in e}$, $N_e := \sum\limits_{i \in e} |V(G_i)|$, and $\mathcal{G}_e = \{G_i \colon i \in e\}$. Let $\mathcal{S} = \{\mathcal{S}_e\}_{e \in E(H)}$. Then we consider $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$, the $(H, \mathcal{T}_{\mathcal{S}})$-join of the hypergraphs in $\mathcal{G}$, where $\mathcal{T}_{\mathcal{S}} = \{{}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]\}_{e \in E(H)}$ in which

$${}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]_{p_1 p_2 \ldots p_{N_e}} = \begin{cases} 1, & \mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_{N_e}) \cap V(G_i) \in S_i \text{ for all } i \in e \\ & \text{with } |\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_{N_e})| \in B_e; \\ 0, & \text{otherwise.} \end{cases}$$

We refer to this hypergraph as the $(H, \mathcal{T}_{\mathcal{S}})$*-join of the hypergraphs in* $\mathcal{G}$ *constrained by* $\mathcal{S}$*.*

The hypergraph defined above can be described as follows: $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$ is the hypergraph with $V(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})) = \bigcup\limits_{i=1}^{k} (V(G_i))$ and

$$E(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})) = \bigcup_{r=1}^{k} (E(G_r)) \bigcup_{e \in E(H)} (E(G_e)),$$

where

$$E(G_e) = \{F \subseteq V(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})) \colon F \cap V(G_i) \in S_i \text{ and } F \notin S_i \ \forall i \in e;$$

$$F \cap V(G_i) = \Phi \ \forall i \notin e \text{ and } |F| \in B_e\}$$

Note 4.2. (i) For each $i \in [k]$, if we take $S_i = \mathcal{P}^*(V(G_i))$ in Definition 4.1, then we get $B_e \subseteq \{|e|, |e|+1, \ldots, \sum_{i\in e}|V(G_i)|\}$. In this case, the operation defined in Definition 4.1 corresponds to the join of non-uniform hypergraphs on a backbone hypergraph, as defined in [13, p. 8].
(ii) For each $i \in [k]$, if we take G_i as a m-uniform hypergraph and $B_e = \{m\}$ in Definition 4.1, we obtain the join of uniform hypergraphs on a backbone hypergraph, as defined in [13, p. 5].

Theorem 4.3. *Let H be a hypergraph with k vertices and $|e| \geq 3$ for every $e \in E(H)$. Let $\mathcal{G} = \{G_i\}_{i=1}^k$, where G_i is a hypergraph on n_i vertices with $rank(G_i) = r$ for all $i \in [k]$. For each edge $e \in E(H)$, we take $\mathcal{S}_e = \{S_i\}_{i\in e}$, where S_i is a nonempty subset of $\mathcal{P}^*(V(G_i))$ and*

$$B_e \subseteq \{|e|, |e|+1, \ldots, \sum_{i\in e}|S_i|\}$$

such that $\max B_e \leq r$. Let $\mathcal{V}_e = (V_i)_{i\in e}$, $\mathcal{S} = \{\mathcal{S}_e\}_{e\in E(H)}$ and let $\mathcal{T}_{\mathcal{S}} = \{_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]\}_{e\in E(H)}$. Then any eigenvalue of G_i is an eigenvalue of $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$.

Furthermore, if (λ, x) is an H-eigenpair (resp. E-eigenpair, Z-eigenpair) of H_i for $i \in [k]$, then (λ, X) is an H-eigenpair (resp. E-eigenpair and Z-eigenpair) of $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$, where $X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{i-1}}, x, \mathbf{0}_{n_{i+1}}, \ldots, \mathbf{0}_{n_k})$.

Proof. Since $\max B_e \leq r$ for each $e \in E(H)$ and $rank(G_i) = r$, we have $rank(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})) = r$. Let $A(G_i) := (a^{(i)}_{t_1 t_2 \ldots t_r})$, where $t_1, t_2, \ldots, t_r \in V(H_i)$ for all $i \in [k]$ and $A(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})) := (a_{i_1 i_2 \ldots i_r})$. We consider an eigenpair $(\lambda, x) \in (\mathbb{C}, \mathbb{C}^{n_t} - \{\mathbf{0}\})$ of G_t.

Let $X = (\mathbf{0}_{n_1}, \ldots \mathbf{0}_{n_{t-1}}, x, \mathbf{0}_{n_{t+1}}, \ldots, \mathbf{0}_{n_k})$. Clearly, X is a $\sum_{i=1}^{k} n_i$-tuple in $\mathbb{C}^{n_1+n_2+\cdots+n_k}$.

For every $j \in \{1, 2, \ldots, \sum_{i=1}^{k} n_i\}$, we have

$$(A(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}}))X^{r-1})_j = \sum_{(j,i_2,i_3,\ldots,i_r)\in E(\mathcal{G}(H,\mathcal{T}_{\mathcal{S}}))} a_{j,i_2,\ldots,i_r} x_{i_2} x_{i_3} \cdots x_{i_r}. \tag{4.1}$$

If $j \in V(G_t)$, then (4.1) can be written as

$$(A(\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})X^{r-1})_j = \sum_{(j i_2 i_3 \ldots i_r)\in E(G_t)} a^{(t)}_{j i_2 \ldots i_r} x_{i_2} x_{i_3} \cdots x_{i_r}$$

$$+\sum_{e\in E(H)}\sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e])} a_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}. \tag{4.2}$$

If $(j,i_2,i_3,\ldots,i_r)\in E({}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e])$ for some $e\in E(H)$, then

$$\mathcal{R}^{\blacktriangledown}(j,i_2,\ldots,i_r)\notin S_i$$

for any $i\in[k]$, since by the definition of an indicating tensor that

$$\mathcal{R}^{\blacktriangledown}(j,i_2,\ldots,i_r)\notin V(H_i)$$

for all $i\in[k]$. Therefore, any one of $i_2,i_3,\ldots,i_r$ is in $V(G_s)$ for $s\neq t$. Thus (4.2) becomes,

$$\begin{aligned}(A(\mathcal{G}(H,\mathcal{T}_{\mathcal{S}}))X^{r-1})_j &= \lambda x_j^{r-1}+0\\ &= \lambda x_j^{r-1}. \end{aligned}\tag{4.3}$$

If $j\in V(G_l)$ for $l\neq t$, then (4.1) can be written as

$$\begin{aligned}(A(\mathcal{G}(H,\mathcal{T}_{\mathcal{S}})X^{r-1})_j &= \sum_{(ji_2i_3\ldots i_r)\in E(G_l)} a^{(l)}_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}\\ &\quad+\sum_{e\in E(H)}\sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e])} a_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}. \end{aligned}\tag{4.4}$$

Since $x_j=0$ for all $j\in V(H_l)$ for $l\neq t$, we have

$$\sum_{(ji_2i_3\ldots i_r)\in E(G_l)} a^{(l)}_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}=0. \tag{4.5}$$

Since $|e|\geq 3$ for every $e\in E(H)$, any one of $i_2,i_3,\ldots,i_r$ is in $V(G_s)$, where $s\neq t,l$. Therefore, the corresponding component of X is zero. Thus, from (4.4) and (4.5), we have

$$\begin{aligned}(A(\mathcal{G}(H,\mathcal{T}_{\mathcal{S}}))X^{r-1})_j &= \sum_{e\in E(H)}\sum_{(j,i_2,i_3,\ldots,i_r)\in E({}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e])} a_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}\\ &= 0. \end{aligned}\tag{4.6}$$

Since $x_j=0$ for all $j\in V(H_l)$ for $l\neq t$, we have $\lambda x_j^{r-1}=0$. Thus, from (4.3) and (4.6), we have $A(\mathcal{H}^*)X^{r-1}=\lambda X^{[r-1]}$. Since (λ,x) is an arbitrary eigenpair of G_t for each $t\in[k]$, it follows that any eigenvalue of G_i is an eigenvalue of $\mathcal{G}(H,\mathcal{T}_{\mathcal{S}})$.

Furthermore, if we proceed the above arguments with the H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ,x) of G_i, then we have (λ,X) as an H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ,x) of $\mathcal{G}(H,\mathcal{T}_{\mathcal{S}})$, since x is a real unit vector. □

For each $i \in [k]$, if we take $S_i = \mathcal{P}^*(V(G_i))$ (resp. $B_e = \{m\}$) in the above theorem, we obtain some eigenvalues and eigenvectors of the join of non-uniform hypergraphs on a backbone hypergraph H (resp. the join of m-uniform hypergraphs on a backbone hypergraph H), as defined in [13], when each edge of H is of cardinality greater than or equal to 3.

The next theorem reveals a relation between the eigenvalues of an m-uniform hypergraph H, and the eigenvalues of a hypergraph $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$.

Theorem 4.4. *Let H be an m-uniform hypergraph on k vertices. Let $\mathcal{G} = \{G_i\}_{i=1}^k$, where G_i is a $r-$ regular, m-uniform hypergraph with $|V(G_i)| = n$ for all $i \in [k]$. Let*

$$\mathcal{S} = \{S_i \colon S_i \text{ is the set of all singleton subsets of } V(G_i) \text{ for all } i \in [k]\}.$$

For each $e \in E(H)$, let $\mathcal{S}_e = \{S_i\}_{i \in e}$ and $\mathcal{V}_e = (V(G_i))_{i \in e}$. Let

$$\mathcal{T}_{\mathcal{S}} = \{{}_mT_{\mathcal{S}_e}[\mathcal{V}_e]\}_{e \in E(H)}.$$

Then for any eigenpair (resp. H-eigenpair) (λ, y) of H, $(r + n^{m-1}\lambda, y \otimes \mathbf{J}_n)$ is an eigenpair (resp. H-eigenpair) of $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$, where $\mathbf{J}_n$ is a $1 \times n$ vector with all of its entries equal to 1.

Proof. For simplicity, in this proof, we denote the hypergraph $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$ by $\mathcal{H}^*$. Let $V(G_i) = \{v_1^i, v_2^i, \dots, v_n^i\}$ for all $i \in [k]$ and let $A(\mathcal{H}^*) := (a_{i_1 i_2 \dots i_m})$. We consider an eigenpair $(\lambda, y) \in (\mathbb{C}, \mathbb{C}^k - \{\mathbf{0}\})$ of H, where $y := (y_1, y_2, \dots, y_k)$. We consider the vector $X = y \otimes \mathbf{J}_n \in \mathbb{C}^{nk}$. We denote X as

$$(y_{11}, y_{12}, \dots, y_{1n}, y_{21}, y_{22}, \dots, y_{2n}, \dots, y_{k1}, y_{k2}, \dots, y_{kn}).$$

For a fixed $p \in [k]$ and $q \in [n]$, the component of $A(\mathcal{H}^*)$ corresponds to the vertex v_q^p is

$$\begin{aligned}(A(\mathcal{H}^*)X^{m-1})_{v_q^p} &= \sum a_{v_q^p v_{q_1}^{p_1} v_{q_2}^{p_2} \dots v_{q_{m-1}}^{p_{m-1}}} y_{p_1 q_1} y_{p_2 q_2} \cdots y_{p_{m-1} q_{m-1}} \\ &= \sum_{\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \dots, v_{q_{m-1}}^{p_{m-1}}\right) \in E(\mathcal{H}^*)} \Delta^m_{p_1 q_1 p_2 q_2 p_{m-1} q_{m-1}},\end{aligned} \tag{4.7}$$

where $\Delta^m_{p_1 q_1 p_2 q_2 p_{m-1} q_{m-1}} := \frac{1}{(m-1)!} y_{p_1 q_1} y_{p_2 q_2} \cdots y_{p_{m-1} q_{m-1}}$, and $p_i \in [k]$, $q_i \in [n]$ for all $i \in [m-1]$.

Since v_q^p is fixed and each $\left\{v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \dots, v_{q_{m-1}}^{p_{m-1}}\right\} \in E(\mathcal{H}^*)$ contributes $(m-1)!$ ordered tuples $\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \dots, v_{q_{m-1}}^{p_{m-1}}\right) \in E(\mathcal{H}^*)$ in the summation in (4.7). Therefore, we obtain

$$(A(\mathcal{H}^*)X^{m-1})_{v_q^p} = \sum_{\left\{v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \dots, v_{q_{m-1}}^{p_{m-1}}\right\} \in E(\mathcal{H}^*)} y_{p_1 q_1} y_{p_2 q_2} \cdots y_{p_{m-1} q_{m-1}}. \tag{4.8}$$

Now by Definition 4.1, the above equation (4.8) can be written as

$$\begin{aligned}
&(A(\mathcal{H}^*)X^{m-1})_{v_q^p} \\
&= \sum_{\substack{p=p_1=\cdots=p_{m-1};\\ \left\{v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right\} \in E(G_p)}} y_{p_1q_1}y_{p_2q_2}\cdots y_{p_{m-1}q_{m-1}} \\
&+ \sum_{(p,p_1,\ldots,p_{m-1})\in E(H)} \sum_{\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right)\in E({}_mT_{\mathcal{S}_e}[\mathcal{V}_e])} \Delta^m_{p_1q_1p_2q_2p_{m-1}q_{m-1}}, \qquad (4.9)
\end{aligned}$$

where $e = \{p, p_1, \ldots, p_{m-1}\}$. Observe that in X, $y_{ij} = y_i$ for all $i \in [k]$, $j \in [n]$ and so (4.9) becomes

$$\begin{aligned}
&(A(\mathcal{H}^*)X^{m-1})_{v_q^p} \\
&= \sum_{\substack{p=p_1=\cdots=p_{m-1};\\ \left\{v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right\} \in E(G_p)}} (y_p)^{m-1} \\
&+ \frac{1}{(m-1)!} \sum_{\{p,p_1,\ldots,p_{m-1}\}\in E(H)} \sum_{\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right)\in E({}_mT_{\mathcal{S}_e}[\mathcal{V}_e])} y_{p_1}y_{p_2}\cdots y_{p_{m-1}}. \qquad (4.10)
\end{aligned}$$

Since the vertex v_q^p is fixed, and S_i is a set of all singleton subsets of $V(G_i)$ for all $i \in [k]$, it follows that $\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right) \in E({}_mT_{\mathcal{S}_e}[\mathcal{V}_e])$ for each $v_{q_i}^{p_i} \in V(G_i)$. Thus,

$$\left|\left\{\left(v_q^p, v_{q_1}^{p_1}, v_{q_2}^{p_2}, \ldots, v_{q_{m-1}}^{p_{m-1}}\right) \in E({}_mT_{\mathcal{S}_e}[\mathcal{V}_e])\right\}\right| = \prod_{i=1}^{m-1} |V(G_i)| = n^{m-1}. \qquad (4.11)$$

Therefore, from (4.10) and (4.11), we have

$$\begin{aligned}
(A(\mathcal{H}^*)X^{m-1})_{v_q^p} &= (y_p)^{m-1} \times deg_{G_p}(v_q^p) \\
&\quad + \frac{n^{m-1}}{(m-1)!} \sum_{(p,p_1,\ldots,p_{m-1})\in E(H)} y_{p_1}y_{p_2}\cdots y_{p_{m-1}} \\
&= r{y_p}^{m-1} + n^{m-1}\lambda {y_p}^{m-1} \\
&= (r + n^{m-1}\lambda){y_p}^{m-1}. \qquad (4.12)
\end{aligned}$$

Since $y_p = y_{pq}$, from (4.12), we have $r + n^{m-1}\lambda$ is an eigenvalue of $A(\mathcal{H}^*)$.

Furthermore, if we proceed the above arguments with the H-eigenpair (λ, y) of H, then we have $(r + n^{m-1}\lambda, X)$ as an H-eigenpair of $\mathcal{G}(H, \mathcal{T}_\mathcal{S})$, since x is a real vector. □

Definition 4.5. ([4]) Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. The lexicographic product $H = H_1 o H_2$ has vertex set $V(H) = V_1 \times V_2$ and edge set

$$\begin{aligned} E(H) &= \{e \subseteq V(H) \colon p_1(e) \in E_1, |p_1(e)| = |e|\} \\ &\cup \{\{x\} \times e_2 \colon x \in V_1, e_2 \in E_2\}, \end{aligned}$$

where $p_1(e)$ denotes the set of all first co-ordinate in each two tuples in e.

Note 4.6. Let H and H' be two hypergraphs and let $|V(H)| = k$. Let $\mathcal{G} = \{G_i \colon G_i = H'\}_{i=1}^{k}$. Let $\mathcal{S} = \{S_i \colon S_i$ is the set of all singleton subsets of $V(G_i)$ for all $i \in [k]\}_{i=1}^{k}$. Then the lexicographic product of H and H' can be viewed as a $(H, \mathcal{T}_{\mathcal{S}})$-join of the hypergraphs in $\mathcal{G}$ constrained by $\mathcal{S}$ (c.f. [14, S. No. 5 in Table 4, p.130]).

As a direct consequence of Note 4.6 and Theorem (4.4), we have the following result.

Corollary 4.7. *Let H_1 and H_2 be m-uniform hypergraphs with H_2 is $r-$regular. If (λ, y) is an eigenpair (resp. H-eigenpair) of H_1, then*

$$(r + n^{m-1}\lambda, y \otimes \mathbf{J}_n)$$

is an eigenpair (resp. H-eigenpair) of the lexicographic product of H_1 and H_2.

Definition 4.8. Let H be a complete graph with vertex set $[k]$, and let $\mathcal{G} = \{G_i\}_{i=1}^{k}$ be a given set of k hypergraphs. For each edge

$$e := \{i, j\} \in E(H),$$

take $\mathcal{V}_e = (V_i)_{i \in e}$, $\mathcal{S}_e = \{S_i^e, S_j^e\}$, where S_i^e is a non-empty subset of $\mathcal{P}^*(V(G_i))$ for all $i \in [k]$, and take $B_e \subseteq \{\beta_{\mathcal{S}_e}, \beta_{\mathcal{S}_e}+1, \ldots, \gamma_{\mathcal{S}_e}\}$. Let $\mathcal{S} = \{\mathcal{S}_e \colon e \in E(H)\}$. Then we consider a $\mathcal{G}(H, \mathcal{T}_{\mathcal{S}})$-join of hypergraphs in $\mathcal{G}$, where

$$\mathcal{T}_{\mathcal{S}} = \{{}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]\}_{e \in E(H)}$$

in which

$${}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]_{p_1 p_2 \ldots p_{N_e}} = \begin{cases} 1, & \mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_{N_e}) \cap V(G_i) \in S_i^e \text{ for all } i \in e \\ & \text{with } |\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_{N_e})| \in B_e; \\ 0, & \text{otherwise.} \end{cases}$$

We denote this hypergraph by $\mathcal{G}^*(H, \mathcal{T}_{\mathcal{S}})$ and refer to it as the *couple join of the hypergraphs in $\mathcal{G}$ constrained by $\mathcal{S}$.*

The hypergraph defined above can be described as follows: $\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})$ is the hypergraph with $V(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})) = \bigcup_{i=1}^{k}(V(G_i))$ and

$$E(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})) = \bigcup_{r=1}^{k}(E(G_r)) \bigcup_{1\le i<j\le k}(E(G_{i,j})),$$

where

$$E(G_{i,j}) = \{e \subseteq V(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))\colon e\cap V(G_p) \in S_p^e \text{ and } e \notin S_p^e \ \forall p = i,j; \\ e\cap V(G_p) = \Phi \ \forall p \neq i,j \text{ and } |e| \in B_e\}$$

for all $1 \le i < j \le k$.

Theorem 4.9. *Let H be a complete graph on k vertices and let $\mathcal{G} = \{G_i\}_{i=1}^k$, where G_i is a hypergraph on n_i vertices with $rank(G_i) = r$. For each edge $e := \{i,j\} \in E(H)$, take $\mathcal{S}_e = \{\mathcal{S}_i^e, \mathcal{S}_j^e\}$ such that $|W| > 1$ for any $W \in S_t^e$, where $\mathcal{S}_t^e$ is a non-empty subset of $\mathcal{P}^*(V(G_t))$ for all $t = i,j$ and take $B_e \subseteq \{\beta_{\mathcal{S}_e}, \beta_{\mathcal{S}_e}+1, \ldots, \gamma_{\mathcal{S}_e}\}$ with $\max B_e \le r$. Let $\mathcal{V}_e = (V_i)_{i\in e}$, $\mathcal{S} = \{\mathcal{S}_e\}_{e\in E(H)}$ and $\mathcal{T}_\mathcal{S} = \{{}_{B_e}T_{\mathcal{S}_e}[\mathcal{V}_e]\}_{e\in E(H)}$. Then any eigenvalue of G_i is an eigenvalue of $\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})$.*

Furthermore, if (λ, x) is an H-eigenpair (resp. E-eigenpair, Z-eigenpair) of H_i for $i \in [k]$, then (λ, X) is an H-eigenpair (resp. E-eigenpair and Z-eigenpair) of $\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})$, where $X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{i-1}}, x, \mathbf{0}_{n_{i+1}}, \ldots, \mathbf{0}_{n_k})$.

Proof. Since $rank(G_i) = r$ and $\max B_e \le r$, as per Definition 4.8, we have $rank(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})) = r$. Let $A(G_i) := (a^{(i)}_{t_1t_2\ldots t_r})$, where $t_1, t_2, \ldots, t_r \in V(G_i)$ for all $i \in [k]$ and $A(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S})) := (a_{i_1i_2\ldots i_r})$, $i_1, i_2 \ldots, i_r \in V(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))$. For any eigenpair $(\lambda, x) \in (\mathbb{C}, \mathbb{C}^{n_t} - \{\mathbf{0}\})$ of G_t, we consider

$$X = (\mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_{t-1}}, x, \mathbf{0}_{n_{t+1}}, \ldots, \mathbf{0}_{n_k}).$$

For each $j \in \{1, 2, \ldots, \sum_{i=1}^{k} n_i\}$, the j-th component of $A(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))X^{r-1}$ is

$$(A(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))X^{r-1})_j = \sum_{(j,i_2,i_3,\ldots,i_r)\in E(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))} a_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}.$$

If $j \in V(G_t)$ and by Definition 4.8, the above equation can be written as

$$(A(\mathcal{G}^*(H,\mathcal{T}_\mathcal{S}))X^{r-1})_j = \sum_{(j,i_2,i_3,\ldots,i_r)\in E(G_t)} a_{ji_2\ldots i_r}x_{i_2}x_{i_3}\cdots x_{i_r}$$

$$+ \sum_{z\in[k]\backslash\{t\}} \sum_{(j,i_2,i_3,\ldots,i_r)\in E(G_{t,z})} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r}. \quad (4.13)$$

Since $j \in V(G_t)$, we have

$$\sum_{(j,i_2,i_3,\ldots,i_r)\in E(G_t)} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} = \sum_{(j,i_2,i_3,\ldots,i_r)\in E(G_t)} a^{(t)}_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} = \lambda x_j^{r-1}. \quad (4.14)$$

If $(j, i_2, i_3, \ldots, i_r) \in E(G_{t,z})$ for some $z \in [k]\backslash\{t\}$, then any one of $i_2, i_3, \ldots, i_r$ is in $V(G_z)$ and since the corresponding component of X is zero, we have

$$\sum_{z\in[k]\backslash\{t\}} \sum_{(j,i_2,i_3,\ldots,i_r)\in E(G_{t,z})} a_{ji_2\ldots i_r} x_{i_2}x_{i_3}\cdots x_{i_r} = 0. \quad (4.15)$$

Substituting (4.14) and (4.15) in (4.13), we get

$$(A(\mathcal{G}^*(H, \mathcal{T}_\mathcal{S}))X^{r-1})_j = \lambda x_j^{r-1}. \quad (4.16)$$

Suppose that $j \in V(G_l)$ for $l \neq t$. If $(j, i_2, i_3, \ldots, i_r) \in E(G_{t,z})$ for some $z \in [k]\backslash\{t\}$, then by the assumption that $|W| > 1$ for all $W \in S_i^e$ for any $e \in E(H)$, it follows that any one of $i_2, i_3, \ldots, i_r$ must belong to $V(G_z)$ for those z. Notice that the components of X corresponding to $V(G_z)$ are zero. Therefore, (4.13) becomes

$$(A(\mathcal{G}^*(H, \mathcal{T}_\mathcal{S}))X^{r-1})_j = 0. \quad (4.17)$$

Since $x_j = 0$ for all $j \in V(G_l)$, $l \neq t$ $\lambda x_j^{r-1} = 0$. Thus from (4.16) and (4.17), we have $A(\mathcal{H}^*)X^{r-1} = \lambda X^{[r-1]}$. Since (λ, x) is an arbitrary eigenpair of G_t for each $t \in [k]$, it follows that any eigenvalue of G_i is an eigenvalue of $\mathcal{G}^*(H, \mathcal{T}_\mathcal{S})$.

Furthermore, if we proceed the above arguments with the H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x) of G_i, then we have (λ, X) as an H-eigenpair (resp. E-eigenpair, Z-eigenpair) (λ, x) of $\mathcal{G}^*(H, \mathcal{T}_\mathcal{S})$, since x is a real unit vector. □

Acknowledgments

The first author gratefully acknowledges the financial support provided by the University Grants Commission (UGC), Government of India, in the form of a Senior Research Fellowship (NTA Ref. No.: 221610053976).

References

1. A. Banerjee, On the spectrum of hypergraphs, *Linear Algebra Appl.*, **614** (2021), 82–110.
2. A. Banerjee, A. Char and B. Mondal, Spectra of general hypergraphs, *Linear Algebra Appl.*, **518** (2017), 14–30.
3. J. Cooper and A. Dutle, Spectra of uniform hypergraphs, *Linear Algebra Appl.*, **436**(9) (2012), 3268–3292.
4. M. Hellmuth, L. Ostermeier and P. F. Stadler, A survey on hypergraph products, *Math. Comput. Sci.*, **6** (2012), 1–32.
5. S. Hu and L. Qi, Algebraic connectivity of an even uniform hypergraph, *J. Comb. Optim.*, **24**(4) (2012), 564–579.
6. L. Kang, L. Liu and E. Shan, The eigenvectors to the p-spectral radius of general hypergraphs, *J. Comb. Optim.*, **38** (2019), 556–569.
7. G. Li, L. Qi and G. Yu, The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory, *Numer. Linear Algebra Appl.*, **20**(6) (2013), 1001–1029.
8. H. Lia and C. Dengb, On the principal eigenvectors of general hypergraphs, *ScienceAsia*, **50**(2) (2024), Article ID: 2024006.
9. K. Pearson and T. Zhang, Eigenvalues of the adjacency tensor on products of hypergraphs, *Int. J. Contemp. Math. Sciences*, **8**(4) (2013), 151–158.
10. L. Qi, H^{+}-eigenvalues of Laplacian tensor and signless Laplacians, *Commun. Math. Sci.*, **12** (2014), 1045–1064.
11. L. Qi and Z. Luo, *Tensor Analysis: Spectral theory and special tensors*, SIAM, Philadelphia, 2017.
12. S. Rota Bulò and M. Pelillo, New bounds on the clique number of graphs based on spectral hypergraph theory, *International Conference on Learning and Intelligent Optimization*, Berlin, Heidelberg: Springer Berlin Heidelberg, (2009), 45–58.
13. A. Sarkar and A. Banerjee, Joins of hypergraphs and their spectra, *Linear Algebra Appl.*, **603** (2020), 101–129.
14. R. Vishnupriya and R. Rajkumar, Tensor join of hypergraphs and its spectra, *Le Matematiche*, **78**(1) (2023), 111–147.
15. J. Xie and A. Chang, On the Z-eigenvlues of the signless Laplacian tensor or an even uniform hypergraph, *Numer. Linear Algebra Appl.*, **20**(6) (2013), 1030–1045.
16. W. Zhang, L. Liu, L. Kang and Y. Bai, Some properties of the spectral radius for general hypergraphs, *Linear Algebra Appl.*, **513** (2017), 103–119.

Ramkumar Vishnupriya
Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram–624 302, Tamil Nadu, India.
Email: rrvmaths@gmail.com

Rajendran Rajkumar
Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram–624 302, Tamil Nadu, India.
Email: rrajmaths@yahoo.co.in

Journal of Algebraic Systems

ON THE SPECTRA OF TENSOR JOIN OF HYPERGRAPHS

R. VISHNUPRIYA AND R. RAJKUMAR

دربارهی طیفهای الحاق تانسوری ابرگرافها

رامکومار ویشنوپریا[۱] و راجندران راجکومار[۲]

[۱,۲]گروه ریاضی، مؤسسه روستایی گاندیگرام (دارای عنوان دانشگاه)، گاندیگرام، تامیل نادو، هند

در این مقاله، برخی ردههای خاص از ابرگرافها را که از طریق الحاق تانسوری ابرگرافها ساخته میشوند بررسی میکنیم؛ بهویژه، الحاق تانسوری ابرگرافها با قید زیرمجموعههای رأسی و نیز $(H, \mathcal{T}_S)$-الحاق ابرگرافها با قید S. با ساختن بردارهای ویژهی متناظر، تعدادی از مقادیر ویژهی تانسور مجاورت این دسته از ابرگرافها را تعیین میکنیم. نشان میدهیم که مقادیر ویژهی تانسور مجاورت ابرگرافهای تشکیلدهنده، خود از جمله مقادیر ویژهی تانسور مجاورت ابرگراف حاصل از الحاق مجموعهای از ابرگرافها نیز هستند. به علاوه، بهعنوان حالتی خاص، برخی از مقادیر ویژه و بردارهای ویژهی تانسور مجاورت را برای الحاق ابرگرافهای غیر یکنواخت بر روی یک ابرگراف پایه H (و بهطور مشابه، برای الحاق ابرگرافهای m-یکنواخت بر روی ابرگراف پایه H) ارائه میکنیم. همچنین، رابطهای میان مقادیر ویژهی تانسور مجاورت یک ابرگراف H و برخی از مقادیر ویژهی تانسور مجاورت $(H, \mathcal{T}_S)$-الحاق ابرگرافها با قید S برقرار میکنیم. با استفاده از این رابطه، تعدادی از مقادیر ویژه و بردارهای ویژهی متناظر آنها را برای تانسور مجاورت حاصلضرب واژگانی دو ابرگراف به دست میآوریم.

کلمات کلیدی: ابرگرافها، الحاق تانسوری، تانسور مجاورت، مقادیر ویژه.