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SOME REMARKS ON WEAKLY S-LASKERIAN MODULES

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Abstract. The rings considered in this paper are commutative with identity and the modules are unitary. We use R to denote a ring, S to denote a multiplicatively closed subset (m.c. subset) of R, and M to denote a module over R. We say that M is an S-Laskerian (resp., a strongly S-Laskerian) R-module if M is an S-finite R-module and for any submodule N of M, either $(N :_R M) \cap S \neq \emptyset$ or there exist $t \in S$ and an S-decomposable (resp., strongly S-decomposable) submodule K of M (depending on N) such that $tN \subseteq K \subseteq N$. We say that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module if M is an S-finite R-module and any S-finite proper submodule of M is an S-Laskerian (resp., a strongly S-Laskerian) R-module. In this paper, we discuss some results on the basic properties of weakly S-Laskerian modules and we extend some of the properties of S-Laskerian modules to weakly S-Laskerian modules.

1. Introduction

The rings considered in this paper are commutative with identity and the modules are unitary. Unless otherwise specified, the modules considered in this paper are nonzero. This paper is motivated by the interesting results proved on weakly S-Noetherian rings by Kim and Lim [10] and the research work of Heinzer and Lantz on Laskerian rings [9]. Let R be a ring. Let M be a module over R. Recall that R is a *weakly Noetherian ring* if each finitely generated proper ideal of R is a Noetherian R-module [11]. We use f.g. for finitely generated. Let S be a multiplicatively closed subset (m.c. subset) of R. We say that M is *S-finite* if $sM \subseteq F$ for some $s \in S$ and some f.g. submodule F of M [1, see pp.4409-4410]. Recall that M is called *S-Noetherian* if every submodule of M is an S-finite module [1]. We say that R is an *S-Noetherian ring* if R is S-Noetherian regarded as a module over R. Anderson and Dumitrescu stated and proved S-variant of several well known results on Noetherian rings to S-Noetherian rings, see [1, Corollaries 5, 7 and Propositions 9, 10]. Motivated by the research work of Mahdou and Hasani on weakly Noetherian rings [11] and by the research work of Anderson and

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Dumitrescu on S-Noetherian rings [1], Kim and Lim introduced the concept of a weakly S-Noetherian ring [10]. We say that R is a *weakly S-Noetherian ring* if any S-finite proper ideal of R is an S-Noetherian module [10]. For several interesting results on weakly S-Noetherian rings, the reader can refer [10]. The concept of an S-primary submodule was introduced by Farshadifar [7]. Recall that a submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be an *S-primary submodule* if there exists $s \in S$ such that whenever $rm \in N$, where $r \in R$ and $m \in M$, then either $sr \in \sqrt{(N :_R M)}$ or $sm \in N$ [7]. Several interesting properties of S-primary submodules of a module were proved in detail by Ansari-Toroghy and Pourmortazavi [2]. We say that an ideal I of R with $I \cap S = \emptyset$ is an *S-primary ideal* if I is an S-primary submodule of R regarded as an R-module. For some results on S-primary ideals of a commutative ring, the reader is referred to [16]. For more information on S-primary ideals of a commutative ring, one can refer [6].

We recall some needed definitions and results from commutative ring theory. The definitions of a Laskerian module and a Laskerian ring are well known, see [4, Exercise 23, p.295]. For the definition of a strongly Laskerian module (resp., strongly Laskerian ring), refer [4, Exerise 28, p.298]. For interesting and inspiring theorems on Laskerian (resp., strongly Laskerian) rings and some related classes of rings, the reader is referred to [9].

An S-primary submodule N of M is said to be *strongly S-primary* if there exist $s' \in S$ and $k \in \mathbb{N}$ such that $s'(\sqrt{(N :_R M)})^k \subseteq (N :_R M)$ [15]. A submodule K of M with $(K :_R M) \cap S = \emptyset$ is said to be *S-decomposable* (resp., *strongly S-decomposable*) if K is a finite intersection of S-primary (resp., strongly S-primary) submodules of M. We say that M is an *S-Laskerian* (resp., *a strongly S-Laskerian*) R-module if M is an S-finite R-module and for each submodule N of M, either $(N :_R M) \cap S \neq \emptyset$ or there exist $t \in S$ (depending on N) and an S-decomposable (resp., a strongly S-decomposable) submodule K of M such that $tN \subseteq K \subseteq N$. It is known that an S-finite R-module M is an S-Laskerian (resp., a strongly S-Laskerian) R-module if and only if each submodule N of M with $(N :_R M) \cap S = \emptyset$, N is S-decomposable (resp., strongly S-decomposable) by [15, Theorem 2.11 ((1) $\Leftrightarrow$ (4))]. If M is S-Noetherian, then each submodule N of M with $(N :_R M) \cap S = \emptyset$, N is S-decomposable [13]. The ring R is said to be an *S-Laskerian* (resp., *a strongly S-Laskerian*) ring if R is S-Laskerian (resp., strongly S-Laskerian) regarded as a module over R. For more information on S-Laskerian (resp., strongly S-Laskerian) rings, refer [19, 18, 17].

In this paper, the concept of a weakly S-Laskerian (resp., weakly strongly S-Laskerian) module is introduced and its basic properties are studied. We say that M is a *weakly S-Laskerian* (resp., *weakly strongly S-Laskerian*) R-module if M is an S-finite R-module and any S-finite proper submodule of M is an S-Laskerian (resp., a strongly S-Laskerian) R-module. We say that R is a *weakly S-Laskerian* (resp., *weakly strongly S-Laskerian*) ring if R is weakly S-Laskerian (resp., weakly strongly S-Laskerian) regarded as a module over R.

For a ring R, we denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. We denote the set of all minimal prime ideals of R by $Min(R)$. We denote the group of units of R by $U(R)$. The cardinality of a set A is denoted by $|A|$. If A is a proper subset of a set B, then it is denoted by $A \subset B$.

This paper consists of three sections. In Section 2, we discuss some results on the basic properties of weakly S-Laskerian (resp., weakly strongly S-Laskerian) modules. In Section 3, we extend some of the properties of S-Laskerian modules to weakly S-Laskerian modules.

2. Some basic properties of weakly S-Laskerian modules

In this paper, unless otherwise specified, we use R to denote a commutative ring with identity, S to denote a m.c. subset of R, and M to denote an unitary R-module.

In this section, we discuss some results on the basic properties of weakly S-Laskerian (resp., weakly strongly S-Laskerian) modules.

Proposition 2.1. *If M is an S-Laskerian (resp., a strongly S-Laskerian) R-module, then M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module.*

Proof. As M is an S-Laskerian (resp., a strongly S-Laskerian) R-module by assumption, M is an S-finite R-module. Any S-finite proper submodule N of M is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.3]. This shows that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. □

The following example is due to Gilmer and Heinzer [8, Example p.16] and it illustrates that the converse of Proposition 2.1 can fail to hold. Recall that R is said to be *quasi-local* if $|Max(R)| = 1$. We denote the nilradical of R by $nil(R)$. We say that R is *reduced* if $nil(R) = (0)$. We denote the power series ring in n variables $X_1, \ldots, X_n$ over R by $R[[X_1, \ldots, X_n]]$.

Example 2.2. For a set $\{X_i\}_{i=1}^{\infty}$ of indeterminates, consider the integral domain $D = \bigcup_{n=1}^{\infty} K[[X_1, \dots, X_n]]$, where K is a field. If I is the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$, then the ring $R = \frac{D}{I}$ with $S = \{1 + I\}$ is a weakly strongly S-Laskerian ring but R is not an S-Laskerian ring.

Proof. For each $i \in \mathbb{N}$, it is convenient to denote $X_i + I$ by x_i. It was already noted, see [8, Example, p.16] that R is a quasi-local reduced ring with $\mathfrak{m} = \sum_{i=1}^{\infty} Rx_i$ as its unique maximal ideal and it was observed that $Min(R)$ is infinite and $Min(R) = \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$. Let A be any f.g. proper ideal of R. Let B be any R-submodule of A. Then B is a f.g. ideal of R by [10, Example 5(1)]. This implies that any f.g. proper ideal of R is a Noetherian R-module and hence, a strongly Laskerian R-module. Note that $1 + I$ is the identity element of R. Let $S = \{1 + I\}$. Then S is a m.c. subset of R. It is clear that a proper ideal of R is S-finite if and only if it is finitely generated. As any f.g. proper ideal of R is a strongly Laskerian R-module, we get that any S-finite proper ideal of R is a strongly Laskerian R-module. Therefore, R is a weakly strongly S-Laskerian ring. Since $Min(R)$ is infinite, it follows that the zero ideal of R does not admit any primary decomposition by [3, Propositions 4.5 and 4.6], so R is not Laskerian. As $S = \{1 + I\}$, $S^{-1}R = R$ is not Laskerian, so R is not S-Laskerian by [15, Corollary 2.13 ((1) $\Rightarrow$ (2))]. Therefore, R is a weakly strongly S-Laskerian ring but R is not an S-Laskerian ring. □

Corollary 2.3. *If M is an S-Noetherian R-module, then M is a weakly strongly S-Laskerian R-module.*

Proof. If M is an S-Noetherian R-module, then M is a strongly S-Laskerian R-module by [15, Corollary 2.15]. Hence, M is a weakly strongly S-Laskerian R-module by Proposition 2.1. □

We deduce the following corollary with the help of [15, Corollary 2.15].

Corollary 2.4. *If R is a weakly S-Noetherian ring, then R is a weakly strongly S-Laskerian ring.*

Proof. If R is a weakly S-Noetherian ring, then for any S-finite proper ideal I of R, I is an S-Noetherian R-module, so I is a strongly S-Laskerian R-module by [15, Corollary 2.15]. Hence, R is a weakly strongly S-Laskerian ring. □

The following example illustrates that the converse of Corollary 2.4 can fail to hold. We denote the polynomial ring in one variable X over R by $R[X]$.

Example 2.5. If $\mathbb{A}$ is the field of algebraic numbers, then $R = \mathbb{Q} + X\mathbb{A}[X]$ is a weakly strongly S-Laskerian ring for any m.c. subset S of R but R is not a weakly $\{1\}$-Noetherian ring.

Proof. As $\mathbb{A}[X]$ is a principal ideal domain (P.I.D.), it is a Noetherian domain. Hence, it is a strongly Laskerian domain. Observe that $X\mathbb{A}[X]$ is an ideal common to both $\mathbb{A}[X]$ and R and $\frac{R}{X\mathbb{A}[X]} \cong \mathbb{Q}$ as rings, so $\frac{R}{X\mathbb{A}[X]}$ is a field. Therefore, R is a strongly Laskerian ring by [12, Theorem 8 and Corollary 9]. Hence, for any m.c. subset S of R, R is a strongly S-Laskerian ring by [15, Corollary 2.14]. Therefore, R is a weakly strongly S-Laskerian ring for any m.c. subset S of R by Proposition 2.1. Note that $\{1\}$ is a m.c. subset of R. Thus, R is a weakly strongly $\{1\}$-Laskerian ring. We claim that R is not a weakly $\{1\}$-Noetherian ring. If R is a weakly $\{1\}$-Noetherian ring, then R is a $\{1\}$-Noetherian ring by [10, Proposition 1(1)], since R is an integral domain but not a field. This implies that each ideal of R is finitely generated. As $\mathbb{A}$ is an infinite algebraic extension of $\mathbb{Q}$, the ideal $X\mathbb{A}[X]$ is not a f.g. ideal of R. Therefore, R is not a weakly $\{1\}$-Noetherian ring. □

Recall that an element $r \in R$ is said to be a *zero-divisor* of M if $rm = 0$ for some $m \in M \setminus \{0\}$. We denote the set of all zero-divisors of M by $Z_R(M)$ or $Z(M)$. The following proposition is motivated by [10, Proposition 1(1)].

Proposition 2.6. *If $r \in R \setminus Z(M)$ is such that $rM \neq M$, then M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module if and only if M is an S-Laskerian (resp., a strongly S-Laskerian) R-module.*

Proof. Assume that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. As M is an S-finite R-module, there exist $s \in S$ and a f.g. submodule F of M such that $sM \subseteq F \subseteq M$. Hence,

$$s(rM) \subseteq rF \subseteq rM.$$

This implies that rM is an S-finite R-module, since rF is a f.g. R-submodule of rM. As $rM \neq M$, we obtain that rM is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Since $r \notin Z(M)$, $f : M \to rM$ defined by $f(m) = rm$ is an isomorphism of R-modules, so M is an S-Laskerian (resp., a strongly S-Laskerian) R-module.

The converse part follows from Proposition 2.1 and it does not need the assumption that $rM \neq M$ for some $r \in R \setminus Z(M)$. □

We say that M is a *multiplication module* provided for each submodule N of M, there exists an ideal I of R such that $N = IM$ [5]. The following proposition is motivated by [10, Proposition 1(2)].

Proposition 2.7. *If M is a multiplication module and if every maximal ideal of R is S-finite, then the following statements are equivalent:*

(1) *M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module;*
(2) *M is an S-finite R-module and every maximal submodule of M is an S-Laskerian (resp., a strongly S-Laskerian) R-module.*

Proof. (1) $\Rightarrow$ (2). Assume that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. By definition, M is an S-finite R-module. Let K be any maximal submodule of M. Since M is a nonzero multiplication module, $K = \mathfrak{m}M$ for some $\mathfrak{m} \in Max(R)$ by [5, Theorem 2.5(ii)]. As M (resp., $\mathfrak{m}$) is an S-finite R-module, there exist $s_1, s_2 \in S$, a f.g. submodule F of M, and a f.g. ideal I of R with $I \subseteq \mathfrak{m}$ such that $s_1M \subseteq F \subseteq M$ and $s_2\mathfrak{m} \subseteq I \subseteq \mathfrak{m}$. Then $s_1s_2 \in S$ and $s_1s_2K = s_1s_2\mathfrak{m}M \subseteq s_1IM \subseteq IF \subseteq \mathfrak{m}M$. As IF is a f.g. R-module, it follows that K is an S-finite R-module. Thus, K is an S-finite proper submodule of M. Therefore, K is an S-Laskerian (resp., a strongly S-Laskerian) R-module.

(2) $\Rightarrow$ (1). Let N be any S-finite proper submodule of M. Then $N \subseteq K$ for some maximal submodule K of M by [5, Theorem 2.5(i)]. By assumption, K is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Hence, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.3]. As M is an S-finite R-module by assumption, it follows that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. □

Observe that R is a multiplication R-module and the set of maximal R-submodules of R equals $Max(R)$. The following example illustrates the previous proposition.

Example 2.8. If $R = K[X] + (1 + XY)K[X, Y]$, where $K[X, Y]$ is the polynomial ring in two variables X, Y over a field K, then R admits at least one $\mathfrak{m} \in Max(R)$ such that $\mathfrak{m}$ is not an S-Laskerian R-module, where $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$.

Proof. Observe that $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of R. It is known that R is not an S-Laskerian ring, see [16, Example 3.8(2)]. By [14, Proposition 2.3(2)], every maximal ideal of R is f.g., so $\mathfrak{m}$ is S-finite for every $\mathfrak{m} \in Max(R)$. Since R is an integral domain and $RX \neq R$, by applying Proposition 2.6 with $M = R$, we get that R is not a weakly S-Laskerian ring. Therefore, $\mathfrak{m}$ is not an S-Laskerian R-module for at least one $\mathfrak{m} \in Max(R)$ by (2) $\Rightarrow$ (1) of Proposition 2.7. □

It is well known that $R \times M$, the direct product of R-modules R and M can be made into a ring by defining multiplication as follows. For any $(r_1, m_1), (r_2, m_2) \in R \times M$, $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. The ring constructed in this way is called the ring obtained by using *Nagata's principle of idealization* and is denoted by $R(+)M$. Note that $R(+)M$ is a commutative ring with identity $(1, 0)$. This construction is quite useful for providing examples and counterexamples.

In the following example, we provide a ring T and a m.c. subset S_1 of T such that T is not a weakly S_1-Laskerian ring and no maximal ideal of T is a weakly S_1-Laskerian T-module.

Example 2.9. If $T = \mathbb{Z}(+)\mathbb{Q}$ and $S_1 = \{(n, 0) \mid n \in \mathbb{Z}\backslash\{0\}\}$, then S_1 is a m.c. subset of T, T is not a weakly S_1-Laskerian ring, and $\mathfrak{m}$ is not a weakly S_1-Laskerian T-module for any $\mathfrak{m} \in Max(T)$.

Proof. As $\mathbb{Z}\backslash\{0\}$ is a m.c. subset of $\mathbb{Z}$, it follows that

$$S_1 = \{(n, 0) \mid n \in \mathbb{Z}\backslash\{0\}\}$$

is a m.c. subset of T. If $n \in \mathbb{N}\backslash\{1\}$, then $(n, 0) \notin Z(T)$ and as $(1, 0) \notin T(n, 0)$, it follows that $T(n, 0) \neq T$. As T is not an S_1-Laskerian ring, by [15, Example 3.16], it follows from Proposition 2.6 that T is not a weakly S_1-Laskerian ring. We use $\mathbb{P}$ to denote the set of all prime numbers. Note that $Max(T) = \{p\mathbb{Z}(+)M \mid p \in \mathbb{P}\}$. Observe that for any prime number p, $p\mathbb{Z}(+)M = T(p, 0)$, so every maximal ideal of T is principal and it is clear that $T(p, 0) \cong T$ as T-modules, since $(p, 0) \notin Z(T)$. Therefore, no maximal ideal of T is a weakly S_1-Laskerian T-module. □

If M is an S-finite R-module, then the following theorem provides some necessary and sufficient conditions such that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. For a submodule N of M,

$$f : N \to S^{-1}N$$

is the usual homomorphism of R-modules defined by $f(x) = \frac{x}{1}$.

Theorem 2.10. *If M is an S-finite R-module, then the following statements are equivalent:*

(1) *M is a weakly S-Laskerian (resp„ weakly strongly S-Laskerian) R-module;*
(2) *If N is any S-finite proper submodule of M, then $S^{-1}N$ is a Laskerian (resp., strongly Laskerian) $S^{-1}R$-module and for any submodule K of N with $(K :_R N) \cap S = \emptyset$, $f^{-1}(S^{-1}K) = (K :_N s)$ for some $s \in S$;*

(3) *For any S-finite proper submodule N of M and any submodule K of N with $(K :_R N) \cap S = \emptyset$, there exist $s \in S$ and primary (resp., strongly primary) submodules $Q_1, \ldots, Q_l$ of N with $(Q_i :_R N) \cap S = \emptyset$ for each $i \in \{1, \ldots, l\}$ such that $(K :_N s) = \bigcap_{i=1}^{l} Q_i$ is a primary (resp., strong primary) decomposition in N.*

Proof. (1) $\Rightarrow$ (2). If N is any S-finite proper submodule of M, then N is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Hence, (2) follows by applying [15, Proposition 2.12 ((1) $\Rightarrow$ (2))] to N.

(2) $\Rightarrow$ (3). If N is any S-finite proper submodule of M, then [15, Proposition 2.12 (2)] holds for N. Therefore, (3) follows by applying [15, Proposition 2.12 ((2) $\Rightarrow$ (3))] to N.

(3) $\Rightarrow$ (1). By hypothesis, M is an S-finite R-module. If N is any S-finite proper submodule of M, then [15, Proposition 2.12(3)] holds for N. Hence, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module follows by applying [15, Proposition 2.12 ((3) $\Rightarrow$ (1))] to N. Therefore, M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. □

3. Some more results on weakly S-Laskerian modules

As in Section 2, we use R to denote a ring, S to denote a m.c. subset of R, and M to denote a module over R. In this section, we try to extend some of the results on S-Laskerian (resp., strongly S-Laskerian) modules to weakly S-Laskerian (resp., weakly strongly S-Laskerian) modules. The following lemma is needed for the proof of Proposition 3.2.

Lemma 3.1. *For a proper submodule N of M, if N and $\frac{M}{N}$ are S-Laskerian (resp., strongly S-Laskerian) R-modules, then M is an S-Laskerian (resp., a strongly S-Laskerian) R-module.*

Proof. As N and $\frac{M}{N}$ are S-Laskerian (resp., strongly S-Laskerian) R-modules by assumption, N and $\frac{M}{N}$ are S-finite R-modules. Hence, there exist $s_1, s_2 \in S$, a f.g. submodule F of N and a f.g. submodule $\frac{L}{N}$ of $\frac{M}{N}$ such that $s_1 N \subseteq F \subseteq N$ and $s_2 \frac{M}{N} \subseteq \frac{L}{N} \subseteq \frac{M}{N}$. Note that $s_2 M \subseteq L$ and there exist $m_1, \ldots, m_k \in L$ such that $\frac{L}{N} = \sum_{i=1}^{k} R(m_i + N)$. Hence, $L = \sum_{i=1}^{k} Rm_i + N$. If $s = s_1 s_2$, then $s \in S$ and

$$sM = s_1 s_2 M \subseteq s_1 L = (s_1 \textstyle\sum_{i=1}^{k} Rm_i) + s_1 N \subseteq (\sum_{i=1}^{k} R(s_1 m_i)) + F \subseteq M.$$

This shows that M is an S-finite R-module. As N is an S-Laskerian (resp., a strongly S-Laskerian) R-module by hypothesis, it follows that $S^{-1}N$ is a Laskerian (resp., strongly Laskerian) $S^{-1}R$-module and for any submodule

K of N with $(K :_R N) \cap S = \emptyset$, there exists $s' \in S$ (depending on K) such that $g^{-1}(S^{-1}K) = (K :_N s')$ by [15, Proposition 2.12 ((1) $\Rightarrow$ (2))], where $g : N \to S^{-1}N$ is the usual homomorphism of R-modules given by $g(n) = \frac{n}{1}$. As $\frac{M}{N}$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module, by [15, Proposition 2.12 ((1) $\Rightarrow$ (2))], we get that $S^{-1}(\frac{M}{N})$ is a Laskerian (resp., strongly Laskerian) $S^{-1}R$-module and for any submodule $\frac{U}{N}$ of $\frac{M}{N}$ with $(\frac{U}{N} :_R \frac{M}{N}) \cap S = \emptyset$, there exists $s'' \in S$ (depending on $\frac{U}{N}$) such that

$$h^{-1}(S^{-1}(\tfrac{U}{N})) = (\tfrac{U}{N} :_{\frac{M}{N}} s''),$$

where $h : \frac{M}{N} \to S^{-1}(\frac{M}{N})$ is the usual homomorphism of R-modules given by $h(m + N) = \frac{m+N}{1}$. Note that $S^{-1}(\frac{M}{N}) \cong \frac{S^{-1}M}{S^{-1}N}$ as $S^{-1}R$-modules by [3, Corollary 3.4(iii)]. As $S^{-1}N$ and $\frac{S^{-1}M}{S^{-1}N}$ are Laskerian (resp., strongly Laskerian) $S^{-1}R$-modules, $S^{-1}M$ is a Laskerian (resp., strongly Laskerian) $S^{-1}R$-module. If W is any submodule of M such that $(W :_R M) \cap S = \emptyset$, then proceeding as in the proof of [15, Theorem 3.4], it can be shown that there exists $\rho \in S$ (depending on W) such that $f^{-1}(S^{-1}W) = (W :_M \rho)$, where $f : M \to S^{-1}M$ is the homomorphism of R-modules given by $f(m) = \frac{m}{1}$. Hence, M is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Proposition 2.12 ((2) $\Rightarrow$ (1))]. □

Proposition 3.2. *Let $\phi : M \to M'$ be an onto homomorphism of R-modules. If M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module, then M' is also a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module.*

Proof. Assume that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. As M is an S-finite R-module, there exist $s_1 \in S$ and a f.g. submodule F of M such that $s_1M \subseteq F \subseteq M$. This implies that $s_1M' = s_1\phi(M) = \phi(s_1M) \subseteq \phi(F) \subseteq \phi(M) = M'$. As $\phi(F)$ is a f.g. submodule of M', it follows that M' is an S-finite R-module. Let N' be an S-finite proper submodule of M'. We claim that N' is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Note that there exist $s_2 \in S$ and a f.g. submodule F' of N' such that $s_2N' \subseteq F' \subseteq N'$. Let $y_1, \ldots, y_k \in F' \subseteq N'$ be such that $F' = \sum_{i=1}^{k} Ry_i$. Since $\phi : M \to M'$ is onto, we can find $x_1, \ldots, x_k \in M$ such that $y_i = \phi(x_i)$ for each $i \in \{1, \ldots, k\}$. Thus, $F' = \sum_{i=1}^{k} R\phi(x_i)$. Observe that $L = \sum_{i=1}^{k} Rx_i$ is a f.g. R-module and $F' = \phi(L)$. As $F' \neq M'$, it follows that $L \neq M$. By assumption on M, L is an S-Laskerian (resp., a strongly S-Laskerian) R-module, so $F' = \phi(L)$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.1].

Observe that $s_2^2N' \subseteq s_2F' \subseteq s_2N'$. As $s_2 \in S$ and s_2F' is a f.g. submodule contained in s_2N', it follows that s_2N' is an S-finite submodule of F'. Since F' is an S-Laskerian (resp., a strongly S-Laskerian) R-module, s_2N' is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.3]. Observe that the mapping $\xi : N' \to s_2N'$ defined by $\xi(y) = s_2y$ is an onto homomorphism of R-modules and $Ker(\xi) = ((0) :_{N'} s_2)$. Let W' be any submodule of $Ker(\xi)$. As $s_2W' \subseteq s_2Ker(\xi) \subseteq (0)$, it follows that each submodule of $Ker(\xi)$ is S-finite. Therefore, $Ker(\xi)$ is an S-Noetherian R-module, so $Ker(\xi)$ is a strongly S-Laskerian R-module by [15, Corollary 2.15]. Since $\frac{N'}{Ker(\xi)} \cong s_2N'$ as R-modules by the fundamental theorem of homomorphism of modules, we get that $\frac{N'}{Ker(\xi)}$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Hence, N' is an S-Laskerian (resp., a strongly S-Laskerian) R-module by Lemma 3.1. This proves that M' is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. □

Let $\phi : R \to T$ be an onto homomorphism of rings. Assume that $Ker(\phi) \cap S = \emptyset$. Note that $0 \notin \phi(S)$ and $\phi(S)$ is a m.c. subset of T. Observe that T can be made into an R-module by defining $rt = \phi(r)t$ for any $r \in R$ and $t \in T$. Since $\phi : R \to T$ is onto, it follows that a nonempty subset J of T is an R-submodule of T if and only if J is an ideal of T. The following lemma is used in the proof of Corollary 3.4.

Lemma 3.3. *Let R, T, ϕ, S be as in the previous paragraph. For any proper ideal J of T, the following statements hold.*

(1) *J is an $\phi(S)$-finite ideal of T if and only if J is an S-finite R-submodule of T.*
(2) *Q is a primary (resp., strongly primary) T-submodule of J if and only if Q is a primary (resp., strongly primary) R-submodule of J.*
(3) *If J' is any ideal of T, then $(J' :_T J) \cap \phi(S) = \emptyset$ if and only if $(J' :_R J) \cap S = \emptyset$.*

Proof. It is already noted that a nonempty subset J of T is an R-submodule of T if and only if J is an ideal of T. Since $\phi : R \to T$ is onto, for any $t_1, \ldots, t_n \in T$, $Tt_1 + \cdots + Tt_n = Rt_1 + \cdots + Rt_n$. Hence, the collection of all f.g. R-submodules of T coincides with the collection of all f.g. ideals of T.

(1) As for any $s \in S$, $\phi(s)J = sJ$, J is an $\phi(S)$-finite ideal of T if and only if there exist $s \in S$ and a f.g. ideal J' of T such that $\phi(s)J \subseteq J' \subseteq J$ if and only if there exist $s \in S$ and a f.g. R-submodule J' of J such that $sJ \subseteq J' \subseteq J$ if and only if J is an S-finite R-submodule of T.

(2) Since $\phi : R \to T$ is onto, it can be shown that for any ideal J' of T, $(J' :_T J) = \phi(J' :_R J)$. Let Q be a primary T-submodule of J. Then $Q \neq J$. Let $r \in R$ and $a \in J$ be such that $ra \in Q$. Then $\phi(r)a \in Q$. If $a \notin Q$, then $\phi(r)^n \in (Q :_T J)$ for some $n \in \mathbb{N}$. This implies that $\phi(r^n)J \subseteq Q$, so $r^n \in (Q :_R J)$. This shows that Q is a primary R-submodule of J. If Q is a strongly primary T-submodule of J, then Q is a primary T-submodule of J and there exists $k \in \mathbb{N}$ such that $(\sqrt{(Q :_T J)})^k \subseteq (Q :_T J)$. Then Q is a primary R-submodule of J. Let $r_1, \dots, r_k \in \sqrt{(Q :_R J)}$. Then $\phi(r_1), \dots, \phi(r_k) \in \sqrt{(Q :_T J)}$. Hence, $(\prod_{i=1}^k \phi(r_i))J \subseteq Q$ and this implies that $\prod_{i=1}^k r_i \in (Q :_R J)$. This proves that $(\sqrt{(Q :_R J)})^k \subseteq (Q :_R J)$. Hence, Q is a strongly primary R-submodule of J. Similarly, it can be shown that if Q is a primary (resp., strongly primary) R-submodule of J, then Q is a primary (resp., strongly primary) T-submodule of J.

(3) Let J' be any ideal of T. Note that $\phi(s)J \subseteq J'$ for some $s \in S$ if and only if $sJ \subseteq J'$. Therefore, $(J' :_T J) \cap \phi(S) = \emptyset$ if and only if $(J' :_R J) \cap S = \emptyset$. □

Corollary 3.4. *With R, T, ϕ, S as in the statement of Lemma 3.3, if R is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) ring, then T is a weakly $\phi(S)$-Laskerian (resp., weakly strongly $\phi(S)$-Laskerian) ring.*

Proof. As T can be made into an R-module by defining $rt = \phi(r)t$ for any $r \in R$ and $t \in T$, $\phi : R \to T$ is an onto homomorphism of R-modules. Assume that R is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) ring. Hence, R is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) regarded as an R-module. It follows from Proposition 3.2 that T is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. Note that $0 \notin \phi(S)$ and $\phi(S)$ is a m.c. subset of T. Let J be any $\phi(S)$-finite proper ideal of T and let K be any ideal of T with $K \subseteq J$ and $(K :_T J) \cap \phi(S) = \emptyset$. Note that J is an S-finite R-submodule of T by Lemma 3.3(1) and the R-submodule K of J satisfies $(K :_R J) \cap S = \emptyset$ by Lemma 3.3(3). Since T is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module, there exist $s \in S$ and primary (resp., strongly primary) R-submodules $Q_1, \dots, Q_l$ of J with $(Q_i :_R J) \cap S = \emptyset$ for each $i \in \{1, \dots, l\}$ such that $(K :_J s) = \bigcap_{i=1}^l Q_i$ is a primary (resp., strong primary) decomposition in J regarded as an R-module by (1) $\Rightarrow$ (3) of Theorem 2.10. Observe that for each $i \in \{1, \dots, l\}$, Q_i is a primary (resp., strongly primary) T-submodule of J by Lemma 3.3(2) and $(Q_i :_T J) \cap \phi(S) = \emptyset$ by Lemma 3.3(3). As $\phi(s) \in \phi(S)$ and $(K :_J s) = (K :_J \phi(s))$, we get that $(K :_J \phi(s)) = \bigcap_{i=1}^l Q_i$ is a primary (resp., strong primary) decomposition in J regarded as an T-module with

$(Q_i :_T J) \cap \phi(S) = \emptyset$ for each $i \in \{1, \ldots, l\}$. Hence, T is a weakly $\phi(S)$-Laskerian (resp., weakly strongly $\phi(S)$-Laskerian) T-module by (3) $\Rightarrow$ (1) of Theorem 2.10. This proves that T is a weakly $\phi(S)$-Laskerian (resp., weakly strongly $\phi(S)$-Laskerian) ring. □

We use the following proposition in the proof of Example 3.6.

Proposition 3.5. *If $M = M_1 \times M_2 \times \cdots \times M_n$ is the direct product of nonzero R-modules $M_1, M_2, \ldots, M_n$, where $n \in \mathbb{N}\backslash\{1\}$, then the following statements are equivalent:*

1. *M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module;*
2. *M_i is an S-Laskerian (resp., a strongly S-Laskerian) R-module for each $i \in \{1, 2, \ldots, n\}$;*
3. *M is an S-Laskerian (resp., a strongly S-Laskerian) R-module.*

Proof. (1) $\Rightarrow$ (2). Let $i \in \{1, 2, \ldots, n\}$. Let $p_i : M \to M_i$ be defined by $p_i(m) =$ the ith coordinate of m. Note that p_i is an onto homomorphism of R-modules. As M is an S-finite R-module, it follows that M_i is an S-finite R-module. Consider the submodule N of M, $N = N_1 \times N_2 \times \cdots \times N_n$ defined by $N_i = M_i$ and $N_j = (0)$ for each $j \in \{1, 2, \ldots, n\}\backslash\{i\}$. Note that $N \cong M_i$ as R-modules and N is an S-finite proper submodule of M. Therefore, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module, since M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module by assumption. Hence, M_i is an S-Laskerian (resp., a strongly S-Laskerian) R-module.

(2) $\Rightarrow$ (3). If $n = 2$, then $N = M_1 \times (0)$ is a submodule of M. Since $N \cong M_1$ as R-modules, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Since $\frac{M}{N} \cong (0) \times M_2$ as R-modules, it follows that $\frac{M}{N}$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module. As N and $\frac{M}{N}$ are S-Laskerian (resp., strongly S-Laskerian) R-modules, M is an S-Laskerian (resp., a strongly S-Laskerian) R-module by Lemma 3.1. Let $n \geq 3$ and assume by induction that the direct product of $n-1$ S-Laskerian (resp., strongly S-Laskerian) R-modules is an S-Laskerian (resp., a strongly S-Laskerian) R-module. If $N = M_1 \times (0) \times (0) \times \cdots \times (0)$, then N is an S-Laskerian (resp., a strongly S-Laskerian) R-module. The R-module $M_2 \times M_3 \times \cdots \times M_n$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module by induction assumption. Observe that $\frac{M}{N} \cong (0) \times M_2 \times M_3 \times \cdots \times M_n$ as R-modules. Thus, N and $\frac{M}{N}$ are S-Laskerian (resp., strongly S-Laskerian) R-modules, so M is an S-Laskerian (resp., a strongly S-Laskerian) R-module by Lemma 3.1.

(3) $\Rightarrow$ (1). This follows by Proposition 2.1. □

We provide the following example to illustrate that a f.g. module over a weakly strongly S-Laskerian ring need not be a weakly S-Laskerian R-module.

Example 3.6. If R is as in the statement of Example 2.2 and $M = R \times R$ is the direct product of R-modules R and R, then R is a weakly strongly $\{1 + I\}$-Laskerian ring but M is not a weakly $\{1 + I\}$-Laskerian R-module.

Proof. In the notation of Example 2.2, $1 + I$ is the identity element of R and $0 + I$ is the zero element of R. Note that

$$M = R \times R = R(1 + I, 0 + I) + R(0 + I, 1 + I)$$

is a f.g. R-module. It is shown in the proof of Example 2.2 that R is a weakly strongly $\{1 + I\}$-Laskerian ring but R is not an $\{1 + I\}$-Laskerian ring. That is, R is not an $\{1 + I\}$-Laskerian R-module. Hence, by (1) $\Rightarrow$ (2) of Proposition 3.5, we get that $M = R \times R$ is not a weakly $\{1 + I\}$-Laskerian R-module. $\square$

For any $\mathfrak{p} \in Spec(R)$, $R \backslash \mathfrak{p}$ is a m.c. subset of R. We say that M is a *weakly* $\mathfrak{p}$*-Laskerian* (resp., *weakly strongly* $\mathfrak{p}$*-Laskerian*) R-module, if M is a weakly $(R \backslash \mathfrak{p})$-Laskerian (resp., weakly strongly $(R \backslash \mathfrak{p})$-Laskerian) R-module. Motivated by [1, Proposition 12] and [10, Proposition 4], we next have the following.

Theorem 3.7. *If M is a f.g. R-module, then the following statements are equivalent:*

(1) *Every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module;*
(2) *Every f.g. proper submodule of M is an S-Laskerian (resp., a strongly S-Laskerian) R-module for any m.c. subset S of R;*
(3) *M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module for any m.c. subset S of R;*
(4) *M is a weakly $\mathfrak{p}$-Laskerian (resp., weakly strongly $\mathfrak{p}$-Laskerian) R-module for any $\mathfrak{p} \in Spec(R)$;*
(5) *M is a weakly $\mathfrak{m}$-Laskerian (resp., weakly strongly $\mathfrak{m}$-Laskerian) R-module for any $\mathfrak{m} \in Max(R)$.*

Proof. (1) $\Rightarrow$ (2). Let N be any f.g. proper submodule of M. Since N is a Laskerian (resp., strongly Laskerian) R-module, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module for any m.c. subset S of R by [15, Corollary 2.14].

(2) $\Rightarrow$ (3). Let S be a m.c. subset of R. Since M is a f.g. R-module by hypothesis, it follows that M is an S-finite R-module. Let N any S-finite proper submodule of M. Note that there exist $s \in S$ and a f.g. submodule F of N such that $sN \subseteq F \subseteq N$. As $F \neq M$, F is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Observe that $s^2N \subseteq sF \subseteq sN$ and sF is a f.g. submodule of sN. Therefore, sN is an S-finite R-module and is a submodule of F, so sN is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.3]. Proceeding as in the proof of Proposition 3.2, it can be shown that $((0) :_N s)$ is an S-Noetherian R-module and hence, a strongly S-Laskerian R-module and $\frac{N}{((0):_N s)} \cong sN$ is an S-Laskerian (resp., a strongly S-Laskerian) R-module. Therefore, N is an S-Laskerian (resp., a strongly S-Laskerian) R-module by Lemma 3.1. This shows that M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module.

(3) $\Rightarrow$ (4). This is clear, since $R\backslash\mathfrak{p}$ is a m.c. subset of R for any $\mathfrak{p} \in Spec(R)$.

(4) $\Rightarrow$ (5). This is clear, since $Max(R) \subseteq Spec(R)$.

(5) $\Rightarrow$ (1). Let N be any f.g. proper submodule of M. Let $\mathfrak{m} \in Max(R)$. Note that N is an $(R\backslash\mathfrak{m})$-finite proper submodule of M. Therefore, N is an $(R\backslash\mathfrak{m})$-Laskerian (resp., a strongly $(R\backslash\mathfrak{m})$-Laskerian) R-module. So, N is a Laskerian (resp., strongly Laskerian) R-module by [15, Proposition 3.8 ((4) $\Rightarrow$ (1))]. □

We know from the proof of Example 2.2 that the ring R considered there satisfies every f.g. proper ideal of R is a Noetherian R-module and hence, a strongly Laskerian R-module but R is not a Laskerian ring. Thus, R as an R-module satisfies all the equivalent statements of Theorem 3.7 but R regarded as an R-module is not Laskerian.

The following theorem provides necessary and sufficient conditions such that every proper submodule of M is a Noetherian R-module.

Theorem 3.8. *The following statements are equivalent:*

(1) *Every proper submodule of M is a Noetherian R-module;*
(2) *Every proper submodule of M is an S-Noetherian R-module for any m.c. subset S of R;*
(3) *Every proper submodule of M is a strongly Laskerian R-module;*
(4) *Every proper submodule of M is a Laskerian R-module;*
(5) *Every proper submodule of M is a strongly S-Laskerian R-module for any m.c. subset S of R;*

(6) *Every proper submodule of M is a weakly strongly S-Laskerian R-module for any m.c. subset S of R;*
(7) *Every proper submodule of M is a weakly strongly $\mathfrak{p}$-Laskerian R-module for any $\mathfrak{p} \in Spec(R)$;*
(8) *Every proper submodule of M is a weakly strongly $\mathfrak{m}$-Laskerian R-module for any $\mathfrak{m} \in Max(R)$;*
(9) *Every proper submodule of M is a weakly $\mathfrak{m}$-Laskerian R-module for any $\mathfrak{m} \in Max(R)$;*
(10) *Every proper submodule of M is an $(R\backslash\mathfrak{m})$-finite R-module for any $\mathfrak{m} \in Max(R)$.*

Proof. (1) ⇒ (2). This is clear, since any Noetherian R-module is an S-Noetherin R-module for any m.c. subset S of R.

(2) ⇒ (5). This follows, since any S-Noetherian R-module is a strongly S-Laskerian R-module for any m.c. subset S of R by [15, Corollary 2.15].

(5) ⇒ (6). This is clear by Proposition 2.1.

(6) ⇒ (7). This is clear, since $R\backslash\mathfrak{p}$ is a m.c. subset of R for any $\mathfrak{p} \in Spec(R)$.

(7) ⇒ (8). This is clear, since $Max(R) \subseteq Spec(R)$.

(8) ⇒ (9). This is clear, since any weakly strongly $\mathfrak{m}$-Laskerian R-module is a weakly $\mathfrak{m}$-Laskerian R-module.

(9) ⇒ (10). Let $\mathfrak{m} \in Max(R)$. By one of the defining conditions of a weakly $(R\backslash\mathfrak{m})$-Laskerian R-module, it follows that any proper submodule of M is an $(R\backslash\mathfrak{m})$-finite R-module.

(10) ⇒ (1). Let N be any proper submodule of M. Let K be any submodule of N. Then K is also a proper submodule of M. As K is an $(R\backslash\mathfrak{m})$-finite R-module for each $\mathfrak{m} \in Max(R)$, it can be shown as in the proof of [15, Proposition 3.8 ((4) ⇒ (1))] that K is a f.g. R-module. This shows that any submodule of N is f.g., so N is a Noetherian R-module.

(1) ⇒ (3). Since any Noetherian R-module is a strongly Laskerian R-module, this implication is clear.

(3) ⇒ (5). This follows, since any strongly Laskerian R-module is a strongly S-Laskerian R-module for any m.c. subset S of R by [15, Corollary 2.14].

(1) ⇒ (4). Since any Noetherian R-module is a Laskerian R-module, this is clear.

(4) ⇒ (9). This follows by [15, Corollary 2.14] and Proposition 2.1. □

The following example illustrates that an R-module M which satisfies any of the equivalent statements of the previous theorem can fail to be a weakly S-Laskerian R-module for any m.c. subset S of R.

Example 3.9. Let p be a fixed prime number. Consider the submodule M of the $\mathbb{Z}$-module $\frac{\mathbb{Q}}{\mathbb{Z}}$ given by $M = \bigcup_{n=1}^{\infty} M_n$, where M_n is the cyclic submodule of M generated by $\frac{1}{p^n} + \mathbb{Z}$. Then each proper submodule of M is a Noetherian $\mathbb{Z}$-module but M is not a weakly S-Laskerian $\mathbb{Z}$-module for any m.c. subset S of $\mathbb{Z}$.

Proof. It is well known that $M_1 \subset M_2 \subset M_3 \subset \cdots$ is a strictly increasing sequence of submodules of M and if N is any nonzero proper submodule of M, then $N = M_n$ for some $n \in \mathbb{N}$, see [3, Example (3), p.74]. As $|M_n| = p^n$ for each $n \geq 1$, it follows that any proper submodule of M is a Noetherian $\mathbb{Z}$-module. Let S be any m.c. subset of $\mathbb{Z}$. Note that $pM = M$ and $qM = M$ for any prime number q with $q \neq p$, so $nM = M$ for any $n \in \mathbb{Z}\backslash\{0\}$. Hence, $sM = M$ for any $s \in S$. Since M is not a f.g. $\mathbb{Z}$-module, it follows that M is not an S-finite $\mathbb{Z}$-module, so M is not a weakly S-Laskerian $\mathbb{Z}$-module. □

Recall that a m.c. subset T of a ring R is said to be a *saturated m.c. subset* of R, if whenever $ab \in T$ for some $a, b \in R$, then both a and b must be in T. For a m.c. subset S of R, recall that the *saturation* of S, denoted by S^* or $\overline{S}$ is defined as the smallest saturated m.c. subset of R that contains S [3, Exercise 7, p.44]. It is not hard to verify that $S^* = \{a \in R \mid ab \in S \text{ for some } b \in R\}$.

The following lemma is used in the proof of Proposition 3.12.

Lemma 3.10. *If M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module, then M is a weakly S'-Laskerian (resp., weakly strongly S'-Laskerian) R-module for any m.c. subset S' of R with $S \subseteq S'$.*

Proof. Since M is an S-finite R-module and $S \subseteq S'$, we get that M is an S'-finite R-module. Consider any N, an S'-finite proper submodule of M. Then there exist $s' \in S'$ and a f.g. submodule F of N such that $s'N \subseteq F \subseteq N$. As F is f.g proper submodule of M, we obtain that F is an S-Laskerian (resp., a strongly S-Laskerian) R-module, since M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module by assumption. Hence, F is an S'-Laskerian (resp., a strongly S'-Laskerian) R-module by [15, Lemma 3.10]. Observe that $s'^2N \subseteq s'F \subseteq s'N$. Therefore, $s'N$ is an S'-finite submodule of F, so $s'N$ is an S'-Laskerian (resp., a strongly S'-Laskerian) R-module by [15, Lemma 3.3]. It can be shown as in the proof of Proposition 3.2 that $((0) :_N s')$ is an S'-Noetherian R-module and $\frac{N}{((0):_N s')} \cong s'N$ is an S'-Laskerian (resp., a strongly S'-Laskerian) R-module. Hence, N is an S'-Laskerian (resp., a strongly S'-Laskerian) R-module by Lemma 3.1. Therefore, M is a weakly S'-Laskerian (resp., weakly strongly S'-Laskerian) R-module. □

We provide the following example to illustrate that the converse of Lemma 3.10 can fail to hold.

Example 3.11. Consider the subring $R = K[X] + (1 + XY)K[X,Y]$ of $T = K[X,Y]$, the polynomial ring in two variables X, Y over a field K. Let $S = \{1\}$ and let $S' = \{(1+XY)^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then R is a weakly strongly S'-Laskerian ring but R is not a weakly S-Laskerian ring.

Proof. Note that S (resp., S') is a m.c. subset of R with $S \subset S'$. It is known that R is a strongly S'-Laskerian ring, see [16, Example 3.8(1)]. Hence, R is a weakly strongly S'-Laskerian ring by Proposition 2.1. It is clear that $\{X^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of R. It was verified, see [16, Example 3.8(2)] that R is not an $\{X^n \mid n \in \mathbb{N} \cup \{0\}\}$-Laskerian ring. Therefore, R is not a Laskerian ring by [15, Corollary 2.14]. As $S = \{1\}$, $S^{-1}R = R$. Hence, $S^{-1}R$ is not a Laskerian ring, so R is not an S-Laskerian ring by [15, Corollary 2.13 ((1) $\Rightarrow$ (2))]. Therefore, R is not a weakly S-Laskerian ring by Proposition 2.6, since R is an integral domain and $RX \neq R$. □

Proposition 3.12. *The following statements are equivalent:*

(1) *M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module;*
(2) *M is a weakly S^*-Laskerian (resp., weakly strongly S^*-Laskerian) R-module.*

Proof. (1) $\Rightarrow$ (2). This follows by Lemma 3.10, since S^* is a m.c. subset of R with $S \subseteq S^*$.

(2) $\Rightarrow$ (1). As M is an S^*-finite R-module, there exist $s^* \in S^*$ and a f.g. submodule F of M such that $s^*M \subseteq F \subseteq M$. Note that $rs^* \in S$ for some $r \in R$. It is clear that $rs^*M \subseteq rF \subseteq M$. Since rF is a f.g. submodule of M, we get that M is an S-finite R-module. Let N be any S-finite proper submodule of M. Then N is an S^*-finite submodule of M. Hence, N is an S^*-Laskerian (resp., a strongly S^*-Laskerian) R-module, so N is an S-Laskerian (resp., a strongly S-Laskerian) R-module by [15, Lemma 3.11 ((2) $\Rightarrow$ (1))]. Therefore, M is a weakly S-Laskerian (resp., weakly strongly S-Laskerian) R-module. □

Note that for any nonzero proper ideal I of R, $1 + I = \{1 + a \mid a \in I\}$ is a m.c. subset of R. The following corollary provides a statement equivalent to the statement (1) of Theorem 3.7 in terms of the m.c. subset $1 + I$.

Corollary 3.13. *If M is a f.g. R-module, then every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module if and only if M is* *a weakly* $1+I$*-Laskerian (resp., weakly strongly* $1+I$*-Laskerian) R-module for each nonzero proper ideal I of R.*

Proof. Assume that every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module. Let I be any nonzero proper ideal of R. As $1+I=\{1+a \mid a\in I\}$ is a m.c. subset of R, we obtain that M is a weakly $1+I$-Laskerian (resp., weakly strongly $1+I$-Laskerian) R-module by (1) $\Rightarrow$ (3) of Theorem 3.7.

Conversely, assume that M is a weakly $1+I$-Laskerian (resp., weakly strongly $1+I$-Laskerian) R-module for each nonzero proper ideal I of R. We can assume that R is not a field, since M is a Noetherian R-module if R is a field. For any $\mathfrak{m}\in Max(R)$, $\mathfrak{m}$ is a nonzero proper ideal of R, so M is a weakly $1+\mathfrak{m}$-Laskerian (resp., weakly strongly $1+\mathfrak{m}$-Laskerian) R-module. It is clear that $1+\mathfrak{m}\subseteq R\backslash\mathfrak{m}$. Therefore, we obtain from Lemma 3.10 that M is a weakly $(R\backslash\mathfrak{m})$-Laskerian (resp., weakly strongly $(R\backslash\mathfrak{m})$-Laskerian) R-module. This is true for any $\mathfrak{m}\in Max(R)$. Hence, by (5) $\Rightarrow$ (1) of Theorem 3.7, we obtain that any f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module. □

If $a\in R$ is such that a is not nilpotent, then $S_a=\{a^n \mid n\in\mathbb{N}\cup\{0\}\}$ is a m.c. subset of R. We use to denote $S_a^{-1}R$ by R_a. If a_1,a_2 are nonzero non-units of R with $Ra_1+Ra_2=R$, then in the following corollary, we provide another statement equivalent to the statement (1) of Theorem 3.7 in terms of S_{a_1} and S_{a_2}.

Corollary 3.14. *If* a_1,a_2 *are nonzero non-units of R with* $Ra_1+Ra_2=R$ *and M is a f.g. R-module, then every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module if and only if M is a weakly* S_{a_i}*-Laskerian (resp., weakly strongly* S_{a_i}*-Laskerian) R-module for each* $i\in\{1,2\}$.

Proof. Assume that every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module. It follows from (1) $\Rightarrow$ (3) of Theorem 3.7 that M is a weakly S_{a_i}-Laskerian (resp., weakly strongly S_{a_i}-Laskerian) R-module for each $i\in\{1,2\}$.

Assume that M is a weakly S_{a_i}-Laskerian (resp., weakly strongly S_{a_i}-Laskerian) R-module for each $i\in\{1,2\}$. Let $\mathfrak{m}\in Max(R)$. Since $Ra_1+Ra_2=R$, it follows that $a_j\notin\mathfrak{m}$ for some $j\in\{1,2\}$. In such a case, $S_{a_j}\subseteq R\backslash\mathfrak{m}$. As M is a weakly S_{a_j}-Laskerian (resp., weakly strongly S_{a_j}-Laskerian) R-module, we obtain from Lemma 3.10 that M is a weakly $(R\backslash\mathfrak{m})$-Laskerian (resp., weakly strongly $(R\backslash\mathfrak{m})$-Laskerian) R-module. This is true for any $\mathfrak{m} \in Max(R)$. Therefore, we obtain from (5) $\Rightarrow$ (1) of Theorem 3.7 that every f.g. proper submodule of M is a Laskerian (resp., strongly Laskerian) R-module. $\square$

Let $n \in \mathbb{N}$ and let $R[X_1, \ldots, X_n]$ be the polynomial ring in n variables $X_1, \ldots, X_n$ over R. It is convenient to denote $R[X_1, \ldots, X_n]$ by T. Note that S is also a m.c. subset of T. In the following theorem, we show that T is an S-Noetherian ring if and only if T is a weakly S-Laskerian ring.

Theorem 3.15. *If $T = R[X_1, \ldots, X_n]$, then the following statements are equivalent:*

(1) *T is an S-Noetherian ring;*
(2) *T is a weakly S-Noetherian ring;*
(3) *T is a weakly strongly S-Laskerian ring;*
(4) *T is a weakly S-Laskerian ring;*
(5) *T is a strongly S-Laskerian ring;*
(6) *T is an S-Laskerian ring.*

Proof. (1) $\Rightarrow$ (2). As any ideal of T is an S-finite T-module, it follows that any S-finite proper ideal of T is an S-Noetherian T-module. Therefore, T is a weakly S-Noetherian ring.

(2) $\Rightarrow$ (3). This follows from Corollary 2.4.

(3) $\Rightarrow$ (5). Note that $X_1 \in T \backslash Z(T)$ and $TX_1 \neq T$. Hence, T is a strongly S-Laskerian ring by Proposition 2.6.

(5) $\Rightarrow$ (6). This is clear.

(6) $\Rightarrow$ (1). This follows from [16, Theorem 3.10 ((2) $\Rightarrow$ (6))].

(2) $\Rightarrow$ (4). This follows from Corollary 2.4.

(4) $\Rightarrow$ (6). This follows from Proposition 2.6, since $X_1 \in T \backslash Z(T)$ is such that $TX_1 \neq T$. $\square$

With T as in the statement of Theorem 3.15, if T is an S-Noetherian ring, then it is clear that R is an S-Noetherian ring. Recall that a m.c. subset S of R is said to be *anti-Archimedean* if $(\bigcap_{n \in \mathbb{N}} Rs^n) \cap S \neq \emptyset$ for each $s \in S$ [1, see p.4411]. For an anti-Archimedean m.c. subset S of R. if R is S-Noetherian, then so is the polynomial ring $R[X_1, \ldots, X_n]$ [1, Proposition 9]. Hence, for an anti-Archimedean m.c. subset S of R, the equivalent statements (1) to (6) of Theorem 3.15 are equivalent to the statement (7) R is an S-Noetherian ring.

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SOME REMARKS ON WEAKLY S-LASKERIAN MODULES

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نکاتی درباره‌ی مدول‌های ضعیفاً S-لاسکری

سوبرامانی ویسوسواران

گروه ریاضی، دانشگاه ساوراشترا، راجکوت، هند

در این مقاله، تمام حلقه‌ها جابه‌جایی و دارای عضو همانی فرض می‌شوند و همه‌ی مدول‌ها یکانی هستند. از R برای نمایش یک حلقه، از S برای نمایش یک زیرمجموعه‌ی ضربی‌بسته از R، و از M برای نمایش یک R-مدول استفاده می‌کنیم. می‌گوییم M یک R-مدول S-لاسکری (به‌ترتیب، قویاً S-لاسکری) است، هرگاه M یک R-مدول S-متناهی باشد و برای هر زیرمدول N از M، $(N :_R M) \cap S \neq \emptyset$، یا عنصری مانند $t \in S$ و یک زیرمدول S-تجزیه‌پذیر (به‌ترتیب، قویاً S-تجزیه‌پذیر) K از M وجود داشته باشند (که به N وابسته‌اند) به‌طوری‌که $tN \subseteq K \subseteq N$. همچنین می‌گوییم M یک R-مدول ضعیفاً S-لاسکری (به‌ترتیب ضعیفاً قویاً S-لاسکری) است، هرگاه M یک R-مدول S-متناهی باشد و هر زیرمدول سره S-متناهی آن، یک R-مدول S-لاسکری (به‌ترتیب، قویاً S-لاسکری) باشد. به‌علاوه، برخی از نتایج مربوط به ویژگی‌های اساسی مدول‌های ضعیفاً S-لاسکری را بررسی می‌کنیم و تعدادی از خواص مدول‌های S-لاسکری را به مدول‌های ضعیفاً S-لاسکری تعمیم می‌دهیم.

کلمات کلیدی: زیرمدول S-اولیه، زیرمدول قویاً S-اولیه، مدول ضعیفاً S-لاسکری، مدول ضعیفاً قویاً S-لاسکری.