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A GENERALIZATION OF SEMINOETHERIAN RINGS AND MODULES

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Abstract. We study those modules in which all submodules contain a nonzero submodule with Noetherian dimension less than or equal to α, where α is the least ordinal number with this property, calling them modules that have enough α-noetherians. Also, we study those modules in which any nonzero factor module contains a nonzero submodule with Noetherian dimension less than or equal to α, where α is the least ordinal number with this property, calling them α-seminoetherians. Our work extends the results of Kourki and Tribak in [14].

1. Introduction

Artinain and Noetherian modules are essential concepts in module theory. In the past decades, they have received much attention from researchers in the field of modules and transferred to some other branch of mathematics. The Krull dimension (resp., Noetherian dimension) measures how close a module is to being Artinian (resp., Noetherian). For the reader's convenience, we now recall these definitions.

The *Krull dimension* of a module was defined by Gordon and Robson [8] in the following way.

Definition 1.1. Let M be a right R-module. The Krull dimension of M, which will be denoted by k-dim M, is defined by transfinite recursion as follows:

(1) If $M = 0$, then k-dim $M = -1$;
(2) If α is an ordinal number and k-dim $M \not< \alpha$, then k-dim $M = \alpha$, provided there is no infinite descending chain $M = M_0 \supsetneq M_1 \supsetneq \dots$ of submodules M_i such that k-dim $(M_{i-1}/M_i) \not< \alpha$ for all $i \in \mathbb{N}$;
(3) It is possible that there is no ordinal α such that k-dim $M = \alpha$. In this case, M has no Krull dimension.

Following [10], the *Noetherian dimension* (the terms *dual Krull dimension* [2], and *N-dimension* [3] are also used) of a module is defined inductively in the following way.

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Definition 1.2. Let M be a right R-module. The Noetherian dimension of M, which will be denoted by n-dim M, is defined as follows:

(1) If $M = 0$, then n-dim $M = -1$;
(2) If α is an ordinal number and n-dim $M \not< \alpha$, then n-dim $M = \alpha$, provided there is no infinite ascending chain $M_0 \subseteq M_1 \subseteq ...$ of submodules M_i such that n-dim $(M_i/M_{i-1}) \not< \alpha$ for all $i \in \mathbb{N}$;
(3) It is possible that there is no ordinal α such that n-dim $M = \alpha$. In this case, M has no Noetherian dimension.

In [14], Kourki and Tribak introduced the following concepts.

Definition 1.3. ([14, Definitions 2.1 and 3.1]) Let M be an R-module.

(1) We say that M *has enough noetherians* if any nonzero submodule of M contains a nonzero noetherian submodule.
(2) We say that M is *seminoetherian* if any nonzero factor module of M contains a nonzero noetherian submodule.

With the help of the Noetherian dimension concept, this paper aims to generalize the results of Kourki and Tribak [14].

Let us give a brief outline of this paper. Section 1 is the introduction. Section 2 of this paper recalls some known facts on modules with Noetherian dimensions. Section 3 is devoted to studying modules that have enough α-noetherians. We say that M *has enough α-noetherians* if every submodule of M contains a nonzero submodule with Noetherian dimension less than or equal to α, where α is the least ordinal number with this property. We investigate some properties of this class of modules and extend some results of [14] related to modules having enough α-noetherians. After some general results, we prove that for an R-module that M has enough β-noetherians with finite dimension, M is $\lambda.f.e.$ for some $\lambda \leq \beta$ if and only if k-dim $M \leq n$-dim M. Section 4, deals with the notion of α-seminoetherian modules. We say M is *α-seminoetherian* if any nonzero factor module of M contains a nonzero submodule with Noetherian dimension less than or equal to α, where α is the least ordinal number with this property. We extend some results of [14] related to seminoetherian modules. We show that a quotient finite dimensional module M is α-seminoetherian for some ordinal number α if and only if it has Noetherian dimension, or equivalently, if and only if it has Krull dimension.

Throughout this paper, R will denote an associative ring with nonzero identity, and all modules will be unital right R-modules. The notation M_R will be used to designate a right R-module M. The notation $N \leq M$ (resp.,

$N < M$) will denote for a right R-module M that N is a submodule (resp., proper submodule) of M, $N \leq^{ess} M$ means that N is an essential submodule of M, and $E(M)$ denotes the injective envelope of M. For an R-module M and an ordinal number α, we let $L_\alpha(M) = \sum\{N \leq M \mid n\text{-dim N} \leq \alpha\}$. Other standard terminology and conventions in module theory follow mainly those in [17].

2. Preliminaries

In this section, we recast some definitions and some standard facts that will be useful for our purposes. We begin with a nice result due to Lemonnier.

Proposition 2.1. ([15, Corollaire 6]) *An R-module M has Noetherian dimension if and only if it has Krull dimension.*

Before stating the next two results, the following definition is required. An R-module M is called *(Goldie) finite dimensional*, if M does not contain a direct sum of an infinite number of non-zero submodules. The module M is called *quotient finite dimensional*, if M/N is (Goldie) finite dimensional for every submodule N of M.

Proposition 2.2. ([1, Proposition 1.3]) *Let $\alpha \geq 0$ be an ordinal number. The following statements are equivalent for an R-module M:*

(1) $k\text{-dim}\, M \leq \alpha$.
(2) *M is quotient finite dimensional and for any $N < M$ there exists P such that $N < P \leq M$ and $k\text{-dim}\,(\frac{P}{N}) \leq \alpha$.*

Proposition 2.3. ([16, Théoreme 6]) *Let $\alpha \geq 0$ be an ordinal number. The following statements are equivalent for an R-module M:*

(1) $n\text{-dim}\, M \leq \alpha$.
(2) *M is quotient finite dimensional and for any $N \subset P \subseteq M$, there exists X with $N \subseteq X \subset P$ with $n\text{-dim}\, \frac{P}{X} \leq \alpha$.*

An R-module M is called *α-critical*, where α is an ordinal number, if $k\text{-dim}\, M = \alpha$ and $k\text{-dim}\, \frac{M}{N} < \alpha$ for all nonzero submodules N of M. An R-module M is called *critical* if M is α-critical for some ordinal number α.

The following is a well-known result, see [17, Lemma 6.2.10] for example.

Proposition 2.4. *Every nonzero module with Krull dimension has a critical submodule.*

We next present two well-known and important results, see [7] and [8].

Lemma 2.5. *If an R-module M has Krull dimension and $M = \sum_{i\in I} N_i$, then* $k\text{-dim}\, M = \sup\{k\text{-dim}\, N_i\}_{i\in I}$.

Corollary 2.6. *Let M be a quotient finite dimensional R-module. If $M = \sum_{i\in I} N_i$ such that each N_i has Krull dimension, then M has Krull dimension and* $k\text{-dim}\, M = \sup\{k\text{-dim}\, N_i\}_{i\in I}$.

Before proceeding, we need the following definition.

Definition 2.7. ([11, Definition 2.2]) Let M be an R-module. For each ordinal α, we define $S_\alpha = \sum_{i\in I} \oplus C_i$, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of M. S_α is called *an α-critical socle* of M. Now a critical socle of M is defined to be a submodule S of M with $S = \sum_{\alpha<\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \le \lambda$. If for some ordinal α, there is no α-critical submodule, then we put $S_\alpha = 0$. Clearly, the sum of any maximal independent family of critical submodules of M is a critical socle of M.

In view of Corollary 2.6, we have the following result.

Corollary 2.8. *Let M be a quotient finite dimensional R-module. If α is an ordinal number, then the α-critical socle of M has Krull dimension. This implies that the critical socle of M has Krull dimension.*

We now turn to the final topic of this section, λ-finitely embedded modules; for background on λ-finitely embedded modules, see [11]. An R-module M is called *finitely embedded* (briefly *f.e.*) if the socle of M is finitely generated and essential in M, see [20] for more details. It is known that a module M is Artinian if and only if every factor module of M is *f.e*, see [20, Proposition 2*]. An R-module M is called *λ-finitely embedded* ("λ-*f.e*" for short) if λ is the least ordinal such that each critical submodule of M is α-critical for some $\alpha \le \lambda$ and M contains a finitely generated essential critical socle (equivalently, M contains an essential critical socle with Krull dimension λ).

Proposition 2.9. ([11, Proposition 2.20]) *Let M be an R-module. Then,* $k\text{-dim}\, M = \alpha$ *if and only if α is the least ordinal such that each factor module of M is λ-f.e. for some $\lambda \le \alpha$.*

We close this section with an easy consequence of Corollary 2.8.

Corollary 2.10. *Let M be a quotient finite dimensional R-module and Q be a quotient module of M. Then, Q is λ-f.e., for some ordinal number λ, if and only if the critical socle of Q is an essential submodule of Q.*

3. Modules that have enough α-noetherians

The following definition generalizes the Kourki and Tribak's definition of modules having enough noetherians, see [14, Definition 2.1].

Definition 3.1. We say an R-module M *has enough α-noetherians* if every submodule of M contains a nonzero submodule with Noetherian dimension less than or equal to α, where α is the least ordinal number with this property. Also, we say an R-module M *has strongly enough α-noetherians* if it has enough β-noetherians for some $\beta \leq \alpha$. (Notice immediately that the modules having enough 0-noetherians are precisely those having enough noetherians.)

Example 3.2. Let M be an R-module and α be an ordinal number.

(1) If $n\text{-dim}\, M = \alpha$, then M has enough β-noetherians, for some $\beta \leq \alpha$.
(2) Following [12, Definition, p. 2760], an R-module M with Noetherian dimension is called *α-atomic* if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$ for all proper submodule N of M. If M is α-atomic and $\alpha = \beta + 1$, then M has enough γ-noetherians for some $\gamma \leq \beta$.
(3) If every prime ideal of R is maximal, then the concept of modules having enough α-noetherians and modules having enough noetherians coincide, see [12, Lemma 3.6] for more details.
(4) Assume M is an Artinian module with $n\text{-dim}\, M = \alpha$ (see [13, Example 1] and [6]). Thus, each non-zero submodule of M contains a simple submodule. This yields that M has enough 0-noetherians. Note that the Noetherian dimension of M may be any ordinal number α.
(5) Assume M is a simple module and I is an index set such that $|I| > \infty$. Put $S = \oplus_{i \in I} M_i$, such that $M_i = M$ for each $i \in I$. Then S has enough 0-Noetherian whereas S does not have Noetherian dimension, see [4].

The following is a generalization of [14, Proposition 2.4].

Proposition 3.3. *Let M be an R-module with finite dimension. The following statements are equivalent:*

(1) *If M has enough β-noetherians then it is $\lambda.f.e$ for some $\lambda \leq \beta$.*
(2) *If $n\text{-dim}\, M \leq \alpha$, then $k\text{-dim}\, M \leq \alpha$ (i.e., $k\text{-dim}\, M \leq n\text{-dim}\, M$).*

Proof. (1) $\Rightarrow$ (2) Let M be an R-module and $n\text{-dim}\, M \leq \alpha$. In view of Proposition 2.3, we infer that M is a quotient finite dimensional R-module. Hence, we deduce that M has enough β-noetherians for some $\beta \leq \alpha$, see Example 3.2(1). From (1), we infer that M is $\lambda.f.e$ for some $\lambda \leq \beta \leq \alpha$. With the help of [12, Lemma 1.2], we conclude that each factor module of

M has Noetherian dimension and n-dim $\frac{M}{N} \leq \alpha$. Thus, each factor module of M is a quotient finite dimensional R-module, see Proposition 2.3. In view of Example 3.2(1), we infer that each factor module of M has enough β-noetherians for some $\beta \leq \alpha$. This implies that each factor module of M is $\lambda.f.e$ for some $\lambda \leq \alpha$. Hence, M has Krull dimension and k-dim $M \leq \alpha$, see Proposition 2.9.

(2) $\Rightarrow$ (1) Assume that M has enough α-noetherians. Let $0 \neq N \leq M$. Thus, there exists $0 \neq X \leq N$, such that n-dim $X \leq \alpha$. Hence, k-dim $X \leq \alpha$. Using Proposition 2.4, we infer that X has a β-critical submodule for some $\beta \leq \alpha$. So, critical socle of M is essential in M. Since M has finite dimension, therefore M is $\lambda.f.e$ for some $\lambda \leq \beta$, see Corollary 2.10, as desired. □

Here, we give a characterization of integral domains which have enough α-noetherians, it is also a counterpart of [14, Proposition 2.5].

Proposition 3.4. *Let I be a nonzero ideal of a ring R with no nonzero zero divisors. The following statements are equivalent:*

(1) *The ring R has enough β-noetherians for some $\beta \leq \alpha$.*
(2) *n-dim $R \leq \alpha$.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq x \in I$. Since R has enough α-noetherians, Rx contains a nonzero submodule L, with Noetherian dimension less than or equal to α. Let $0 \neq y \in L$. It is clear that n-dim $Ry \leq n$-dim $L \leq \alpha$. Since $Ry \subseteq L \subseteq Rx \subseteq I$, we deduce that $Ry \cong R$. Hence, we have n-dim $R \leq \alpha$.
(2) $\Rightarrow$ (1) It is clear. □

Proposition 3.5. *Let N be a submodule of an R-module M. If M has enough α-noetherians, then N has enough β-noetherians for some $\beta \leq \alpha$.*

Proof. Let $X \leq N$, then $X \leq M$. Since M has enough α-noetherians, we infer that there exists a submodule X' of X such that n-dim $X' \leq \alpha$. Hence, N has enough $\beta-$noetherians for some $\beta \leq \alpha$. □

A sufficient condition for the converse of Proposition 3.5 to hold, is given in the following result.

Proposition 3.6. *If N is an essential submodule of M and N has enough α-noetherians, so does M.*

Proof. Let $0 \neq X \leq M$, then $0 \neq N \cap X$. Since N has enough α-noetherians, we infer that there exists a submodule $0 \neq T \leq X \cap N$ such that n-dim $T \leq \alpha$. Therefore, M has enough β-noetherians for some $\beta \leq \alpha$. Using Proposition 3.5, we deduce $\alpha = \beta$. □

With the help of Propositions 3.5 and 3.6, we make the following.

Corollary 3.7. *An R-module M has enough α-noetherians if and only if $E(M)$ has enough α-noetherians.*

Proposition 3.8. *The following conditions are equivalent for an R-module M:*

(1) *M has enough α-noetherians.*
(2) *$L_\alpha(M) \leq^{ess} M$.*

Proof. (1) $\Rightarrow$ (2) Assume (1). Hence, every nonzero submodule N of M contains a nonzero submodule K such that $n\text{-}\dim K \leq \alpha$. So, $K \in L_\alpha(M)$. Thus, for every nonzero submodule N of M, we have $N \cap L_\alpha(M) \neq 0$. This yeilds that $L_\alpha(M) \leq^{ess} M$.

(2) $\Rightarrow$ (1) Assume (2). By Proposition 3.6, we infer that $L_\alpha(M)$ has enough α-noetherians. □

Proposition 3.9. *Let M be an R module and $N \leq M$. If N has enough β_1-noetherians and $\frac{M}{N}$ has enough β_2-noetherians, then M has enough α-noetherians where $\alpha \leq \sup\{\beta_1, \beta_2\}$.*

Proof. Let L be a nonzero submodule of M. If $L \cap N \neq 0$, then we infer that $L \cap N$ contains a nonzero submodule K, such that $n\text{-}\dim K \leq \beta_1$, since N has enough β_1-noetherians. Therefore, K is a nonzero submodule of L and $n\text{-}\dim K \leq \beta_1$. Suppose that $L \cap N = 0$. Let f be the R-homomorphism $f : L \longrightarrow \frac{M}{N}$ defined by $f(x) = x + N$. Since $Kerf = L \cap N = 0$, we conclude that f is a monomorphism. Thus, L is isomorphic with a nonzero submodule of $\frac{M}{N}$. By Proposition 3.5, L has enough γ-noetherians for some $\gamma \leq \beta_2$. This implies that L contains a nonzero submodule X and $n\text{-}\dim X \leq \beta \leq \beta_2 \leq \sup\{\beta_1, \beta_2\}$. Therefore, M has enough α-noetherians such that $\alpha \leq \sup\{\beta_1, \beta_2\}$. □

Proposition 3.9 yields the following result.

Corollary 3.10. *Let $M_1, M_2, \ldots, M_n$ be a finite family of R-modules and α be an ordinal number. Then, each M_i has enough β_i-noetherians for some $\beta_i \leq \alpha$ if and only if $M = \bigoplus_{i=1}^n M_i$ has enough α-noetherians such that $\alpha \leq \sup\{\beta_i\}_{i=1}^n$.*

Proof. Assume M_i has enough β_i-noetherians for each i. We proceed by induction on n. Let $M = M_1 \bigoplus M_2$. Since $\frac{M}{M_1} = \frac{M_1 \bigoplus M_2}{M_1} \cong M_2$. By Proposition 3.9, M has enough α-noetherians such that $\alpha \leq \sup\{\beta_1, \beta_2\}$. Let

$M = \bigoplus_{i=1}^{t} M_i$ has enough α-noetherians such that $\alpha \leq \{\beta_i\}_{i=1}^{t}$ for each $t < n$. Now let $M = \bigoplus_{i=1}^{n} M_i$. Thus

$$\frac{M}{M_n} = \frac{M_1 \bigoplus M_2 \bigoplus \ldots \bigoplus M_n}{M_n} \cong M_1 \bigoplus M_2 \bigoplus \ldots \bigoplus M_{n-1}.$$

Hence in view of Proposition 3.9, we infer that M has enough α-noetherians such that $\alpha \leq \sup\{\beta_i\}_{i=1}^{n}$. The converse by Proposition 3.5 and the fact that each M_i has a submodule of M is clear. □

Proposition 3.11. *Let $\{M_i\}_{i \in I}$ be a family of R-modules. If the direct sum $\bigoplus M_i$ has enough α-noetherians, then each M_i has enough β_i-noetherians for some $\beta_i \leq \alpha$. Moreover, if each M_i has enough β_i-noetherians, then $\bigoplus M_i$ has enough α-noetherians such that $\alpha \leq \sup\{\beta_i | i \in I\}$.*

Proof. Suppose $M = \bigoplus M_i$ has enough α-noetherians. By Proposition 3.5 and the fact that each M_i is a submodule of M, we see that M_i has enough β_i-noetherians for some $\beta_i \leq \alpha$ for all $i \in I$. Suppose that each M_i has enough β_i-noetherians. Let x be a nonzero element of M. There exists a finite subset $J \subseteq I$ such that $Rx \leq \bigoplus_{i \in J} M_i$. Using Corollary 3.10 and Proposition 3.5, we deduce that $\bigoplus_{i \in J} M_i$ has enough γ-noetherians for some $\gamma \leq \sup\{\beta_i, i \in J\}$. Hence, Rx contains a nonzero submodule X such that $n\text{-dim}\, X \leq \sup\{\beta_i\}_{i \in J} \leq \sup\{\beta_i\}_{i \in I}$ and we are done. □

An infinite direct product of modules having enough α_i-noetherians need not be enough α-noetherians, see [14, Example 2.8]. However, the following shows that any product of rings having enough α-noetherians inherits the property.

Proposition 3.12. *Let $\{R_i\}_{i \in I}$ be a family of rings and let $R = \prod R_i$. The following are equivalent:*

1. *The ring R has enough α-noetherians.*
2. *The ring R_i has enough β_i-noetherians for all $i \in I$ and for some β_i such that $\beta_i \leq \alpha$ and $\alpha = \sup\{\beta_i\}_{i \in I}$.*

Proof. (1) ⇒ (2) For each $j \in J$, let $K_j = \{(x_i) \in R | x_i = 0 \text{ for all } i \neq j\}$. It is clear that the ring R_j has enough α-noetherians if and only if the ideal K_j of R has enough α-noetherians. By Proposition 3.5, every ideal of R has enough β_i-noetherians for some $\beta_i \leq \alpha$. In particular, each K_j has enough β_j-noetherians for some $\beta_j \leq \alpha$. Consequently, R_j has enough β_j-noetherians for all $j \in J$.

(2) ⇒ (1) Since R_j has enough β_j-noetherians for all $j \in I$, each K_j is an ideal of R and it has enough β_j-noetherians. By Proposition 3.11, $\bigoplus_{i \in I} K_i$ is

an ideal of R and it has enough α-noetherians, for some $\alpha \leq \sup\{\beta_i\}$. Since $\bigoplus_{i\in I} K_i = \bigoplus_{i\in I} R_i \leq^{ess} R$, we infer that R has enough α-noetherians for some $\alpha \leq \sup\{\beta_i\}_{i\in I}$ by Proposition 3.5. Since each $\beta_i \leq \alpha$, we infer that $\alpha = \sup\{\beta_i\}_{i\in I}$, as desired. □

Let R be a ring and M a bimodule over R. The *trivial extension* of R and M is $R \propto M = \{(a,x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by

$$(a,x)(b,y) = (ab, ay + xb).$$

In fact, $R \propto M$ is isomorphic to the subring $\left\{\begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R, x \in M\right\}$ of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$.

Theorem 3.13. *Let M an R-module. The following statements are equivalent:*

(1) *The ring $R \propto M$ has enough β-noetherians for some $\beta \leq \alpha$.*
(2) $\mathrm{Ann}_R(M)$ *and M_R have enough β-noetherians for some $\beta \leq \alpha$.*
(3) $\mathrm{Ann}_R(M) \bigoplus M_R$ *has enough β-noetherians for some $\beta \leq \alpha$.*

Proof. (1) ⇒ (2) Let N be a nonzero submodule of M. Then $(0,N)$ is a nonzero ideal of $R \propto M$ and hence it contains a nonzero ideal E such that $n\text{-dim}\, E \leq \alpha$. It is easy to see that $E = (0,K)$ for some nonzero submodule K of M such that $K \subseteq N$. Moreover, it is clear that $n\text{-dim}\, K \leq \alpha$. Therefore, M has enough β-noetherians for some $\beta \leq \alpha$. We show that $\mathrm{Ann}_R(M)$ has enough β-noetherians for some $\beta \leq \alpha$, consider a nonzero ideal I of R which is contained in $\mathrm{Ann}_R(M)$. Then $(I,0)$ is a nonzero ideal of $R \propto M$, so it contains a nonzero ideal L such that $n\text{-dim}\, L \leq \alpha$. Obviously, $L = (I',0)$ for some nonzero ideal I' of R with $I' \subseteq I$ and $n\text{-dim}\, I' \leq \alpha$. Hence (2) follows.

(2) ⇒ (1) Define

$$H = \{I \leq R \mid n\text{-dim}\, I \leq \alpha \text{ and } I \subseteq \mathrm{Ann}_R(M)\}$$

and

$$W = \{N \leq M \mid n\text{-dim}\, N \leq \alpha\}.$$

Clearly, for each $I \in H$ and $N \in W$, $(I,0)$ and $(0,N)$ are ideals of $R \propto M$ such that $n\text{-dim}\,(I,0) \leq \alpha$ and $n\text{-dim}\,(0,N) \leq \alpha$, and hence so is

$$(I,N) = (I,0) + (0,N).$$

Note that

$$(L_\alpha(\mathrm{Ann}_R(M)), L_\alpha(M)) = (\Sigma_{I\in H} I, \Sigma_{N\in w} N) = \Sigma_{I\in H}\Sigma_{N\in w}(I, N).$$

Hence, $(L_\alpha(\mathrm{Ann}_\alpha(M), L_\alpha(M)) \leq L_\alpha(R \propto M)$. Thus, we have

$$L_\alpha(\mathrm{Ann}_R(M)) \leq^{ess} \mathrm{Ann}_R(M)$$

and $L_\alpha(M) \leq^{ess} M$ by Proposition 3.8. This yields that

$$(L_\alpha(\mathrm{Ann}_R(M)), L_\alpha(M)) \leq^{ess} R \propto M.$$

Therefore $L_\alpha(R \propto M) \leq^{ess} R \propto M$. Consequently, T has enough α-noetherians by Proposition 3.8.

(2) ⇒ (3) It follows from Proposition 3.11. □

Remark 3.14. In view of Proposition 3.5 and Theorem 3.13, we deduce that if R and M have enough α-noetherians, then so does $R \propto M$. However, the converse is not true in general, see[14, Remark 2.17].

We close this section with the following.

Corollary 3.15. *Let M be a finitely generated R-module. The following statements are equivalent:*

(1) *$R \propto M$ has enough α-noetherians.*
(2) *R has enough β_1-noetherians, M has enough β_2-noetherians, where β_1 and β_2 are ordinal numbers, and $\alpha \leq \sup\{\beta_1, \beta_2\}$.*

Proof. (1) ⇒ (2) Assume (1). By Theorem 3.13, we infer that M has enough β_1-noetherians for some $\beta_1 \leq \alpha$ and $\mathrm{Ann}_R(M)$ has enough β_2-noetherians for some $\beta_2 \leq \alpha$. Let f be the R-homomorphism $f : R \longrightarrow M^n$ defined by $f(r) = (rm_1, rm_2, \ldots, rm_n)$. Since $Kerf = \mathrm{Ann}_R(M)$, $\frac{R}{\mathrm{Ann}_R(M)}$ is isomorphic with a nonzero submodule of M^n for some non-negative integer n. Thus, the R-module $\frac{R}{\mathrm{Ann}_R(M)}$ has enough β_2-noetherians for some $\beta_2 \leq \alpha$, see Proposition 3.5. Therefore, R has enough α-noetherians such that $\alpha \leq \sup\{\beta_1, \beta_2\}$ by Proposition 3.9.

(2) ⇒ (1) Assume (2). Hence, $\mathrm{Ann}_R(M)$ has enough β_1-noetherians, see Proposition 3.5. Using Theorem 3.13, the proof is complete. □

4. A generalization of seminoetherian and semiartinian modules

An R-module M is said to be *seminoetherian* if any nonzero factor module of M contains a nonzero Noetherian submodule, see [14] for more details. An R-module M is called a *Loewy module* (also known as *semiartinian module*) if any nonzero factor module of M contains a simple submodule, see [20] for more details.

With the Noetherian dimension concept in hand, we generalize the notions of seminoetherian and semiartinian modules here.

Definition 4.1. We say an R-module M is *α-seminoetherian* if any nonzero factor module of M contains a nonzero submodule with Noetherian dimension less than or equal to α and α is the least ordinal number with this property. We also say an R-module M is *strongly α-seminoetherian* if it is β-seminoetherian for some $\beta \leq \alpha$. Clearly, 0-seminoetherian modules are precisely seminoetherian modules.

Definition 4.2. We say an R-module M is *α-semiartinian*, if any nonzero factor module of M contains a nonzero submodule which it is β-critical for some ordinal number $\beta \leq \alpha$ and α is the least ordinal number with this property. We also say an R-module M is *strongly α-semiartinian* if it is β-semiartinian for some $\beta \leq \alpha$. Clearly, 0-semiartinian modules are just semiartinian modules.

Following [18], an R-module M is called *locally Noetherian* (resp., *locally Artinian*) if every finitely generated submodule of M is Noetherian (resp., Artinian). We will also consider later the following generalization of these definitions.

Definition 4.3. We say an R-module M is

(1) *locally α-Noetherian*, if every nonzero finitely generated submodule of M has Noetherian dimension less than or equal to α and α is the least ordinal number with this property.
(2) *locally α-Artinian*, if every nonzero finitely generated submodule of M has Krull dimension less than or equal to α and α is the least ordinal number with this property.
(3) *strongly locally α-Noetherian* if M is locally α-Noetherian for some ordinal number $\beta \leq \alpha$.
(4) *strongly locally α-Artinian* if M is locally α-Artinian for some ordinal number $\beta \leq \alpha$.

Just for the record, we make the following easy facts.

Example 4.4. (1) If every prime ideal of R is maximal, then every α-seminoetherian R-module is seminoetherian.
(2) Every locally α-Noetherian module is β-seminoetherian for some $\beta \leq \alpha$.
(3) Every locally α-Artinian module is β-semiartinian for some $\beta \leq \alpha$.

Remark 4.5. Let M be an R-module. If α is an ordinal number, then

$$L_\alpha(M) = \sum \{N \leq M \mid n\text{-dim } \mathrm{N} \leq \alpha\}$$

is a unique largest locally β-Noetherian submodule for some $\beta \leq \alpha$. Therefore, every R-module M has a unique largest locally β-Noetherian submodule for some $\beta \leq \alpha$.

Remark 4.6. Obviously, the class of strongly locally α-Noetherian and strongly locally α-Artinian modules are closed under submodules, factor modules, and arbitrary direct sums.

A class of modules is called *hereditary torsion* if it is closed under submodules, factor modules, direct sums, and module extensions.

Remark 4.7. Let R be a ring. The following statements hold:

(1) The class of all strongly α-semiartinian R-modules is the smallest hereditary torsion class containing class of strongly locally α-Artinian modules.
(2) The class of all strongly α-seminoetherian R-modules is the smallest hereditary torsion class containing strongly locally α-Noetherian modules.

Proof. The proofs follow from [18, Propositions VI.2.5 and VI.3.3] and Remark 4.6. □

Before state the next result, let us recall that an R-module M is said to be *uniserial* if its submodules are linearly ordered by inclusion.

Proposition 4.8. *Let M be a uniserial module. If N is α_1-seminoetherian and $\frac{M}{N}$ is α_2-seminoetherian, then M is α-seminoetherian, where*

$$\alpha \leq \sup\{\alpha_1, \alpha_2\}.$$

Proof. Let $K \lneqq M$. If $K \leq N$, then there exists a submodule L of M such that $K < L \leq N$ and $n\text{-dim}\,\frac{L}{K} \leq \alpha_1$. Otherwise, we have $N \leq K$. This yields $\frac{K}{N} \leq \frac{M}{N}$. So, there exists $\frac{K}{N} < \frac{L}{N} \leq \frac{M}{N}$, where $n\text{-dim}\,\frac{\frac{L}{N}}{\frac{K}{N}} = n\text{-dim}\,\frac{L}{K} \leq \alpha_2$. □

Corollary 4.9. *Let M be a uniserial R-module. The following statements are equivalent:*

(1) *$R \propto M$ is an α-seminoetherian ring.*
(2) *R is β-seminoetherian and M is λ-seminoetherian for some $\beta, \lambda \leq \alpha$ and $\alpha \leq \sup\{\beta, \lambda\}$.*

Proof. (1) $\Rightarrow$ (2) Consider the ideal $I = (0, M)$ of $R \propto M$. We have $I^2 = 0$ and $\frac{R \propto M}{I} \cong R$ (as rings). Since $\frac{R \propto M}{I} \cong R$, we deduce that R is β-seminoetherian for some $\beta \leq \alpha$. Since M is a submodule of $R \propto M$, M is λ-seminoetherian by Remark 4.7.

(2) $\Rightarrow$ (1) Consider the $I = (0, M)$ is an ideal of $R \propto M$ and we have $\frac{R \propto M}{I} \cong R$. By Proposition 4.8, we conclude that $R \propto M$ is α-seminoetherian for some $\alpha \leq \sup\{\beta, \lambda\}$. □

Here we give two lemmas.

Lemma 4.10. *If an R-module M is α-seminoetherian, then M is β-semiartinian for some ordinal number β.*

Proof. Let M be α-seminoetherian and let N be a proper submodule of M. Thus, there exists a submodule K of M, such that $N < K \leq M$ and $n\text{-dim}\, \frac{K}{N} \leq \alpha$. Therefore, $\frac{K}{N}$ has Krull dimension, see Proposition 2.1. So, $\frac{K}{N}$ has a β-critical submodule for some ordinal number β, see Proposition 2.4. This implies that M is β-semiartinian for some ordinal number β. □

Lemma 4.11. *If an R-module M is β-semiartinian, then M is α-seminoetherian for some ordinal number α.*

Proof. Let $0 \neq N < M$. Since M is β-semiartinian, we infer that there exists a submodule K of M such that $N < K \leq M$ and $\frac{K}{N}$ is γ-critical for some $\gamma \leq \beta$. Hence, $\frac{K}{N}$ has Noetherian dimension, see Proposition 2.1. So, M is α-seminoetherian for some ordinal number α. □

An R-module N is called *M-generated* if it is a homomorphic image of a direct sum of copies of M. An R-module N is said to be *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. Let R-Mod denotes the category of all R-modules. We denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are R-modules subgenerated by M.

Theorem 4.12. *The following are equivalent for a nonzero R-module M:*

(1) *M is α-seminoetherian.*
(2) *N is β-seminoetherian for all $N \in \sigma[M]$ for some $\beta \leq \alpha$.*
(3) *For all $0 \neq N \in \sigma[M]$, N contains a nonzero submodule K and Noetherian dimension K is less than or equal to α.*
(4) *Every nonzero factor module of M has enough β-noetherians for some $\beta \leq \alpha$ and α is the least ordinal number with this property.*
(5) *For all $N \in \sigma[M]$, $L_\alpha(N) \leq^{ess} N$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) It is clear by Remark 4.7.

(3) $\Rightarrow$ (5) Let $0 \neq N \in \sigma[M]$ and $0 \neq K \leq N$. Thus, $K \in \sigma[M]$, so K contains a nonzero submodule with Noetherian dimension less than or equal to α. This implies that N has enough β-noetherians for some $\beta \leq \alpha$. By Proposition 3.8, we deduce $L_\beta(N) \leq^{ess} N$. We note that $\beta \leq \alpha$ implies $L_\beta(N) \leq L_\alpha(N)$. Hence, we have $L_\alpha(N) \leq^{ess} N$.

(5) $\Rightarrow$ (4) Let N be a proper submodule of M. Since $\frac{M}{N} \in \sigma[M]$, we infer that $L_\alpha(\frac{M}{N}) \leq^{ess} \frac{M}{N}$. Proposition 3.8 completes the proof.

(4) $\Rightarrow$ (1) It is clear. □

The following is an easy consequence of Theorem 4.12.

Corollary 4.13. *For a ring R the following statements are equivalent:*

(1) *R is an α-seminoetherian ring.*
(2) *Every R-module is β-seminoetherian for some $\beta \leq \alpha$.*
(3) *Every nonzero R-module has enough β-noetherians for some $\beta \leq \alpha$.*
(4) *For any R-module M, $L_\alpha(M) \leq^{ess} M$.*

The following is a counterpart of [14, Remark 3.8].

Remark 4.14. (1) From Theorem 4.12 and Corollary 4.13, we see that the class of modules (rings) having enough α-noetherians contains the class of α-seminoetherian modules (rings).
(2) By Corollary 4.13((1) $\Leftrightarrow$ (2)), it easy to see that a ring R is α-seminoetherian if and only if $\frac{R}{I}$ is a β-seminoetherian ring for any ideal I of R.

Here, we give some results about quotient finite dimensional modules.

Lemma 4.15. *The following are equivalent for an R-module M:*

(1) *M is quotient finite dimensional α-seminoetherian for some ordinal number α.*
(2) *n-dim $M \leq \beta$ for some ordinal number β.*
(3) *k-dim $M \leq \gamma$ for some ordinal number γ.*

Proof. (1) $\Rightarrow$ (2) Assume (1). For any $N < M$, there exists a submodule P of M such that $N < P \leq M$ and $n\text{-dim}\,\frac{P}{N} \leq \alpha$. By Proposition 2.1, we have $k\text{-dim}\,\frac{P}{N} \leq \beta$ for some ordinal number β. By Proposition 2.2, we infer that k-dim $M \leq \beta$. We conclude that n-dim $M \leq \gamma$ for some ordinal number γ by Proposition 2.1.

(2) $\Leftrightarrow$ (3) It follows from Proposition 2.1.

(2) ⇒ (1) It is clear by Proposition 2.1 and the fact that $n\text{-dim}\, M_R = \alpha$ implies M_R is β-seminoetherian for some $\beta \leq \alpha$. □

Lemma 4.16. *An R-module M is quotient finite dimensional α-semiartinian if and only if $k\text{-dim}\, M \leq \alpha$.*

Proof. Assume that M is quotient finite dimensional α-semiartinian. For any $N < M$, there exists P such that $N < P \leq M$ and $k\text{-dim}\, \frac{P}{N} \leq \alpha$. Hence, we have $k\text{-dim}\, M \leq \alpha$, see Proposition 2.2. The converse is clear. □

Lemma 4.16 yields the following result.

Corollary 4.17. *A quotient finite dimensional module M is α-seminoetherian if and only if it has Krull dimension.*

Proposition 4.18. *The following are equivalent for a ring R:*

1. *Every α-seminoetherian R-module is β-semiartinian, where $\beta \leq \alpha$.*
2. *Every quotient finite dimensional module M that has enough β-noetherians, where $\beta \leq \alpha$, is $\lambda.f.e$ for some $\lambda \leq \beta$.*
3. *If $n\text{-dim}\, M \leq \alpha$, then $k\text{-dim}\, M \leq \alpha$ (i.e., $k\text{-dim}\, M \leq n\text{-dim}\, M$).*

Proof. (2) ⇔ (3) It follows from Proposition 3.3.

(1) ⇒ (3) Let M be an R-module such that $n\text{-dim}\, M \leq \alpha$. Thus, M is α_1-seminoetherian for some $\alpha_1 \leq \alpha$, so M is β-semiartinian for some $\beta \leq \alpha$. Hence, we have $k\text{-dim}\, M \leq \alpha$ by [1, Proposition 1.3].

(3) ⇒ (1) Let M be α-seminoetherian. Then, for each $N < M$, there exists a submodule K of M such that $N < K \leq M$ and $n\text{-dim}\, \frac{K}{N} \leq \alpha$. Therefore, we have $k\text{-dim}\, \frac{K}{N} \leq \alpha$. Hence, M is β-semiartinian for some $\beta \leq \alpha$. □

Following [19], a ring R is called *amen* (Artinian modules equal Noetherian modules) if R satisfies the conditions (1) Every Artinian R-module is Noetherian, and (2) Every Noetherian R-module is Artinian.

In view of [9, Theorem 3.2], we have the following result.

Corollary 4.19. *Let R be an amen ring. Then $n\text{-dim}\, M_R \leq \alpha$ if and only if M_R is quotient finite dimensional α-seminoetherian.*

Corollary 4.20. *Let R be an amen ring. Then M_R is α-seminoetherian if and only if M_R is α-semiartinian.*

Proof. Let M be α-seminoetherian. Then, for each submodule N of M, there exists a submodule $\frac{X}{N}$ of $\frac{M}{N}$ such that $n\text{-dim}\, \frac{X}{N} \leq \alpha$. Since R is an amen ring, we infer that $k\text{-dim}\, \frac{X}{N} \leq \alpha$. Therefore, $\frac{X}{N}$ has a β-critical submodule for some

$\beta \leq \alpha$, see Proposition 2.4. Hence, M is β-semiartinian for some $\beta \leq \alpha$. Conversely, let M be $\beta-$semiartinian. Then every nonzero factor module of M contains a γ-critical submodule for some ordinal number $\gamma \leq \beta$. Thus, there exists a submodule $\frac{X}{N}$ of $\frac{M}{N}$ such that k-dim $\frac{X}{N} = \gamma \leq \beta$. Since R is an amen ring, we infer that n-dim $\frac{X}{N} \leq \beta$. Hence, M is α-seminoetherian for some $\alpha \leq \beta$. This implies that $\alpha = \beta$, as desired. □

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A GENARALIZATION OF SEMINOETHERIAN RINGS AND MODULES

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تعمیمی از حلقه‌ها و مدول‌های نیم‌نوتری

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در این مقاله مدول‌هایی که همه زیرمدول‌های آن شامل زیرمدول غیرصفر با بعد نوتری کمتر یا مساوی α، که α کوچکترین عدد ترتیبی با این خاصیت است، را مطالعه و بررسی می‌کنیم. این مدول‌ها را مدول‌های دارای α-نوتری کافی می‌نامیم. هم‌چنین، مدول‌هایی که هر مدول خارج‌قسمتی غیرصفر از آن‌ها شامل زیرمدولی غیرصفر با بعد نوتری کمتر یا مساوی α است که α کوچکترین عدد ترتیبی با این خاصیت می‌باشد، مدول‌های α-نیم‌نوتری می‌نامیم و آن‌ها را مورد بررسی قرار می‌دهیم. کار ما در واقع تعمیمی از نتایج کورکی و تریباک در مرجع [۱۴] می‌باشد.

کلمات کلیدی: بعد کرول، بعد نوتری، مدول‌های نیم‌نوتری، نوتری‌های کافی.