

R-IDEALS OF ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT. The concepts have been introduced in Almost Distributive Lattices(ADLs), namely, *R*-ideals and λ -ideals. A set of conditions has been identified that are equivalent to converting an *E*-ideal into an *R*-ideal. Moreover, it has been derived that for any *E*-ideal, there exists a homomorphism with a dual dense kernel, which is itself an *R*-ideal. The characterization of λ -ideals in terms of *R*-ideals and congruences has been established. Additionally, equivalent conditions have been established to demonstrate that the space of all prime λ -ideals forms a Hausdorff space.

1. INTRODUCTION

In [8], Swamy and Rao developed an Almost Distributive Lattice(ADL), who established a connection between ideals in ADLs and distributive lattices. They proved that an ADL's collection of principal ideals forms a distributive lattice.

Additionally, *w*-filters are presented in ADLs and their properties are examined in [4]. The notion of *E*-ideals is given in an ADL and their properties are explored in [3]. The purpose of this work is to investigate the properties of dual dense elements and *E*-ideals within ADLs, namely *R*-ideals and λ -ideals. A set of equivalent conditions for an *E*-ideal can be transformed into an *R*-ideal is established. In addition, it is shown that there is a homomorphism between *E*-ideals within ADLs such that the dual dense kernel of this homomorphism is an *R*-ideal. Furthermore, a sufficient condition given for each ADL to be regarded as relatively complemented, expressed in terms of *R*-ideals. A set of equivalent conditions is established for an ADL to convert into a Boolean algebra. Finally, some characteristics of the space of all prime λ -ideals of an ADL are studied topologically.

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2. PRELIMINARIES

The definitions and significant results from [6, 8] are gathered and given in this part; these will be needed during the entire document.

Definition 2.1. [8] An algebraic structure $(\mathcal{L}, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is an Almost Distributive Lattice (ADL) with zero if it satisfies the conditions given below:

- (1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$;
- (2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
- (3) $(a \vee b) \wedge b = b$;
- (4) $(a \vee b) \wedge a = a$;
- (5) $a \vee (a \wedge b) = a$;
- (6) $0 \wedge a = 0$, for any $a, b, c \in \mathcal{L}$.

If $(\mathcal{L}, \vee, \wedge, 0)$ is an ADL, for any $x, y \in \mathcal{L}$, define $x \leq y$ if and only if $x = x \wedge y$ (or equivalently, $x \vee y = y$), then \leq is a partial ordering on \mathcal{L} . An element ν in \mathcal{L} is said to be *maximal* if it is maximal with respect to the partial ordering \leq on \mathcal{L} . That is, for any $a \in \mathcal{L}$, $m \leq a \Rightarrow m = a$. The set of all maximal elements of an ADL is denoted by $\mathcal{M}_{Max.elts}$.

In Swamy's work[8], it is noted that an ADL satisfies almost all properties of a distributive lattice except the right distributivity of \vee over \wedge , commutativity of \vee , commutativity of \wedge . Any one of these properties make an ADL \mathcal{L} a distributive lattice. A nonempty subset G of \mathcal{L} is called an *ideal* (respectively a *filter*) of \mathcal{L} , if $a \vee b, a \wedge x \in G$ (respectively $a \wedge b, x \vee a \in G$) for all $a, b \in G$ and all $x \in \mathcal{L}$. A proper ideal(filter) \mathcal{U} of \mathcal{L} is called a prime ideal(filter) if, for any $x, y \in \mathcal{L}$, $x \wedge y \in \mathcal{U}(x \vee y \in \mathcal{U}) \Rightarrow x \in \mathcal{U}$ or $y \in \mathcal{U}$. A proper ideal(filter) \mathcal{V} of \mathcal{L} is said to be maximal if it is not properly contained in any proper ideal(filter) of \mathcal{L} . It can be observed that every maximal ideal(filter) of \mathcal{L} is a prime ideal(filter). For any subset \mathcal{V} of \mathcal{L} the smallest ideal containing \mathcal{V} is given by $(\mathcal{V}] := \{(\bigvee_{i=1}^n a_i) \wedge x \mid a_i \in \mathcal{V}, x \in \mathcal{L} \text{ and } n \in \mathbb{N}\}$. If $\mathcal{V} = \{a\}$, we write $(a]$ instead of $(\mathcal{V}]$ and such an ideal is called the principal ideal of \mathcal{L} . In a similar way, for each $\mathcal{V} \subseteq \mathcal{L}$, $[\mathcal{V}) := \{x \vee (\bigwedge_{i=1}^n a_i) \mid a_i \in \mathcal{V}, x \in \mathcal{L} \text{ and } n \in \mathbb{N}\}$. If $\mathcal{V} = \{a\}$, we write $[a)$ instead of $[\mathcal{V})$ and such a filter is called the principal filter of \mathcal{L} .

For any $a, b \in \mathcal{L}$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. Hence the set $(\mathcal{P}G(\mathcal{L}), \vee, \cap)$ of all principal ideals of \mathcal{L} is a sublattice of the

distributive lattice $(\mathfrak{G}(\mathcal{L}), \vee, \cap)$ of all ideals of \mathcal{L} . Also, we have that the set $(\mathfrak{F}(\mathcal{L}), \vee, \cap)$ of all filters of \mathcal{L} is a bounded distributive lattice. In an ADL[7], observe that the prime ideal \mathcal{U} of \mathcal{L} can only exist if $\mathcal{L} \setminus \mathcal{U}$ is a prime filter of \mathcal{L} .

For each non-void subset \mathcal{V} of \mathcal{L} , the set

$$\mathcal{V}^+ = \{x \in \mathcal{L} \mid a \vee x \in \mathcal{M}_{\text{Max.elts}}, \text{ for all } a \in \mathcal{V}\}$$

is a filter of \mathcal{L} . Generally, for every $a \in \mathcal{L}$, $\{a\}^+ = (a)^+$, where $(a) = [a]$. The *dual annihilator* of an element $a \in \mathcal{L}$ is defined as the set

$$(a)^+ = \{x \in \mathcal{L} \mid x \vee a \in \mathcal{M}_{\text{Max.elts}}\}.$$

If $(\rho)^+ = \mathcal{M}_{\text{Max.elts}}$, then an element $\rho \in \mathcal{L}$ is considered dual dense. Within \mathcal{L} , the set E is the set of dual dense elements. An ideal of an ADL \mathcal{L} can be obtained by the set E . According to [3], an ideal \mathcal{H} of an ADL \mathcal{L} is called an *E-ideal* if $E \subseteq \mathcal{H}$. The smallest *E-ideal* is E . For any subset \mathcal{V} of \mathcal{L} , define $(\mathcal{V}, E) = \{x \in \mathcal{L} \mid a \wedge x \in E \text{ for all } a \in \mathcal{V}\}$. It is noted that $(\mathcal{L}, E) = E$ and $(E, E) = \mathcal{L}$. Furthermore, for any subset \mathcal{V} of \mathcal{L} , $E \subseteq (\mathcal{V}, E)$. For every $a \in \mathcal{L}$, $(\{a\}, E)$ is denoted as (a, E) . Therefore, $(\nu, E) = E$ for any $\nu \in \mathcal{M}_{\text{Max.elts}}$. (\mathcal{V}, E) is an *E-ideal* in \mathcal{L} for each subset \mathcal{V} of \mathcal{L} .

Lemma 2.2. [3] *Given two subsets \mathcal{V}, \mathcal{S} of an ADL \mathcal{L} , the following holds*

- (1) $\mathcal{V} \subseteq \mathcal{S} \Rightarrow (\mathcal{S}, E) \subseteq (\mathcal{V}, E)$;
- (2) $\mathcal{V} \subseteq ((\mathcal{V}, E), E)$;
- (3) $(\mathcal{V}, E) = (((\mathcal{V}, E), E), E)$;
- (4) $\mathcal{V} \subseteq E \Leftrightarrow (\mathcal{V}, E) = \mathcal{L}$.

Proposition 2.3. [3] *Any ideals \mathcal{H}, G of \mathcal{L} can have the following:*

- (1) $E = ((\mathcal{H}, E), E) \cap (\mathcal{H}, E)$;
- (2) $\mathcal{H} \cap G \subseteq E \Rightarrow \mathcal{H} \subseteq (G, E)$;
- (3) $((\mathcal{H} \vee G), E) = (\mathcal{H}, E) \cap (G, E)$;
- (4) $((\mathcal{H} \cap G), E), E) \subseteq ((\mathcal{H}, E), E) \cap ((G, E), E)$.

The idea that $((x), E) = (x, E)$ is obvious. It follows that $(0, E) = \mathcal{L}$. The previously noted observations directly lead to the corollary that follows.

Corollary 2.4. [3] *Given any $a, b, c \in \mathcal{L}$, the following holds*

- (1) $a \leq b$ implies $(b, E) \subseteq (a, E)$;
- (2) $((a \vee b), E) = (a, E) \cap (b, E)$;
- (3) $((a \wedge b), E), E) = ((a, E), E) \cap ((b, E), E)$;
- (4) $(a, E) = \mathcal{L}$ if and only if $a \in E$.

Proposition 2.5. [3] *Any E -ideal \mathcal{H} of \mathcal{L} can have the following:*

- (1) $(\mathcal{H}, E) \cap \mathcal{H} = E$;
- (2) $(\mathcal{H}, E) = \bigcap \{\mathcal{U} \mid \mathcal{U} \text{ is a prime } E\text{-ideal, } \mathcal{H} \not\subseteq \mathcal{U}\}$;
- (3) *Given any ideal G of \mathcal{L} where $\mathcal{H} \cap G = \emptyset$, there is a prime E -ideal \mathcal{U} of \mathcal{L} such that $\mathcal{U} \cap G = \emptyset$ and $\mathcal{H} \subseteq \mathcal{U}$.*

Consider x not being an element of the E -ideal \mathcal{H} of \mathcal{L} . Then, there is a prime E -ideal \mathcal{U} such that $x \notin \mathcal{U}$ and $\mathcal{H} \subseteq \mathcal{U}$. A prime E -ideal \mathcal{U} is called minimal if there is no other prime E -ideal \mathcal{W} such that $\mathcal{W} \subset \mathcal{U}$. It is noted that a prime E -ideal \mathcal{U} of \mathcal{L} is minimal if and only if for every $x \in \mathcal{U}$, there exists $y \notin \mathcal{U}$ such that $x \wedge y \in E$.

3. R -IDEALS OF ADLS

The notion of R -ideals in ADLs is presented in this section. A set of equivalent conditions is established for an E -ideal in an ADL to become an R -ideal.

Definition 3.1. An ideal \mathcal{H} of \mathcal{L} is said to be an R -ideal if $\mathcal{H} = ((\mathcal{H}, E), E)$.

It is evident that (\mathcal{V}, E) is an R -ideal for every subset \mathcal{V} of \mathcal{L} . Clearly, both E and \mathcal{L} are R -ideals as well. For any ideal \mathcal{H} , we have that every R -ideal is also an E -ideal. Additionally, for any $\mathcal{G} \subseteq E$, it holds that

$$\mathcal{L} = (E, E) \subseteq (\mathcal{G}, E).$$

Consequently, $((\mathcal{G}, E), E) = (\mathcal{L}, E) = E$, demonstrating that E is the smallest R -ideal in \mathcal{L} . The set $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ indicate the set of all R -ideals of \mathcal{L} . The following proof establishes that the set $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ constitutes a complete Boolean algebra.

Theorem 3.2. *In an ADL \mathcal{L} , $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ forms a complete Boolean algebra.*

Proof. Clearly, that $(\mathcal{I}_{\mathcal{R}}(\mathcal{L}), \subseteq)$ forms a poset, where \subseteq denotes set inclusion. Let $\mathcal{H}, G \in \mathcal{I}_{\mathcal{R}}(\mathcal{L})$. It follows that

$$((\mathcal{H}, E), E) \cap ((G, E), E) = (((\mathcal{H} \cap G), E), E)$$

is the infimum of \mathcal{H} and G in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$. Define the binary operation \sqcup on $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ as:

$$\mathcal{H} \sqcup G = ((\mathcal{H}, E) \cap (G, E), E).$$

It's easy to see $((\mathcal{H}, E) \cap (G, E), E)$ serves as the supremum for \mathcal{H} and G in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$. Clearly, E and \mathcal{L} are the least and greatest elements in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$, respectively. Thus, $(\mathcal{I}_{\mathcal{R}}(\mathcal{L}), \cap, \sqcup, E, \mathcal{L})$ forms a bounded distributive lattice.

Furthermore, for any $\mathcal{H} \in \mathcal{I}_{\mathcal{R}}(\mathcal{L})$, we have

$$\mathcal{H} \cap (\mathcal{H}, E) = ((\mathcal{H}, E), E) \cap (\mathcal{H}, E) = E$$

and $\mathcal{H} \sqcup (\mathcal{H}, E) = ((\mathcal{H}, E) \cap ((\mathcal{H}, E), E), E) = (E, E) = \mathcal{L}$. Therefore, (\mathcal{H}, E) is the unique complement of \mathcal{H} in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$. Thus, $(\mathcal{I}_{\mathcal{R}}(\mathcal{L}), \cap, \sqcup, E, \mathcal{L})$ is a complete Boolean algebra. \square

The R -ideal is (x, E) for any $x \in \mathcal{L}$. Thus, in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$, the supremum of any pair of R -ideals (x, E) and (y, E) is given by

$$\begin{aligned} (x, E) \sqcup (y, E) &= (((x, E), E) \cap ((y, E), E)), E) \\ &= (((x \wedge y, E), E), E) \\ &= (x \wedge y, E) \end{aligned}$$

Also their infimum in $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ is $(x, E) \cap (y, E) = (x \vee y, E)$. It brings us to the next result, which follows directly from the previous observation.

Theorem 3.3. *The set $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$, consisting of all R -ideals of the form (x, E) for $x \in \mathcal{L}$, forms a lattice $\langle \mathcal{I}_{\mathcal{R}}(\mathcal{L}), \cap, \sqcup \rangle$ and is a sublattice of the distributive lattice $\langle \mathcal{I}_{\mathcal{R}}(\mathcal{L}), \cap, \sqcup \rangle$ of all R -ideals of \mathcal{L} . Additionally, $\mathcal{I}_{\mathcal{R}}(\mathcal{L})$ has a greatest element $\mathcal{L} = (\rho, E)$ for any $\rho \in E$ and a smallest element $(m, E) = E$, where $m \in \mathcal{M}_{Max.elts}$.*

Theorem 3.4. *Let \mathcal{H} be an E -ideal of an ADL \mathcal{L} . Then $\mathcal{H} \vee (\mathcal{H}, E) = \mathcal{L}$ if and only if \mathcal{H} is an R -ideal and $(\mathcal{H}, E) \vee ((\mathcal{H}, E), E) = \mathcal{L}$.*

Proof. Suppose $\mathcal{H} \vee (\mathcal{H}, E) = \mathcal{L}$ for each E -ideal \mathcal{H} of L . Therefore

$$\begin{aligned} ((\mathcal{H}, E), E) &= ((\mathcal{H}, E), E) \cap \mathcal{L} \\ &= ((\mathcal{H}, E), E) \cap (\mathcal{H} \vee (\mathcal{H}, E)) \\ &= ((\mathcal{H}, E), E) \cap \mathcal{H} \vee ((\mathcal{H}, E), E) \cap (\mathcal{H}, E) \\ &= \mathcal{H} \vee E \\ &= \mathcal{H}. \end{aligned}$$

Hence \mathcal{H} is an R -ideal. Also $(\mathcal{H}, E) \vee ((\mathcal{H}, E), E) = (\mathcal{H}, E) \vee \mathcal{H} = \mathcal{L}$. Clearly, the converse is also true. \square

A similar set of conditions is presented in the next result for characterizing a prime E -ideal as a minimal prime E -ideal.

Theorem 3.5. *In an ADL \mathcal{L} , the following assumptions are equivalent:*

- (1) *each prime E -ideal is minimal;*

- (2) $(x] \vee (x, E) = \mathcal{L}$ for each $x \in \mathcal{L}$;
 (3) $(x, E) \vee ((x, E), E) = \mathcal{L}$ and $(x] = ((x, E), E)$, for each $x \in \mathcal{L}$.

Proof. (1) \Rightarrow (2) Let us assume all prime E -ideals are minimum. Suppose $x \in \mathcal{L}$. There is a prime ideal \mathcal{U} such that $(x] \vee (x, E) \subseteq \mathcal{U}$ if $(x] \vee (x, E) \neq \mathcal{L}$. \mathcal{U} is an E -ideal since (x, E) is an E -ideal. The hypothesis states that \mathcal{U} is minimum. But according to Proposition 2.5(2), since $(x, E) \subseteq \mathcal{U}$, we discover that $x \notin \mathcal{U}$, which results in a contradiction. $(x] \vee (x, E) = \mathcal{L}$ as a result.

(2) \Rightarrow (3) It is obvious from Theorem 3.4.

(3) \Rightarrow (1) Assume condition (3). Let \mathcal{U} be a prime E -ideal. Suppose there exists another prime E -ideal \mathcal{W} such that $\mathcal{W} \subset \mathcal{U}$. Choose $x \in \mathcal{U} \setminus \mathcal{W}$. Due to $x \notin \mathcal{W}$, which follows that $(x, E) \subseteq \mathcal{W}$. Since $x \in \mathcal{U}$, applying the assumed condition yields

$$\mathcal{L} = ((x, E), E) \vee (x, E) = (x] \vee (x, E) \subseteq \mathcal{U} \vee \mathcal{W} = \mathcal{U},$$

which leads to a contradiction. Hence, \mathcal{U} must be a minimal prime E -ideal of \mathcal{L} . \square

If $(\mathcal{H}, E) = E$, then an ideal \mathcal{H} in an ADL \mathcal{L} is called dual condensed. It is observed that the set of all dual condensed ideal of an ADL is a sublattice to the lattice of all ideals of \mathcal{L} . It is clear that proper dual condensed ideal never be an R -ideal. Furthermore, all R -ideals are E -ideals. Nevertheless, not all E -ideals of ADLs have to be an R -ideals. However, the subsequent theorem provides a set of equivalent conditions that allow each E -ideal of \mathcal{L} to become an R -ideal.

Theorem 3.6. *Consider an ADL \mathcal{L} , where each proper ideal is non-dual condensed. Then, all of the following arguments are equivalent:*

- (1) each E -ideal is an R -ideal;
 (2) each prime E -ideal is an R -ideal;
 (3) each prime E -ideal is minimal;
 (4) each prime E -ideal is maximal.

Proof. (1) \Rightarrow (2) It's clear.

(2) \Rightarrow (3) Assume every prime E -ideal is also an R -ideal. Let \mathcal{U} be a prime E -ideal in \mathcal{L} . Then $((\mathcal{U}, E), E) = \mathcal{U}$. Suppose \mathcal{U} is not a minimal prime E -ideal. This implies there is a prime E -ideal \mathcal{W} such that $\mathcal{W} \subset \mathcal{U}$. Choose $x \in \mathcal{U} \setminus \mathcal{W}$. Let $a \in (\mathcal{U}, E)$. Since $x \in \mathcal{U}$, we have $a \wedge x \in E$ and thus $a \wedge x \in \mathcal{W}$. Given \mathcal{W} is prime and $x \notin \mathcal{W}$, it follows that $a \in \mathcal{W} \subset \mathcal{U}$. Therefore, $(\mathcal{U}, E) \subseteq \mathcal{U} \subseteq ((\mathcal{U}, E), E)$. Consequently, $(\mathcal{U}, E) = (\mathcal{U}, E) \cap ((\mathcal{U}, E), E) = E$.

This leads to $\mathcal{U} = ((\mathcal{U}, E), E) = \mathcal{L}$, which contradicts our initial assumption. Hence, \mathcal{U} must indeed be a minimal prime E -ideal of \mathcal{L} .

(3) \Rightarrow (4) This is obvious as each maximal E -ideal is prime.

(4) \Rightarrow (1) Assume (4). Let \mathcal{H} be a non dual dense ideal. Clearly $\mathcal{H} \subseteq ((\mathcal{H}, E), E)$. Let $x \in ((\mathcal{H}, E), E)$. Then we get $(\mathcal{H}, E) \subseteq (x, E)$. Suppose $x \notin \mathcal{H}$. Then there is a prime E -ideal \mathcal{U} such that $\mathcal{H} \subseteq \mathcal{U}$, $x \notin \mathcal{U}$. According to condition (4), \mathcal{U} being maximal means $\mathcal{U} \vee (x] = \mathcal{L}$. Therefore

$$(\mathcal{U}, E) \cap (x, E) = (\mathcal{U} \vee (x], E) = (\mathcal{L}, E) = E.$$

Thus $(\mathcal{U}, E) = (\mathcal{U}, E) \cap (\mathcal{H}, E) = E$, which is a contradiction. Hence $x \in \mathcal{H}$ and thus $((\mathcal{H}, E), E) \subseteq \mathcal{H}$. Therefore \mathcal{H} is an R -ideal of \mathcal{L} . \square

For any ideal \mathcal{H} of an ADL \mathcal{L} , let $Hom_{\mathcal{L}}(\mathcal{H})$ denote the set of all homomorphisms defined on \mathcal{H} . It is evident that $Hom_{\mathcal{L}}(\mathcal{H})$ forms a distributive lattice with respect to point-wise operations. The following proposition can be straightforwardly verified.

Proposition 3.7. *For any ideal \mathcal{H} of \mathcal{L} and $\eta \in \mathcal{L}$, define $\phi_{\eta} : \mathcal{H} \rightarrow \mathcal{H}$ by $\phi_{\eta}(x) = x \wedge \eta$ for any $x \in \mathcal{H}$. Then:*

- (1) ϕ_{η} is a homomorphism;
- (2) $\phi_{\eta \wedge \mu} = \phi_{\eta} \wedge \phi_{\mu}$ for $\eta, \mu \in \mathcal{L}$;
- (3) $\phi_{\eta \vee \mu} = \phi_{\eta} \vee \phi_{\mu}$ for $\eta, \mu \in \mathcal{L}$.

Let \mathcal{H} be an E -ideal of \mathcal{L} . We call a homomorphism $v : \mathcal{H} \rightarrow \mathcal{H}$ dual dense-valued if $v(x)$ is dual dense for every $x \in \mathcal{H}$. The collection of all dual dense-valued homomorphisms in $Hom_{\mathcal{L}}(\mathcal{H})$ is denoted by $\mathcal{E}(\mathcal{H})$. It is evident that the identity element of $Hom_{\mathcal{L}}(\mathcal{H})$ belongs to $\mathcal{E}(\mathcal{H})$. Specifically, the map $\mathbf{1} : \mathcal{H} \rightarrow \mathcal{H}$ defined by $\mathbf{1}(x) = x$ for all $x \in \mathcal{H}$ is a dual dense-valued homomorphism, hence $\mathbf{1} \in \mathcal{E}(\mathcal{H})$. Furthermore, it can be observed that $\mathcal{E}(\mathcal{H})$ forms an ideal of $Hom_{\mathcal{L}}(\mathcal{H})$. Additionally, for any $\rho \in E$, the map Φ_{ρ} belongs to $\mathcal{E}(\mathcal{H})$.

Definition 3.8. Consider \mathcal{H} as an E -ideal of \mathcal{L} , and let $\kappa : \mathcal{L} \rightarrow Hom_{\mathcal{L}}(\mathcal{H})$ be a homomorphism. We define the dual dense-kernel of the homomorphism κ , denoted by $Ker^E(\kappa)$, as follows: $Ker^E(\kappa) = \{x \in \mathcal{L} \mid \kappa(x) \in \mathcal{E}(\mathcal{H})\}$. Additionally, define the map $\Phi_{\mathcal{H}} : \mathcal{L} \rightarrow Hom_{\mathcal{L}}(\mathcal{H})$ by $\Phi_{\mathcal{H}}(\eta) = \Phi_{\eta}$ for every $\eta \in \mathcal{L}$.

Lemma 3.9. *Suppose that \mathcal{H} is the E -ideal of an ADL \mathcal{L} , and that the homomorphism $\kappa : \mathcal{L} \rightarrow Hom_{\mathcal{L}}(\mathcal{H})$. In \mathcal{L} , $Ker^E(\kappa)$ is an ideal.*

Theorem 3.10. *For every E -ideal \mathcal{H} of an ADL \mathcal{L} , we obtain*

$$\text{Ker}^E(\Phi_{\mathcal{H}}) = (\mathcal{H}, E)$$

Thus, (\mathcal{H}, E) can be regarded as a homomorphism's dual dense-kernel.

Proof. Suppose $\eta \in \text{Ker}^E(\Phi_{\mathcal{H}})$. We get $\Phi_{\eta} \in \mathcal{E}(\mathcal{H})$, thus $x \wedge \eta = \Phi_{\eta}(x)$ is dual dense for all $x \in \mathcal{H}$. Hence $\eta \in (\mathcal{H}, E)$. Conversely, let $\eta \in (\mathcal{H}, E)$. Then $x \wedge \eta \in E$ for all $x \in \mathcal{H}$. Thus Φ_{η} is dual dense for all $x \in \mathcal{H}$. Thus $\Phi_{\mathcal{H}}(\eta) = \Phi_{\eta} \in \mathcal{E}(\mathcal{H})$. Hence $\eta \in \text{Ker}^E(\Phi_{\mathcal{H}})$. \square

Theorem 3.11. *If each E -ideal is an R -ideal, then every pair of prime E -ideals are incomparable.*

Proof. Suppose every E -ideal is an R -ideal. If there are two distinct prime E -ideals \mathcal{U}, \mathcal{W} with $\mathcal{U} \subset \mathcal{W}$. Now choose $\tau \in \mathcal{W} \setminus \mathcal{U}$. For each $x \in (\mathcal{W}, E)$, we have $x \wedge \tau \in E \subseteq \mathcal{U}$. As $\tau \notin \mathcal{U}$ and \mathcal{U} is prime, we obtain $x \in \mathcal{U}$. Which gives $(\mathcal{W}, E) \subseteq \mathcal{U} \subseteq \mathcal{W}$. Therefore $(\mathcal{W}, E) = \mathcal{W} \cap (\mathcal{W}, E) = E$. As E -ideals are all R -ideals, \mathcal{W} is also an R -ideal. Therefore $\mathcal{W} = ((\mathcal{W}, E), E) = (E, E) = \mathcal{L}$, it leads a contradiction. It follows that every two prime E -ideals are not comparable. \square

The next theorem provides a necessary condition for an ADL to become relatively complemented in terms of R -ideals.

Theorem 3.12. *Consider an ADL \mathcal{L} , where each principal ideal is an E -ideal. Then \mathcal{L} is relatively complemented if each E -ideal is an R -ideal.*

Proof. Let \mathcal{L} be an ADL, where each principal ideal is an E -ideal. Additionally, suppose that every E -ideal in \mathcal{L} is an R -ideal and that \mathcal{L} is not relatively complemented. Consequently, there exists $a, b, c \in \mathcal{L}$ such that $b < c < a$ and c does not have a complement within the interval $[b, a]$. Consider the set $G = \{x \in \mathcal{L} \mid (c \vee x) \wedge a = a\}$. It is to be verified easily that G forms a filter in \mathcal{L} . Next we define the ideal E as $G \vee [c]$. Suppose $b \in E$. Then, b can be expressed as $b = c \wedge j$ for some $j \in G$. Now

$$\begin{aligned} b &= a \wedge b \\ &= a \wedge (c \wedge j) \\ &= c \wedge (a \wedge j) \end{aligned} \quad \longrightarrow (1)$$

Again we get the following:

$$\begin{aligned} c \vee (j \wedge a) &= (c \vee j) \wedge (c \vee a) \\ &= (c \vee j) \wedge a \end{aligned}$$

$$= a \quad \text{because } j \in G \quad \longrightarrow (2)$$

As we can see from(1) and (2), we have $j \wedge a$ is a relative complement of c in $[b, a]$, which leads a contradiction. Thus $b \notin E$. Hence $(b) \cap E = \emptyset$. Since (b) is an E -ideal, by Proposition 2.5(3), there is a prime E -ideal \mathcal{U} of \mathcal{L} such that $(b) \subseteq \mathcal{U}$, $\mathcal{U} \cap E = \emptyset$. Hence $\mathcal{U} \cap G = \emptyset$. Now

$$\begin{aligned} E \cap \mathcal{U} = \emptyset &\Rightarrow \{G \vee [c]\} \cap \mathcal{U} = \emptyset \\ &\Rightarrow G \cap \mathcal{U} = \emptyset, \mathcal{U} \cap [c] = \emptyset. \end{aligned}$$

Suppose $\mathcal{H} = (c] \vee \mathcal{U}$. we obtained easily that \mathcal{H} is an E -ideal of \mathcal{L} . If $a \in \mathcal{H}$, then

$$\begin{aligned} a \in (c] \vee \mathcal{U} &\Rightarrow b = c \vee \rho \text{ for some } \rho \in \mathcal{U} \\ &\Rightarrow \rho \in G. \end{aligned}$$

Thus, if $\rho \in \mathcal{U} \cap G$, it leads to a contradiction. Therefore, $a \notin \mathcal{H}$, which implies that $\mathcal{H} \cap [a) = \emptyset$. According to Proposition 2.5(3), there must exist a prime E -ideal \mathcal{W} such that $\mathcal{H} \subseteq \mathcal{W}$ and $[a) \cap \mathcal{W} = \emptyset$. This leads to $\mathcal{U} \subset \mathcal{H} \subseteq \mathcal{W}$, indicating that \mathcal{U} and \mathcal{W} are distinct prime E -ideals with $\mathcal{U} \subset \mathcal{W}$. Consequently, there exist two prime E -ideals that are comparable. By the above theorem, this implies that \mathcal{L} is relatively complemented. \square

4. λ -IDEALS OF ADLS

Here introduces λ -ideals in Almost Distributive Lattices, characterizing them through the use of R -ideals and congruences. Additionally, equivalent conditions are provided for an ADL to become a Boolean algebra.

Definition 4.1. An ideal \mathcal{H} of \mathcal{L} is said to be a λ -ideal if $x \in \mathcal{H}$ implies $((x, E), E) \subseteq \mathcal{H}$ for every $x \in \mathcal{L}$.

For every $x \in E$, it is obvious that $(x, E) = \mathcal{L}$ and which gives

$$((x, E), E) = E.$$

Hence E is a λ -ideal and additionally, it is the smallest λ -ideal in an ADL \mathcal{L} . Each R -ideal is a λ -ideal. For, let us consider an R -ideal \mathcal{H} . Let $x \in \mathcal{H}$. Therefore $((x, E), E) \subseteq ((\mathcal{H}, E), E) = \mathcal{H}$. Hence \mathcal{H} is a λ -ideal. But the converse is not true. Because $((\mathcal{H}, E), E) = (E, E) = \mathcal{L} \neq \mathcal{H}$, a λ -ideal \mathcal{H} having the property $(\mathcal{H}, E) = E$ is not an R -ideal.

Proposition 4.2. Each minimal prime E -ideal of an ADL is a λ -ideal.

Proof. For any ADL \mathcal{L} , let \mathcal{U} be its minimal prime E -ideal. Assume $x \in \mathcal{U}$. We obtain that, for some $y \notin \mathcal{U}$, $x \wedge y \in E$, since \mathcal{U} is minimum. Let $\delta \in ((x, E), E)$. It follows that $(x, E) \subseteq (\delta, E)$. From this, $y \in (\delta, E)$. Consequently, since $y \notin \mathcal{U}$, $\delta \in ((\delta, E), E) \subseteq (y, E) \subseteq \mathcal{U}$. Which gives $(x, E), E \subseteq \mathcal{U}$. Thus \mathcal{U} is a λ -ideal of \mathcal{L} . \square

Definition 4.3. For any ideal \mathcal{H} of an ADL \mathcal{L} , specify an extension of \mathcal{H} as follows:

$$\mathcal{H}^\epsilon = \{x \in \mathcal{L} \mid (a, E) \subseteq (x, E) \text{ for some } a \in \mathcal{H}\}.$$

It is obvious that $E^\epsilon = E$ and $E \subseteq \mathcal{H}^\epsilon$. Additionally, the definition mentioned above directly leads to the following result.

Lemma 4.4. *The following properties hold to every two ideals \mathcal{H}, G of \mathcal{L} :*

- (1) $\mathcal{H} \subseteq G \Rightarrow \mathcal{H}^\epsilon \subseteq G^\epsilon$;
- (2) $(\mathcal{H} \cap G)^\epsilon = \mathcal{H}^\epsilon \cap G^\epsilon$;
- (3) $(\mathcal{H}^\epsilon)^\epsilon = \mathcal{H}^\epsilon$.

Proposition 4.5. *The smallest λ -ideal containing \mathcal{H} is \mathcal{H}^ϵ for each ideal \mathcal{H} of an ADL \mathcal{L} .*

Proof. Clearly, $E \subseteq \mathcal{H}^\epsilon$. Let $x, y \in \mathcal{H}^\epsilon$. Therefore there is $a, b \in \mathcal{H}$ which satisfies $(b, E) \subseteq (y, E)$ and $(a, E) \subseteq (x, E)$. It gives

$$(a \vee b, E) = (a, E) \cap (b, E) \subseteq (x, E) \cap (y, E) = (x \vee y, E).$$

Hence $x \vee y \in \mathcal{H}^\epsilon$. Now, let $x \in \mathcal{H}^\epsilon$. Then there exists $a \in \mathcal{H}$ such that $(a, E) \subseteq (x, E)$. Let $y \in \mathcal{L}$. Now $(a, E) \subseteq (x, E) \subseteq (x \wedge y, E)$. Therefore \mathcal{H}^ϵ is an ideal of \mathcal{L} . Obviously, $\mathcal{H} \subseteq \mathcal{H}^\epsilon$. Let $x \in \mathcal{H}^\epsilon$ and $\delta \in ((x, E), E)$. Then there exists $a \in \mathcal{H}$ such that $(a, E) \subseteq (x, E) \subseteq (\delta, E)$. Therefore

$$((x, E), E) \subseteq \mathcal{H}^\epsilon$$

and hence \mathcal{H}^ϵ is a λ -ideal of \mathcal{L} . Suppose G is a λ -ideal of \mathcal{L} with $\mathcal{H} \subseteq G$. Let $x \in \mathcal{H}^\epsilon$. Therefore $(a, E) \subseteq (x, E)$ for some $a \in \mathcal{H} \subseteq G$. As G is a λ -ideal of \mathcal{L} , we obtain $x \in ((x, E), E) \subseteq ((a, E), E) \subseteq G$. Therefore $\mathcal{H}^\epsilon \subseteq G$. Thus \mathcal{H}^ϵ is the smallest λ -ideal of \mathcal{L} containing \mathcal{H} . \square

The following result makes it clear that $\mathcal{H} = \mathcal{H}^\epsilon$ is the only condition under which an ideal \mathcal{H} is a λ -ideal. Therefore, in \mathcal{L} , E is the smallest λ -ideal. It is evident from the preceding two results that the class $\mathcal{G}^\lambda(\mathcal{L})$ of all λ -ideals of an ADL \mathcal{L} forms a complete distributive lattice with respect the operations given by

$$\mathcal{H} \wedge G = \mathcal{H} \cap G \quad \text{and} \quad \mathcal{H} \vee G = (\mathcal{H} \vee G)^\epsilon$$

for any $\mathcal{H}, G \in \mathcal{G}^\lambda(\mathcal{L})$ in which the smallest element is E .

Theorem 4.6. *Assume that any ideal of \mathcal{L} is \mathcal{H} . Define a binary relation $\Theta(\mathcal{H})$ on \mathcal{L} , for all $x, y \in \mathcal{L}$, it is as follows:*

$$(x, y) \in \Theta(\mathcal{H}) \Leftrightarrow \{E \vee (x)\} \cap (a, E) = \{E \vee (y)\} \cap (a, E)$$

for some $a \in \mathcal{H}$. Then $\Theta(\mathcal{H})$ is a congruence on \mathcal{L} .

Proof. It is obvious that on \mathcal{L} , $\Theta(\mathcal{H})$ is an equivalence relation. Let us assume $(x, y) \in \Theta(\mathcal{H})$. Then, there is $a \in \mathcal{H}$ such that

$$\{E \vee (x)\} \cap (a, E) = \{E \vee (y)\} \cap (a, E).$$

For every $c \in \mathcal{L}$, we get

$$\begin{aligned} \{E \vee (x \vee c)\} \cap (a, E) &= \{E \vee (x)\} \vee \{E \vee (c)\} \cap (a, E) \\ &= \{E \vee (y)\} \vee \{E \vee (c)\} \cap (a, E) \\ &= \{E \vee (y \vee c)\} \cap (a, E) \end{aligned}$$

Therefore $(x \vee c, y \vee c) \in \Theta(\mathcal{H})$. Again,

$$\begin{aligned} \{E \vee (x \wedge c)\} \cap (a, E) &= \{E \vee \{(x] \cap (c)\}\} \cap (a, E) \\ &= \{\{E \vee (x)\} \cap (a, E)\} \cap \{(c] \cap (a, E)\} \\ &= \{\{E \vee (y)\} \cap (a, E)\} \cap \{(c] \cap (a, E)\} \\ &= \{E \vee (y \wedge c)\} \cap (a, E) \end{aligned}$$

Hence $(x \wedge c, y \wedge c) \in \Theta(\mathcal{H})$. Therefore $\Theta(\mathcal{H})$ is a congruence on \mathcal{L} . □

Lemma 4.7. *Let \mathcal{L} be an ADL. For any $x \in \mathcal{L}$, the following properties hold:*

- (1) $\{E \vee (x)\} \cap (x, E) = E$;
- (2) $\{E \vee ((x], E), E)\} = ((x, E), E)$.

Proof. (1)

$$\begin{aligned} \{E \vee (x)\} \cap (x, E) &= \{E \vee (x)\} \cap \{\mathcal{L} \cap (x, E)\} \\ &= \{E \vee (x)\} \cap \{(E, E) \cap (x, E)\} \\ &= \{E \vee (x)\} \cap (E \vee (x), E) \\ &= E \end{aligned}$$

as $E \vee (x]$ is an E -ideal.

(2)

$$(\{\{E \vee (x)\}, E), E) = (\{(E, E) \cap (x, E)\}, E)$$

$$\begin{aligned}
&= (\{\mathcal{L} \cap (x, E)\}, E) \\
&= ((x, E), E).
\end{aligned}$$

□

Proposition 4.8. Define the dual dense-kernel $Ker^E\Theta(\mathcal{H})$ of the congruence $\Theta(\mathcal{H})$ for every ideal \mathcal{H} of an ADL \mathcal{L} as follows:

$$Ker^E\Theta(\mathcal{H}) = \{x \in \mathcal{L} \mid \{E \vee (x)\} \cap (a, E) = E \text{ for some } a \in \mathcal{H}\}.$$

Then $Ker^E\Theta(\mathcal{H})$ is an ideal in \mathcal{L} such that $\mathcal{H} \subseteq Ker^E\Theta(\mathcal{H})$.

Proof. Clearly $E \subseteq Ker^E\Theta(\mathcal{H})$. Let $x, y \in Ker^E\Theta(\mathcal{H})$. Then

$$\{E \vee (x)\} \cap (a, E) = \{E \vee (y)\} \cap (b, E) = E$$

for some $a, b \in \mathcal{H}$. Now

$$\begin{aligned}
\{E \vee (x \vee y)\} \cap ((a \vee b), E) &= \{E \vee (x) \vee E \vee (y)\} \cap (a, E) \cap (b, E) \\
&= \{(E \vee (x)) \cap (a, E) \cap (b, E)\} \\
&\quad \vee \{(E \vee (y)) \cap (a, E) \cap (b, E)\} \\
&= \{E \cap (b, E)\} \vee \{E \cap (a, E)\} \\
&= E
\end{aligned}$$

Hence $x \vee y \in Ker^E\Theta(\mathcal{H})$. Again, let $x \in Ker^E\Theta(\mathcal{H})$. Then there exists $a \in \mathcal{H}$ such that $\{E \vee (x)\} \cap (a, E) = E$. Let $y \in \mathcal{L}$. Then

$$\{E \vee (x \wedge y)\} \cap (a, E) \subseteq \{E \vee (x)\} \cap (a, E) = E.$$

Hence $x \wedge y \in Ker^E\Theta(\mathcal{H})$. Therefore $Ker^E\Theta(\mathcal{H})$ is an ideal. Now, let $x \in \mathcal{H}$. From the above lemma, we get $x \in Ker^E\Theta(\mathcal{H})$. Therefore $\mathcal{H} \subseteq Ker^E\Theta(\mathcal{H})$. □

In the following, the λ -ideals are characterized.

Theorem 4.9. Let \mathcal{H} be an ideal in an ADL \mathcal{L} . Then the following are equivalent:

- (1) \mathcal{H} is a λ -ideal;
- (2) $\mathcal{H} = Ker^E\Theta(\mathcal{H})$;
- (3) for $x, y \in \mathcal{L}$, $(x, E) = (y, E)$ and $x \in \mathcal{H}$ imply that $y \in \mathcal{H}$;
- (4) $x \in \mathcal{H} \Leftrightarrow x \in ((a, E), E)$ for some $a \in \mathcal{H}$.

Proof. (1) \Rightarrow (2) Assume (1). Obviously, $\mathcal{H} \subseteq Ker^E\Theta(\mathcal{H})$. Let $x \in Ker^E\Theta(\mathcal{H})$. Then $\{E \vee (x)\} \cap (a, E) = E$ for some $a \in \mathcal{H}$. It follows that

$$x \in E \vee (x) \subseteq ((a, E), E) \subseteq \mathcal{H}$$

(because of \mathcal{H} is a λ -ideal). Therefore $Ker^E\Theta(\mathcal{H}) \subseteq \mathcal{H}$. Hence

$$\mathcal{H} = Ker^E\Theta(\mathcal{H}).$$

(2) \Rightarrow (3) Assume (2). Let $a, b \in \mathcal{L}$ such that $(a, E) = (b, E)$. If $a \in \mathcal{H}$, then there is $\delta \in \mathcal{H}$ such that $\{E \vee [a]\} \cap (\delta, E) = E$. Next, we get

$$\begin{aligned} \{E \vee [a]\} \cap (\delta, E) = E &\Rightarrow ((\{E \vee [a]\}, E), E) \cap (\delta, E) = ((E, E), E) = E \\ &\Rightarrow ((a, E), E) \cap (\delta, E) = E \\ &\Rightarrow ((b, E), E) \cap (\delta, E) = E \\ &\Rightarrow \{E \vee [b]\} \cap (\delta, E) \\ &\quad \subseteq ((\{E \vee [b]\}, E), E) \cap (\delta, E) = E \\ &\Rightarrow b \in Ker^E\Theta(\mathcal{H}) = \mathcal{H} \end{aligned}$$

(3) \Rightarrow (4): Assume (3). Suppose $x \in \mathcal{H}$. It is evident that $x \in ((x, E), E)$. Once again let $x \in ((a, E), E)$ for some $a \in \mathcal{H}$. Thus,

$$((x, E), E) \subseteq ((a, E), E),$$

which leads that $((x, E), E) = ((x, E), E) \cap ((a, E), E) = (((x \wedge a), E), E)$. Thus $x \wedge a \in \mathcal{H}$ and $(x, E) = ((x \wedge a), E)$. Hence $x \in \mathcal{H}$, as by condition (3).

(4) \Rightarrow (1): Assume (4). Suppose $x \in \mathcal{H}$. There is $a \in \mathcal{H}$ such that $x \in ((a, E), E)$. $\delta \in ((x, E), E)$. So, for this $a \in \mathcal{H}$, we obtain that $\delta \in ((x, E), E) \subseteq ((a, E), E)$. Thus, $\delta \in \mathcal{H}$ is obtained by condition (4). Hence $((x, E), E) \subseteq \mathcal{H}$. Thus \mathcal{H} is a λ -ideal of \mathcal{L} . \square

The following theorem is a direct consequence.

Theorem 4.10. *There are some equivalent statements in an ADL \mathcal{L} :*

- (1) \mathcal{L} is a Boolean algebra;
- (2) each ideal is an E -ideal;
- (3) each ideal is a λ -ideal;
- (4) each prime ideal is a λ -ideal.

Every minimal prime E -ideal is a prime λ -ideal, as previously noted. It is not true in reverse. However, a sufficient criterion for a prime λ -ideal to become a minimal prime E -ideal can be established.

Proposition 4.11. *Every prime λ -ideal is a minimal prime E -ideal if each $(x, E), x \in \mathcal{L}$ is a principal ideal.*

Proof. Assume that $(x, E), x \in \mathcal{L}$ is a principal ideal and \mathcal{U} is any prime λ -ideal of \mathcal{L} . Let $x \in \mathcal{U}$. Then by the hypothesis, there exists $y \in \mathcal{L}$ such that $(x, E) = (y]$. Therefore $x \wedge y \in E$. Hence

$$((x \vee y), E) = (x, E) \cap (y, E) = (x, E) \cap ((x, E), E) = E.$$

Thus $x \vee y \notin \mathcal{U}$, it gives that $y \notin \mathcal{U}$. Hence \mathcal{U} is a minimal prime E -ideal. \square

Theorem 4.12. *An ADL \mathcal{L} has the following equivalent conditions:*

- (1) every λ -ideal is a principal ideal;
- (2) each (x, E) is a principal ideal and every minimal prime E -ideal is dual non-condensed;
- (3) every prime λ -ideal is a principal ideal.

Proof. (1) \Rightarrow (2) It is sufficient to prove that any minimal prime E -ideal is non-dual condensed since each (x, E) is a λ -ideal. Consider a minimal prime E -ideal, \mathcal{U} . It is observed by Proposition 4.2, that \mathcal{U} is a λ -ideal. By our assumption, we get that $\mathcal{U} = (a]$ for some $a \in \mathcal{L}$. Let us assume that $(\mathcal{U}, E) = E$. Therefore $\mathcal{L} = ((a, E), E) \subseteq \mathcal{U}$, which is contradictory, since $(a, E) = E$. Thus $(\mathcal{U}, E) \neq E$.

(2) \Rightarrow (3) Assume condition (2). Let \mathcal{U} be a prime λ -ideal of \mathcal{L} . Given that each (x, E) is a principal ideal, by the earlier proposition, \mathcal{U} must be a minimal prime E -ideal with the property that $(\mathcal{U}, E) \neq E$. Therefore, there exists some $x \notin E$ such that $x \in (\mathcal{U}, E)$. Consequently, we have $\mathcal{U} \subseteq ((\mathcal{U}, E), E) \subseteq (x, E)$. Conversely, consider any $\delta \in (x, E)$. This implies that $t \wedge x \in E \subseteq \mathcal{U}$. Given that \mathcal{U} is prime and $x \notin (x, E) = \mathcal{U}$, which gives that $\delta \in \mathcal{U}$. Hence, $\mathcal{U} = (x, E)$. Therefore, according to condition (2), $\mathcal{U} = (x, E)$ is a principal ideal.

(3) \Rightarrow (1) Consider each prime λ -ideal is a principle ideal. Consider \mathcal{H} is a λ -ideal of \mathcal{L} . Let \mathcal{H} not be principle. Consider the collection

$$\Sigma = \{G \mid G \text{ is a } \lambda\text{-ideal which is not a principal ideal} \}.$$

obviously $\mathcal{H} \in \Sigma$. Assume for some $i \in \Delta$ that $a \in G_i$. Thus, for some $i \in \Delta$, $(a] \subseteq G_i$. Alternatively, $G_i \subseteq \bigcup G_i = (a]$. Thus, for some $i \in \Delta$, $G_i = (a]$, which is contradictory. Therefore, an upper bound for $\{G_i\}_{i \in \Delta}$ in Σ is $\bigcup G_i$. Because of Zorn's Lemma, let \mathcal{Y} be a maximal element of Σ containing \mathcal{H} . Assume that $x \notin \mathcal{Y}$, $y \notin \mathcal{Y}$. Consequently we obtained $\mathcal{Y} \subset \{\mathcal{Y} \vee (x)\}^\epsilon$ and $\mathcal{Y} \subset \{\mathcal{Y} \vee (y)\}^\epsilon$. Therefore, for some $b, c \in \mathcal{L}$, $\{\mathcal{Y} \vee (x)\}^\epsilon = (b]$ and $\{\mathcal{Y} \vee (y)\}^\epsilon = (c]$. Thus, $\{\mathcal{Y} \vee (x \wedge y)\}^\epsilon = \{\mathcal{Y} \vee (x)\}^\epsilon \cap \{\mathcal{Y} \vee (y)\}^\epsilon = (b] \cap (c] = (b \wedge c]$. Suppose $x \wedge y \in \mathcal{Y}$. Then $\mathcal{Y} = \mathcal{Y}^\epsilon = (b \wedge c]$, which is in conflict with (3). Hence \mathcal{H} a principal ideal. \square

5. TOPOLOGICAL CHARACTERIZATION OF THE SPACE OF PRIME λ -IDEALS

In this section, we study the topological properties of the set all prime λ -ideals in an Almost Distributive Lattice and establish a collection of equivalent conditions for the space of all prime λ -ideals in an ADL forms a Hausdorff space.

In an ADL \mathcal{L} , represent the class of all λ -ideals of \mathcal{L} by $Spec^\lambda(\mathcal{L})$.

Theorem 5.1. *Given a λ -ideal \mathcal{H} , let \mathcal{F} be a filter of \mathcal{L} with $\mathcal{H} \cap \mathcal{F} = \phi$. Then there is $\mathcal{U} \in Spec^\lambda(\mathcal{L})$ such that $\mathcal{H} \subseteq \mathcal{U}$, $\mathcal{U} \cap \mathcal{F} = \phi$.*

Proof. Take

$$\mathfrak{F} = \{G \mid G \text{ is an } \lambda\text{-ideal, } G \cap \mathcal{F} = \phi \text{ and } \mathcal{H} \subseteq G\}.$$

It is clear that $\mathcal{H} \in \mathfrak{F}$, and \mathfrak{F} achieves Zorn's Lemma's hypothesis. Hence choose a maximal element \mathcal{U} in \mathfrak{F} . Let $x, y \in \mathcal{L}$ with $x \notin \mathcal{U}$, $y \notin \mathcal{U}$. Then $\mathcal{U} \subset \mathcal{U} \vee (y)^\epsilon = \{\mathcal{U} \vee (x)\}^\epsilon$, $\mathcal{U} \subset \mathcal{U} \vee (y)^\epsilon = \{\mathcal{U} \vee (y)\}^\epsilon$. As \mathcal{U} is maximal, $\{\mathcal{U} \vee (x)\}^\epsilon \cap \mathcal{F} \neq \phi$, $\mathcal{F} \cap \{\mathcal{U} \vee (y)\}^\epsilon \neq \phi$. Choose $a \in \{\mathcal{U} \vee (x)\}^\epsilon \cap \mathcal{F}$ and $b \in \{\mathcal{U} \vee (y)\}^\epsilon \cap \mathcal{F}$. It follows that $a \wedge b \in \mathcal{F}$,

$$\begin{aligned} a \wedge b &\in \{\mathcal{U} \vee (x)\}^\epsilon \cap \{\mathcal{U} \vee (y)\}^\epsilon = \{\{\mathcal{U} \vee (x)\} \cap \{\mathcal{U} \vee (y)\}\}^\epsilon \\ &= \{\mathcal{U} \vee (x \wedge y)\}^\epsilon. \end{aligned}$$

If $x \wedge y \in \mathcal{U}$, then $a \wedge b \in \mathcal{U}^\epsilon = \mathcal{U}$ and $a \wedge b \in \mathcal{F}$. Which gives that $\mathcal{F} \cap \mathcal{U} \neq \phi$. It leads a contradiction. Therefore $x \wedge y \notin \mathcal{U}$. Thus \mathcal{U} is prime. □

Corollary 5.2. *Let \mathcal{H} be a λ -ideal of \mathcal{L} and $x \notin \mathcal{H}$. Then there is $\mathcal{U} \in Spec^\lambda(\mathcal{L})$ such that $x \notin \mathcal{U}$ and $\mathcal{H} \subseteq \mathcal{U}$.*

Corollary 5.3. *For every λ -ideal \mathcal{H} , $\mathcal{H} = \cap\{\mathcal{U} \mid \mathcal{U} \in Spec^\lambda(\mathcal{L}) \text{ and } \mathcal{H} \subseteq \mathcal{U}\}$.*

Corollary 5.4. *E is equal to the intersection of all the members of $Spec^\lambda(\mathcal{L})$.*

For every $\mathcal{V} \subseteq \mathcal{L}$, $\mathcal{K}'(\mathcal{V}) = \{\mathcal{U} \in Spec^\lambda(\mathcal{L}) \mid \mathcal{V} \not\subseteq \mathcal{U}\}$. In specific, for $x \in \mathcal{L}$, $\mathcal{K}'(x) = \{\mathcal{U} \in Spec^\lambda(\mathcal{L}) \mid x \notin \mathcal{U}\}$.

Lemma 5.5. *Every $x, y \in \mathcal{L}$ gives us*

- (i) $Spec^\lambda(\mathcal{L}) = \bigcup_{x \in \mathcal{L}} \mathcal{K}'(x)$;
- (ii) $\mathcal{K}'(x \wedge y) = \mathcal{K}'(x) \cap \mathcal{K}'(y)$;
- (iii) $\mathcal{K}'(x \vee y) = \mathcal{K}'(x) \cup \mathcal{K}'(y)$;
- (iv) $\mathcal{K}'(x) = \phi \Leftrightarrow x \in E$;
- (v) $\mathcal{K}'(x) = Spec^\lambda(\mathcal{L}) \Leftrightarrow x \in \mathcal{M}_{Max.elts}$.

It is simple to see from the above lemma that a topology on $\text{Spec}^\lambda(\mathcal{L})$ has as its basis $\{\mathcal{K}'(x)\}$, $x \in \mathcal{L}$. This topology is referred to as a hull-kernel topology and it produces the following topological property:

Theorem 5.6. *In $\text{Spec}^\lambda(\mathcal{L})$, the basis $\{\mathcal{K}'(x) \mid x \in \mathcal{L}\}$ is the set of all compact open sets.*

Proof. Let $x \in \mathcal{L}$ and let \mathcal{V} be a subset of \mathcal{L} with $\mathcal{K}'(x) \subseteq \bigcup_{y \in \mathcal{V}} \mathcal{K}'(y)$, forming an open cover of $\mathcal{K}'(x)$. Let \mathcal{H} represent the ideal generated by \mathcal{V} . Suppose $x \notin \mathcal{H}^\epsilon$. According to Corollary 5.2, there is a prime λ -ideal \mathcal{U} such that $\mathcal{H}^\epsilon \subseteq \mathcal{U}$ and $x \notin \mathcal{U}$. Hence, $\mathcal{U} \in \mathcal{K}'(x) \subseteq \bigcup_{y \in \mathcal{V}} \mathcal{K}'(y)$. This implies there exists some $y \in \mathcal{V}$ such that $y \notin \mathcal{U}$, contradicting $\mathcal{H} \subseteq \mathcal{H}^\epsilon \subseteq \mathcal{U}$. Therefore, $x \in \mathcal{H}^\epsilon$. Hence, there exists $a \in \mathcal{H}$ such that $x \in ((a, E), E)$. Since \mathcal{H} is generated by \mathcal{V} , we can express $a = a_1 \vee a_2 \vee \dots \vee a_n$ for some $a_1, a_2, \dots, a_n \in \mathcal{V}$. Therefore, $x \in ((a, E), E) = ((a_1 \vee a_2 \vee \dots \vee a_n, E), E)$. Clearly, $\mathcal{K}'(x) \subseteq \bigcup_{i=1}^n \mathcal{K}'(a_i)$, which is a finite subcover of $\mathcal{K}'(x)$. Thus, $\mathcal{K}'(x)$ is compact in $\text{Spec}^\lambda(\mathcal{L})$.

To prove that every compact open subset of $\text{Spec}^\lambda(\mathcal{L})$ is of the form $\mathcal{K}'(x)$ for some $x \in \mathcal{L}$, let \mathcal{G} be a compact open subset of $\text{Spec}^\lambda(\mathcal{L})$. As \mathcal{G} is open, we can write $\mathcal{G} = \bigcup_{a \in \mathcal{V}} \mathcal{K}'(a)$ for some $\mathcal{V} \subseteq \mathcal{L}$. Given that \mathcal{G} is compact, there exist $a_1, a_2, \dots, a_n \in \mathcal{V}$ such that

$$\mathcal{G} = \bigcup_{i=1}^n \mathcal{K}'(a_i) = \mathcal{K}'\left(\bigvee_{i=1}^n a_i\right)$$

Thus, $\mathcal{G} = \mathcal{K}'(x)$ for some $x \in \mathcal{L}$. □

A maximal λ -ideal refers to the maximal element within the collection of all proper λ -ideals of an ADL. Since the set of all λ -ideals constitutes a distributive lattice, it follows that every maximal λ -ideal is also a prime λ -ideal. In the subsequent discussion, we derive a set of equivalent conditions for a prime λ -ideal to qualify as a minimal prime E -ideal.

Theorem 5.7. *The following assertions are equivalent in an ADL \mathcal{L} :*

- (1) every prime λ -ideal is a minimal prime E -ideal;
- (2) $\text{Spec}^\lambda(\mathcal{L})$ is T_1 -space;
- (3) every prime λ -ideal is maximal;
- (4) every prime λ -ideal is minimal;
- (5) for each $x \in \mathcal{L}$, $(x, E) \vee ((x, E), E) = \mathcal{L}$;
- (6) $\text{Spec}^\lambda(\mathcal{L})$ is a Hausdorff space;
- (7) for any $x, y \in \mathcal{L}$, there exists $z \in \mathcal{L}$ such that $x \wedge z \in E$ and

$$\mathcal{K}'(y) \cap \{Spec^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)\} = \mathcal{K}'(y \wedge z).$$

Proof. (1) \Rightarrow (2) Assume (1). Let \mathcal{U} and \mathcal{W} be two prime λ -ideals of \mathcal{L} with $\mathcal{U} \neq \mathcal{W}$. Since \mathcal{U} and \mathcal{W} are minimal, it follows that $\mathcal{W} \not\subseteq \mathcal{U}$ and $\mathcal{U} \not\subseteq \mathcal{W}$. Choose $x \in \mathcal{U} \setminus \mathcal{W}$ and $y \in \mathcal{W} \setminus \mathcal{U}$. Consequently, $\mathcal{W} \in \mathcal{K}'(x) \setminus \mathcal{K}'(y)$ and $\mathcal{U} \in \mathcal{K}'(y) \setminus \mathcal{K}'(x)$. Thus, $Spec^\lambda(\mathcal{L})$ is T_1 -space.

(2) \Rightarrow (3) Suppose that $Spec^\lambda(\mathcal{L})$ is T_1 -space. Let \mathcal{U} be a prime λ -ideal of \mathcal{L} . Assume there exists a maximal λ -ideal \mathcal{W} in \mathcal{L} such that $\mathcal{U} \subset \mathcal{W}$. Since $Spec^\lambda(\mathcal{L})$ is T_1 -space, there must be two basic open sets, $\mathcal{K}'(x)$ and $\mathcal{K}'(y)$, such that $\mathcal{W} \in \mathcal{K}'(x) \setminus \mathcal{K}'(y)$ and $\mathcal{U} \in \mathcal{K}'(y) \setminus \mathcal{K}'(x)$. However, since $x \in \mathcal{U} \subset \mathcal{W}$, we get $\mathcal{W} \notin \mathcal{K}'(x)$, which leads to a contradiction. Therefore, \mathcal{U} must be a maximal λ -ideal.

(3) \Rightarrow (4) It's obvious.

(4) \Rightarrow (5) Assume (4). Consequently, each prime λ -ideal is a minimal prime E -ideal. If there exists some $x \in \mathcal{L}$ such that $(x, E) \vee ((x, E), E) \neq \mathcal{L}$, there must exist a prime λ -ideal \mathcal{U} such that

$$((x, E), E) \vee ((x, E), E) \subseteq \mathcal{U}.$$

Hence, $x \in ((x, E), E) \subseteq \mathcal{U}$. Since \mathcal{U} is a minimal prime E -ideal and $((x, E), E) \subseteq \mathcal{U}$, it follows that $x \notin \mathcal{U}$, which is a contradiction. Therefore, $(x, E) \vee ((x, E), E) = \mathcal{L}$.

(5) \Rightarrow (6) Assume condition (5). Let \mathcal{U} and \mathcal{W} be two distinct elements of $Spec^\lambda(\mathcal{L})$. Choose $x \in \mathcal{U}$ such that $x \notin \mathcal{W}$. By hypothesis, we have $(x, E) \vee ((x, E), E) = \mathcal{L}$. Therefore, there exists a maximal element $m \in (x, E) \vee ((x, E), E) = \{(x, E) \vee ((x, E), E)\}^e$. Hence, $(a, E) \subseteq (m, E)$ for some $a \in (x, E) \vee ((x, E), E)$. Thus, $a = \eta \vee \mu$ for some $\eta \in (x, E)$ and $\mu \in ((x, E), E)$. Therefore, $\eta \wedge x \in E$. Suppose $\eta \in \mathcal{U}$. As \mathcal{U} is a λ -ideal, we have $((\eta, E), E) \subseteq \mathcal{U}$. Now, $(\eta, E) \cap (\mu, E) = ((\eta \vee \mu), E) = (a, E) = E$ implies that $(\mu, E) \subseteq ((\eta, E), E) \subseteq \mathcal{U}$. Furthermore, $\mu \in ((x, E), E)$ implies $((x, E), E) \subseteq (\mu, E) \subseteq \mathcal{U}$. As $x \in \mathcal{U}$, we obtain $\mathcal{L} = (x, E) \vee ((x, E), E) \subseteq \mathcal{U}$, which leads to a contradiction. So that, $\eta \notin \mathcal{U}$. Therefore $\mathcal{U} \in \mathcal{K}'(\eta)$. Similarly, $\mathcal{W} \in \mathcal{K}'(x)$. Since $x \wedge \eta \in E$, we have

$$\mathcal{K}'(x) \cap \mathcal{K}'(\eta) = \mathcal{K}'(x \wedge \eta) = \emptyset.$$

Therefore, $Spec^\lambda(\mathcal{L})$ is Hausdorff.

(6) \Rightarrow (7) Assume that $Spec^\lambda(\mathcal{L})$ is Hausdorff. This implies that $\mathcal{K}'(a)$ is a compact subset of $Spec^\lambda(\mathcal{L})$ for each $a \in \mathcal{L}$. Consequently, $\mathcal{K}'(a)$ is a clopen (both closed and open) subset of $Spec^\lambda(\mathcal{L})$. Suppose $x, y \in \mathcal{L}$ with $x \neq y$. Then $\mathcal{K}'(y) \cap (Spec^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x))$ is a compact subset of the compact space

$\mathcal{K}'(y)$. As $\mathcal{K}'(y)$ is open in $\text{Spec}^\lambda(\mathcal{L})$, $\mathcal{K}'(y) \cap (\text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x))$ is a compact open subset of $\text{Spec}^\lambda(\mathcal{L})$. According to Theorem 5.6, there is $z \in \mathcal{L}$ such that

$$\mathcal{K}'(z) = \mathcal{K}'(y) \cap \{\text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)\}$$

Therefore, $\mathcal{K}'(y) \cap (\text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)) = \mathcal{K}'(y) \cap \mathcal{K}'(z) = \mathcal{K}'(y \wedge z)$. Additionally, $\mathcal{K}'(x \wedge z) = \mathcal{K}'(x) \cap \mathcal{K}'(z) = \emptyset$ because $\mathcal{K}'(x) \cap \mathcal{K}'(y) = \emptyset$ due to $x \neq y$ and $\mathcal{K}'(z) \subseteq \text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)$. Therefore, $x \wedge z \in E$.

(7) \Rightarrow (1) Assume (7). Let \mathcal{U} be a prime λ -ideal of \mathcal{L} . Choose $x, y \in \mathcal{L}$ such that $x \in \mathcal{U}$ and $y \notin \mathcal{U}$. By (7), there is $z \in \mathcal{L}$ such that $x \wedge z \in E$ and

$$\mathcal{K}'(y) \cap \{\text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)\} = \mathcal{K}'(y \wedge z)$$

Thus, $\mathcal{U} \in \mathcal{K}'(y) \cap (\text{Spec}^\lambda(\mathcal{L}) \setminus \mathcal{K}'(x)) = \mathcal{K}'(y \wedge z)$. If $z \in \mathcal{U}$, then $y \wedge z \in \mathcal{U}$, which contradicts $\mathcal{U} \in \mathcal{K}'(y \wedge z)$. Hence, $z \notin \mathcal{U}$. Therefore, for each $x \in \mathcal{U}$, there exists $z \notin \mathcal{U}$ such that $x \wedge z \in E$. This shows that \mathcal{U} is a minimal prime E -ideal of \mathcal{L} . \square

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R -IDEALS OF ALMOST DISTRIBUTIVE LATTICES

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R -ایده‌آل‌های شبکه‌های تقریباً توزیع‌پذیر

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گروه ریاضی، کالج مهندسی باپاتلا، باپاتلا، آندرا پرادش، هند

در این مقاله، دو مفهوم جدید در شبکه‌های تقریباً توزیع‌پذیر (ADL) با نام‌های R -ایده‌آل‌ها و λ -ایده‌آل‌ها معرفی شده‌اند. مجموعه‌ای از شرایط شناسایی شده است که همگی معادل با این هستند که یک E -ایده‌آل به یک R -ایده‌آل تبدیل شود. همچنین نشان داده شده است که برای هر E -ایده‌آل، یک هم‌ریختی با هسته‌ی چگال دوگان وجود دارد که یک R -ایده‌آل نیز می‌باشد. به علاوه، مشخصه‌سازی λ -ایده‌آل‌ها بر حسب R -ایده‌آل‌ها و هم‌نهشتی‌ها ارائه شده است. همچنین شرایط معادل مختلفی به دست آمده‌اند که نشان می‌دهند فضای تمامی λ -ایده‌آل‌های اول، یک فضای هاسدورف است.

کلمات کلیدی: R -ایده‌آل، E -ایده‌آل اول مینیمال، λ -ایده‌آل، عنصر چگال دوگان، شبکه تقریباً توزیع‌پذیر دارای متمم نسبی، فضای هاسدورف.