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# THE BANASCHEWSKI-SIOEN NUCLEUS ON AN ALGEBRAIC FRAME

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Abstract. In their construction of the Stone-Čech compactification using only ring ideals (as opposed to $\ell$-ideals), Banaschewski and Sioen [6] define a certain nucleus on the coherent frame $\mathrm{RId}(A)$ of radical ideals of a commutative ring $A$. In this paper we extend this nucleus to any algebraic frame that has a dense compact element and in which the meet of two compact elements is compact. Of course, not every such algebraic frame is coherent, so the extension is indeed a genuine extension. We then study some properties of this nucleus which are not considered in [6].

## Introduction

Throughout the paper, the term "ring" means a commutative ring with nonzero identity. Ring homomorphisms preserve the identity, and the category of rings with their homomorphisms is denoted by **CRing**. Another category that is going to play a prominent role is **FIPFrm**, the objects of which are algebraic frames in which the meet of two compact elements is compact, and the morphisms are the frame homomorphisms which take compact elements to compact elements.

Let us call the nucleus defined in [6] the *Banaschewski-Sioen nucleus*. We will recall how it is defined at the appropriate time. Since, as shown in [3], coherent frames are (up to isomorphism) precisely the algebraic frames $\mathrm{RId}(A)$ for rings $A$, the Banaschewski-Sioen nucleus has thus actually been defined for all coherent frames. The chain

$$\Omega = \{0 < 1 < 2 < \cdots < \top\},$$

consisting of the non-negative integers with their usual order and given a top element, is an algebraic frame with a dense compact element (actually, each of the nonzero elements strictly below $\top$ is compact) and in which the meet of two elements is compact. Since $\top$ is not compact, $\Omega$ is an object in **FIPFrm** which is not a coherent frame. So, the nucleus we define here (taking a cue from how the Banaschewski-Sioen nucleus is defined in [6]) covers more objects than coherent frames.

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The outlay of the paper is as follows. In the Preliminaries we recall some basic results in algebraic frames and fix notation. Following that, in Section 2 we define the Banaschewski nucleus (which we denote by $\kappa\colon L \to L$) and show, as our first major result, that it is actually a dense nucleus and preserves directed joins (Theorem 2.4). We have said the nucleus we introduce extends that defined in [6]. That is actually not precise. We have only been able to prove (Theorem 2.6) that if $A$ is a reduced ring which satisfies Quentel's condition (C) [27], which requires every finitely generated ideal consisting entirely of zerodivisors to have nonzero annihilator, then our nucleus on $\mathrm{RId}(A)$ coincides with that of [6]. The reader may recall that condition (C) is styled "Property (A)" in Huckckaba's book [15]. Rings with this property abound, and they include rings of continuous functions (classical and localic), which are the rings considered in [6].

In Section 3, we study some properties of the Banaschewski-Sioen nucleus which are not considered in [6]. They include the result that the sublocale $\mathrm{Fix}(\kappa)$ induced by this nucleus if *fitted*, in the sense that it is an intersection of open sublocales of $L$ (Theorem 3.10). We identify the prime elements of the frame $\mathrm{Fix}(\kappa)$ as precisely those of $L$ which are above no dense compact element of $L$ (Theorem 3.12). A consequence of this is that the minimal primes of $\mathrm{Fix}(\kappa)$ are precisely the minimal primes of $L$ (Corollary 3.14).

Section 4 is motivated by the work of Martínez and Zenk [23]. In our case, this boils down to characterizing morphisms $h\colon L \to M$ in the category of algebraic frames with the FIP and possessing units, for which there is a morphism $\bar{h}\colon \kappa L \to \kappa M$ in **FIPFrm** such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\quad h \quad} & M \\ {\scriptstyle \kappa_L}\downarrow & & \downarrow{\scriptstyle \kappa_M} \\ \mathrm{Fix}(\kappa_L) & \xrightarrow{\quad \bar{h} \quad} & \mathrm{Fix}(\kappa_M) \end{array}$$

where the downward morphisms are the frame homomorphisms induced by the Banaschewski-Sioen nuclei on $L$ and $M$, respectively.

In [28], Simmons defines what he calls a curious nucleus on a frame $L$, and then uses the curious nucleus to define another nucleus on the frame $\mathfrak{J}L$ of ideals of $L$. In the penultimate of our paper we follow Simmons' style and define a nucleus on $\mathfrak{J}L$ which is (roughly speaking) based on the Banaschewski-Sioen nucleus on $L$. We denote it by $\vartheta\colon \mathfrak{J}L \to \mathfrak{J}L$. The major result is that if $L$ is a zipped algebraic frame (in the sense that every dense element is above a dense compact element [9]), then the Banaschewski-Sioen nucleus on $\mathfrak{J}L$ coincides with $\vartheta$ (Corollary 5.9).

The final chapter is a little miscellany on some operators that arise naturally in the course of the study of the Banaschewski-Sioen nucleus. We show that if $L$ is a Noetherian frame (meaning that every element is compact), then these operators coincide, and among things, as a mapping $L \to \mathfrak{J}L$, they preserve the Heyting operation (Theorem 6.4).

## 1. Preliminaries

1.1. **The category FIPFrm.** Throughout these preliminaries, $L$ denotes a frame. We refer to [17] and [26] for the general information regarding frames. If $a \le b$ in $L$ we shall at times say $a$ is below $b$ or $b$ is above $a$. An element $a \in L$ is *compact* if, for any $S \subseteq L$, $a \le \bigvee S$ implies that there is a finite $T \subseteq S$ with $a \le \bigvee T$. We denote by $\mathfrak{k}(L)$ the set of all compact elements of $L$. If every element of $L$ is the join of the compact elements below it, then $L$ is an *algebraic frame*. If $a \wedge b \in \mathfrak{k}(L)$ for every $a, b \in \mathfrak{k}(L)$, then $L$ has the *finite intersection property on compact elements*. In such a case we will say $L$ has the FIP. An algebraic frame with the FIP is called a *coherent frame* if its top element is compact.

A frame homomorphism between algebraic frames is called a *coherent map* if it maps compact elements to compact elements. As in [24], we write **FIPFrm** for the category whose objects are algebraic frames with the FIP and whose morphisms are coherent maps. Its subcategory consisting of coherent frames is denoted **CohFrm**. If $h\colon L \to M$ is a surjective coherent map, then

$$\mathfrak{k}(M) = \{h(c) \mid c \in \mathfrak{k}(L)\},$$

so that if $L$ has the FIP, then so does $M$.

The *pseudocomplement* of an element $a$ of $L$ is the element

$$a^* = \bigvee\{x \in L \mid x \wedge a = 0\}.$$

If $L$ is an algebraic frame, it has become standard to denote the pseudocomplement of an $a \in L$ by $a^\perp$, and to refer to this element as the *polar* of $a$. By a polar of $L$ is then meant any element of the form $x^\perp$ for $x \in L$. We shall henceforth use this notation and terminology

We shall freely use without comment the following well-known properties:

$$a \le b \implies b^\perp \le a^\perp, \quad a^\perp = a^{\perp\perp\perp}, \quad \Big(\bigvee_i a_i\Big)^\perp = \bigwedge_i a_i^\perp,$$

and

$$(a \wedge b)^{\perp\perp} = a^{\perp\perp} \wedge b^{\perp\perp}.$$

If $a^{\perp} = 0$ then $a$ is said to be *dense*. This is the case precisely when $a \wedge x \neq 0$ for every $0 \neq x \in L$. For any $a \in L$, $a \vee a^{\perp}$ is dense. If $a \vee a^{\perp} = 1$ then $a$ is said to be *complemented*. If $a = a^{\perp\perp}$ then $a$ is said to be a *regular* element. This is the case precisely when $a$ is a polar. A frame homomorphism $h\colon L \to M$ is called *dense* if for any $a \in L$, $h(a) = 0$ implies $a = 0$. A dense onto frame homomorphism $h\colon L \to M$ commutes with pseudocomplementation, in the sense that $h(a^{\perp}) = h(a)^{\perp}$ for every $a \in L$. A consequence of this is that a dense onto frame homomorphism sends dense elements to dense elements.

1.2. **Rings.** As already mentioned in the Introduction, throughout the paper the term "ring" means a commutative ring with $1 \neq 0$, and ring homomorphisms preserve the identity. We denote by **CRing** the category of rings with identity-preserving ring homomorphisms. Let $A$ be a ring. We write $\langle S \rangle$ for the ideal of $A$ generated by a set $S \subseteq A$. If $S = \{a_1, \dots, a_n\}$, we abbreviate as $\langle a_1, \dots, a_n \rangle$. The *annihilator* of a set $S \subseteq A$ is the ideal

$$\operatorname{Ann}(S) = \{a \in A \mid ax = 0 \text{ for every } x \in S\}.$$

We write $\operatorname{Nzd}(A)$ for the set of *non-zerodivisors* of $A$. That is,

$$\operatorname{Nzd}(A) = \{a \in A \mid \text{for any } x \in A, ax = 0 \text{ implies } x = 0\}.$$

The reader will likely be aware that non-zerodivisors are called *regular elements* by some authors.

The *radical* of an ideal $I$ of a ring $A$ is the ideal

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

If $I = \sqrt{I}$, then $I$ is said to be a *radical ideal*. It is well known that an ideal is a radical ideal if and only whenever it contains the square of an element, then it contains the element. We denote the *nilradical* of $A$ by $\operatorname{Nil}(A)$, and recall that

$$\operatorname{Nil}(A) = \sqrt{\{0\}} = \{x \in A \mid x^n = 0 \text{ for some positive integer } n\}.$$

If $\operatorname{Nil}(A) = \{0\}$, then $A$ is said to be *reduced*. This is the case precisely when $A$ contains no nonzero nilpotent elements.

The lattice $\operatorname{RId}(A)$, partially ordered by inclusion, is an object of **FIPFrm**. It is, in fact, a coherent frame (see, for instance, [2]), whose compact elements are precisely the radicals of finitely generated ideals. Its top element is $A$ and its bottom element is $\operatorname{Nil}(A)$. To avoid over-usage of the symbols 0 and 1, we shall at times write the bottom (resp. top) element of a frame by $\bot$ (resp. $\top$), with subscripts when it is desirable to do so.

If $A$ is a reduced ring, then the bottom of $\mathrm{RId}(A)$ is $\mathsf{O}$. When dealing with the frame $\mathrm{RId}(A)$, a convenient notation from [3] is to set $[a] = \sqrt{\langle a\rangle}$, for any $a \in A$. Then $\bot_{\mathrm{RId}(A)} = [0]$ and $\top_{\mathrm{RId}(A)} = [1]$. With this notation, we then have that, for any $I \in \mathrm{RId}(A)$,

$$I \in \mathfrak{k}(\mathrm{RId}(A)) \qquad \Longleftrightarrow \qquad I = [a_1] \vee \cdots \vee [a_n],$$

for some finite set $\{a_1, \ldots, a_n\} \subseteq A$. Observe that

$$[a_1] \vee \cdots \vee [a_n] = \sqrt{\langle a_1, \ldots, a_n\rangle},$$

and that $[a] \wedge [b] = [ab]$, for all $a, b \in A$.

## 2. The Banaschewski-Sioen nucleus

Recall that a *nucleus* on a frame $L$ is a mapping $j\colon L \to L$ such that, for all $a, b \in L$:

$$a \le j(a), \quad j(j(a)) = j(a), \quad j(a \wedge b) = j(a) \wedge j(b).$$

Then the set

$$\mathrm{Fix}(j) = \{a \in L \mid j(a) = a\} = \{j(a) \mid a \in L\}$$

is a frame, with:

- the partial order inherited from $L$;
- meets calculated as in $L$; and
- joins $\bigsqcup$ given by $\bigsqcup S = j(\bigvee S)$, for any $S \subseteq \mathrm{Fix}(j)$.

The mapping $L \to \mathrm{Fix}(j)$, given by $x \mapsto j(x)$, is an onto frame homomorphism. We shall, by abuse of notation, name it $j\colon L \to \mathrm{Fix}(j)$. This does not lead to confusion because the codomain indicates whether we view $j$ as a nucleus or as a frame homomorphism.

In [6], the authors define a mapping $k\colon \mathrm{RId}(A) \to \mathrm{RId}(A)$ by the equation

$$k(J) = \{a \in A \mid ar \in J \text{ for some } r \in \mathrm{Nzd}(A)\},$$

and show that $k$ is a nucleus on $\mathrm{RId}(A)$. Let us call this nucleus the *Banaschewski-Sioen nucleus* on $\mathrm{RId}(A)$. Recall that a *unit* in an algebraic frame is a dense compact element. We denote by $\mathfrak{u}(L)$ the set of units of $L$. If $\mathfrak{u}(L) \neq \emptyset$, then $L$ is said to *possess units*. Not every algebraic frame with the FIP possesses units. For instance, the powerset of any infinite set is an algebraic frame with the FIP which does not possess units.

Taking a cue from [6], we wish to define a nucleus on any algebraic frame with the FIP and possessing units in such a way that when specialized to

RId$(A)$ it agrees with the one just described, for a large class of rings. Our guiding rule of thumb in extending ring-theoretic results to algebraic frames is that non-zerodivisors in rings correspond to units in algebraic frames, and multiplication corresponds to binary meet. So, we formulate the following definition.

**Definition 2.1.** Let $L$ be an algebraic frame with the FIP and possessing units. We define a mapping $\kappa\colon L \to L$ by

$$\kappa(a) = \bigvee\{c \in \mathfrak{k}(L) \mid c \wedge u \leq a \text{ for some } u \in \mathfrak{u}(L)\}.$$

Our first goal is to show that $\kappa$ is a nucleus on $L$, with some properties which we now recall. We shall at times adorn $\kappa$ with a subscript if there is more than one frame under consideration. In [24], Martinéz and Zenk say a nucleus $\gamma\colon L \to L$ on an algebraic frame $L$ is *inductive* if it satisfies the equality

$$\gamma(a) = \bigvee\{\gamma(c) \mid c \in \mathfrak{k}(L), c \leq a\}$$

for each $a \in L$. This is the case precisely when it preserves directed joins. They then show that if $L$ is an algebraic frame with the FIP and $\gamma$ is an inductive nucleus on $L$, then $\mathrm{Fix}(\gamma)$ is also an algebraic frame with the FIP. The set of its compact elements is

$$\mathfrak{k}(\mathrm{Fix}(\gamma)) = \{\gamma(c) \mid c \in \mathfrak{k}(L)\}.$$

In consequence, the frame homomorphism $\gamma\colon L \to \mathrm{Fix}(\gamma)$ is a coherent map. We should mention that, in [1], Banaschewski says a nucleus is *finitary* if it preserves directed joins.

Recall that a nucleus $j\colon L \to L$ is said to be *dense* if $j(0) = 0$. Of course, this is the case precisely when the frame homomorphism $j\colon L \to \mathrm{Fix}(L)$ is dense. Since surjective dense frame homomorphisms send dense elements to dense elements, if $\gamma\colon L \to L$ is a dense inductive nucleus on an algebraic frame $L$, then for any $u \in \mathfrak{u}(L)$, we have that $\gamma(u) \in \mathfrak{u}(\mathrm{Fix}(\gamma))$, so that $\mathrm{Fix}(\gamma)$ possesses units if $L$ possesses units.

Let $L$ be an algebraic frame with the FIP and possessing units. To facilitate calculations in proving some desired results about the mapping $\kappa\colon L \to L$, let us set, for each $x \in L$,

$$\mathsf{K}(x) = \{c \in \mathfrak{k}(L) \mid c \wedge u \leq x \text{ for some } u \in \mathfrak{u}(L)\}.$$

Then, of course, $\kappa(x) = \bigvee \mathsf{K}(x)$.

The following lemma will be useful below. Since our frames have the FIP, $\mathfrak{k}(L)$ is a distributive lattice with a bottom but not necessarily a top.

**Lemma 2.2.** *Let $L$ be an algebraic frame with the FIP and possessing units. Then:*

(a) *For any $a \in L$, $\mathsf{K}(a)$ is an ideal of $\mathfrak{k}(L)$.*
(b) *For any $a \leq b$ in $L$, $\mathsf{K}(a) \subseteq \mathsf{K}(b)$.*

*Proof.* (a) Since $L$ possesses units, it is clear that $0 \in \mathsf{K}(a)$. Equally clear is that $\mathsf{K}(a)$ is a downset in the poset $\mathfrak{k}(L)$. Now let $c_1, c_2$ be elements of $\mathsf{K}(a)$. Pick $u_1, u_2 \in \mathfrak{u}(L)$ such that $c_i \wedge u_i \leq a$ for each $i = 1, 2$. Since the meet of two dense elements is dense, and since $L$ has the FIP, $u_1 \wedge u_2$ is a unit in $L$ such that

$$\begin{aligned}(c_1 \vee c_2) \wedge (u_1 \wedge u_2) &= (c_1 \wedge u_1 \wedge u_2) \vee (c_2 \wedge u_1 \wedge u_2)\\ &\leq (c_1 \wedge u_1) \vee (c_2 \vee u_2)\\ &\leq a \vee a\\ &= a,\end{aligned}$$

which shows that $c_1 \vee c_2 \in \mathsf{K}(a)$. Therefore $\mathsf{K}(a)$ is an ideal of $\mathfrak{k}(L)$.
(b) This is immediate. □

Although $\mathsf{K}(a)$ is an ideal of $\mathfrak{k}(L)$, as a subset of $L$ it is only a directed set. As an upshot of this, we record the following lemma which we will use often. If a compact element in a frame is below the join of some directed subset of the frame, then it is below some element of that subset. Since, for any $a \in L$, $\mathsf{K}(a)$ is an ideal of $\mathfrak{k}(L)$, we have the following.

**Lemma 2.3.** *Let $L$ be an algebraic frame with the FIP and possessing units. For any $a \in L$ and $c \in \mathfrak{k}(L)$ we have the equivalence*

$$c \leq \kappa(a) \iff c \in \mathsf{K}(a).$$

**Theorem 2.4.** *If $L$ is an algebraic frame with the FIP and possessing units, then $\kappa\colon L \to L$ is a dense inductive nucleus on $L$.*

*Proof.* We show first that $\kappa$ is expansive, meaning that every element is below its image under $\kappa$. Let $a \in L$, and consider any $c \in \mathfrak{k}(L)$ with $c \leq a$. Take any $u \in \mathfrak{u}(L)$. The inequalities $c \wedge u \leq c \leq a$ imply that $c \in \mathsf{K}(a)$, and hence $c \leq \kappa(a)$. Since $a$ is the join of the compact elements below it, it follows that $a \leq \kappa(a)$.

To show that $\kappa$ preserves binary meets, we first observe that whenever $x \leq y$ in $L$, then $\kappa(x) \leq \kappa(y)$ because $\mathsf{K}(x) \subseteq \mathsf{K}(y)$ by Lemma 2.2. Consequently, for any $a, b \in L$, we have the inequality

$$\kappa(a \wedge b) \leq \kappa(a) \wedge \kappa(b).$$

To reverse it, consider any compact $w \leq \kappa(a) \wedge \kappa(b)$. Then $w \leq \kappa(a)$ and $w \leq \kappa(b)$. By Lemma 2.3, $w \in \mathsf{K}(a)$ and $w \in \mathsf{K}(b)$. So we can find $u, v \in \mathfrak{u}(L)$ such that $w \wedge u \leq a$ and $w \wedge v \leq b$. Thus,

$$w \wedge (u \wedge v) = (w \wedge u) \wedge (w \wedge v) \leq a \wedge b.$$

Since $u \wedge v \in \mathfrak{u}(L)$, we deduce that $w \in \mathsf{K}(a \wedge b)$, and so $w \leq \kappa(a \wedge b)$. Since $L$ is an algebraic frame, it follows that $\kappa(a) \wedge \kappa(b) \leq \kappa(a \wedge b)$, and hence $\kappa(a \wedge b) = \kappa(a) \wedge \kappa(b)$.

Next we show that $\kappa(\kappa(a)) = \kappa(a)$. Since $a \leq \kappa(a)$, as shown above, the inequality $\kappa(a) \leq \kappa(\kappa(a))$ holds because, as we saw above, $\kappa$ preserves order. To see the reverse inequality, let $c$ be a compact element of $L$ with $c \leq \kappa(\kappa(a))$. Then, by Lemma 2.3, $c \in \mathsf{K}(\kappa(a))$. There is therefore an element $u \in \mathfrak{u}(L)$ such that $c \wedge u \leq \kappa(a)$. Repeating the argument, but now to the compact element $c \wedge u$ and the fact $c \wedge u \leq \kappa(a)$, we have that $c \wedge u \in \mathsf{K}(a)$, and so there is a $v \in \mathfrak{u}(L)$ such that $(c \wedge u) \wedge v \leq a$, which implies $c \wedge (u \wedge v) \leq a$. Since $u \wedge v \in \mathfrak{u}(L)$, it follows that $c \in \mathsf{K}(a)$, so that $c \leq \kappa(a)$. Since $L$ is an algebraic frame, we conclude $\kappa(\kappa(a)) \leq \kappa(a)$, which yields the desired equality $\kappa(\kappa(a)) = \kappa(a)$. In all, then, $\kappa$ is a nucleus on $L$.

We now show that $\kappa$ is dense. Let $c \in \mathsf{K}(0)$. Then $c \wedge u = 0$ for some $u \in \mathfrak{u}(L)$. Since $u$ is dense, this implies $c = 0$, and so $\kappa(0) = \bigvee \mathsf{K}(0) = 0$, showing that $\kappa$ is dense.

Finally, let $D$ be a directed subset of $L$ and $c$ be a compact element of $L$ with $c \leq \kappa(\bigvee D)$. By Lemma 2.3, $c \in \mathsf{K}(\bigvee D)$, and so, we can find $u \in \mathfrak{u}(L)$ such that $c \wedge u \leq \bigvee D$. Since $D$ is directed and $c \wedge u$ is compact, there is a $d_0 \in D$ such that $c \wedge u \leq d_0$. This implies

$$c \leq \kappa(d_0) \leq \bigvee \{\kappa(d) \mid d \in D\} = \bigvee \kappa[D],$$

whence $\kappa(\bigvee D) \leq \bigvee \kappa[D]$, and hence $\kappa(\bigvee D) = \bigvee \kappa[D]$ because the reverse inequality holds as $\kappa$ preserves order. Thus, $\kappa$ is inductive. □

We now show that, for a large class of rings (which includes every $C(X)$), $\kappa_{\mathrm{RId}(A)}$ does indeed agree with the Banaschewski-Sioen nucleus on $\mathrm{RId}(A)$. We first recall some ring-theoretic terminology. Quentel [27] says a ring *satisfies condition* (C) if whenever $I$ is a finitely generated ideal of the ring consisting entirely of zerodivisors, then $\mathrm{Ann}(I) \neq \{0\}$. Condition (C) is called "Property A" in Huckaba's book [15]. For use below, note that if $A$ is a reduced ring and $J$ any ideal of $A$, then $\mathrm{Ann}(J) = \mathrm{Ann}(\sqrt{J})$. The following lemma is proved in [11]. We include a proof since the cited paper has not yet been published. Recall that the bottom element of $\mathrm{RId}(A) = [0]$.

**Lemma 2.5.** *Let $A$ be a reduced ring and $r \in A$. Then $r \in \mathrm{Nzd}(A)$ iff $[r] \in \mathfrak{u}(\mathrm{RId}(A))$.*

*Proof.* Assume that $r \in \mathrm{Nzd}(A)$, and consider any $x \in A$ such that $[r] \wedge [x] = [0]$. Then $[rx] = [0]$, which implies that $rx = 0$ since $A$ is reduced. Therefore $x = 0$ since $r$ is a non-zerodivisor. Therefore $[x] = [0]$. Since any $J \in \mathrm{RId}(A)$ is expressible as $J = \bigvee_{x \in J}[x]$, if $[r] \wedge J = [0]$, then

$$[0] = [r] \wedge \bigvee_{x \in J}[x] = \bigvee_{x \in J}\big([r] \wedge [x]\big),$$

which implies $[r] \wedge [x] = [0]$ for each $x \in J$. By the preceding calculation, this implies $[x] = [0]$ for each $x \in J$, whence $J = [0]$. Therefore $[r]$ is a dense element in $\mathrm{RId}(A)$, and since $[r]$ is a compact element in $\mathrm{RId}(A)$, it follows that $[r] \in \mathfrak{u}(\mathrm{RId}(A))$.

Conversely, suppose that $[r] \in \mathfrak{u}(\mathrm{RId}(A))$. Consider any $a \in A$ with $ra = 0$. Then $[r] \wedge [a] = [0]$, which implies $[a] = [0]$ since $[r]$ is dense. Therefore $a = 0$, showing that $r$ is a non-zerodivisor. □

The left-to-right implication in this lemma actually holds even if the ring is not reduced, as shown in [11].

**Theorem 2.6.** *If $A$ is a reduced ring satisfying condition* (C)*, then $\kappa_{\mathrm{RId}(A)}$ coincides with the Banaschewski-Sioen nucleus on* $\mathrm{RId}(A)$*.*

*Proof.* Let $J$ be a radical ideal of $A$ and $a \in k(J)$. Then there is a non-zerodivisor $r$ of $A$ such that $ar \in J$. Writing in frame language, this implies $[a] \wedge [r] = [ar] \leq J$. Since $[a] \in \mathfrak{k}(\mathrm{RId}(A))$ and since $[r] \in \mathfrak{u}(\mathrm{RId}(A))$ by Lemma 2.5, it follows that $[a] \leq \kappa_{\mathrm{RId}(A)}(J)$. Since the partial order in $\mathrm{RId}(A)$ is inclusion, $[a] \subseteq \kappa_{\mathrm{RId}(A)}(J)$, so that $a \in \kappa_{\mathrm{RId}(A)}(J)$. Therefore $k(J) \subseteq \kappa_{\mathrm{RId}(A)}(J)$.

To show the reverse inclusion, let $a \in \kappa_{\mathrm{RId}(A)}(J)$. Then $[a] \leq \kappa_{\mathrm{RId}(A)}(J)$. Since $[a]$ is a compact element in $\mathrm{RId}(A)$, Lemma 2.3 tells us that $[a] \in \mathsf{K}(J)$. There is therefore an $H \in \mathfrak{u}(\mathrm{RId}(A))$ such that $[a] \wedge H \leq J$. Since $H$ is a compact element in $\mathrm{RId}(A)$, there are finitely many elements $u_1, \ldots, u_n$ in $A$ such that $H = \sqrt{\langle u_1, \ldots, u_n \rangle}$. Since $H$ is a dense element in $\mathrm{RId}(A)$, $H^{\perp} = [0]$, which, in light of the fact that $A$ is a reduced ring, implies $\mathrm{Ann}(H) = \{0\}$. Again, since $A$ is reduced, this implies

$$\mathrm{Ann}(\langle u_1, \ldots, u_n \rangle) = \{0\}.$$

Since $A$ satisfies condition (C), the ideal $\langle u_1, \ldots, u_n \rangle$ contains a non-zero-divisor, $r$, say. Then, of course, $[r] \leq H$, and so $[a] \wedge [r] \leq J$ because

$[a] \wedge H \leq J$. Thus, $[ar] \leq J$, hence $ar \in J$. It follows therefore that $a \in k(J)$, showing that $\kappa_{\mathrm{RId}(A)}(J) \subseteq k(J)$, and so $\kappa_{\mathrm{RId}(A)}(J) = k(J)$. Since $J$ is an arbitrary element of $\mathrm{RId}(A)$, we have that $\kappa_{\mathrm{RId}(A)} = k$. □

## 3. Some properties of the Banaschewski-Sioen nucleus

As customary, if $j$ is a nucleus on $L$, we shall also write $jL$ in place of $\mathrm{Fix}(j)$. Recall that a nucleus $j\colon L \to L$ is said to be *codense* if, for any $x \in L$, the equality $j(x) = 1$ implies $x = 1$. In [1, Lemma 2], Banaschewski shows that if a nucleus $j\colon L \to L$ is codense or inductive, then $jL$ is compact whenever $L$ is compact. Since $\kappa\colon L \to L$ is an inductive nucleus, we therefore have the following corollary. Note that a coherent frame possesses units because its top element is a unit.

**Corollary 3.1.** *If $L$ is a coherent frame, then $\kappa L$ is a coherent frame.*

For use below, let us record the following lemma.

**Lemma 3.2.** *In an algebraic frame with the FIP and possessing units, every element above a unit is mapped to the top by the Banaschewski-Sioen nucleus.*

*Proof.* Let $L$ be such a frame, and suppose that $a \in L$ is above a unit, $u$, say. Let $c \in \mathfrak{k}(L)$. Then $c \wedge u \leq u \leq a$, which then implies $c \leq \kappa(a)$. Since this is true for every compact element, and since the frame is algebraic, so that the join of all compact elements is the top element of $L$, it follows that $1 \leq \kappa(a)$, whence $\kappa(a) = 1$. □

We can deduce from this lemma that, in general, $\kappa\colon L \to L$ need not be codense, even if $L$ is coherent. To see this, let $L$ be a finite chain with more than two elements. Then $L$ is a coherent frame and each nonzero element of $L$ is a unit (and is hence above a unit) and hence is mapped to the top by $\kappa$. Thus, for any two distinct nonzero elements $x$ and $y$ in $L$, $\kappa(x) = \kappa(y)$. It follows that $\kappa$ is not codense because if $x \in L \smallsetminus \{0,1\}$ (which is possible since $L$ has more than two elements), then $\kappa(x) = 1$.

There is actually more.

**Proposition 3.3.** *Let $L$ be a coherent frame. Then, for any $a \in L$, $\kappa(a) = 1$ iff $a$ is above some unit of $L$.*

*Proof.* Having already observed that elements above a unit are mapped to the top, we prove only one implication. So, suppose that $a \in L$ is such that $\kappa(a) = 1$. Since 1 is compact with $1 \leq \kappa(a)$, Lemma 2.3 implies that $1 \in \mathsf{K}(a)$, and so there exists some $u \in \mathfrak{u}(L)$ such that $1 \wedge u \leq a$. Therefore $u \leq a$. □

As a consequence of this result, we can characterize the coherent frames on which the Banaschewski-Sioen nucleus is codense.

**Corollary 3.4.** *If $L$ is a coherent frame, then $\kappa\colon L \to L$ is codense iff $\mathfrak{u}(L) = \{1\}$.*

*Proof.* Suppose that $\kappa$ is codense, and let $u \in \mathfrak{u}(L)$. Then $u$ is above a unit of $L$ (namely, itself), and so, by Proposition 3.3, $\kappa(u) = 1$, which implies $u = 1$ since $\kappa$ is codense. Therefore $\mathfrak{u}(L) = \{1\}$.

Conversely, suppose that $\mathfrak{u}(L) = \{1\}$. Let $a \in L$ be such that $\kappa(a) = 1$. By Proposition 3.3, $a$ is above a unit of $L$, and since 1 is the only unit $L$ has, it follows that $a = 1$, showing that $\kappa$ is codense. □

**Example 3.5.** The four-element Boolean algebra is an example of a coherent frame with exactly one unit; its top element.

We shall now give a sufficient condition for $\kappa L$ to coincide with $L$. Let us first observe that a dense nucleus on any frame is one-one on regular elements. Let us state this as a lemma, for ease of reference. Note that we do not assume that the frames for which we assert this to be true are algebraic. As such, we shall use the $(-)^*$-notation for pseudocomplements.

**Lemma 3.6.** *Let $j\colon M \to M$ be a dense nucleus on a frame $M$. For any $x, y \in M$, the equality $j(x^*) = j(y^*)$ implies $x^* = y^*$.*

*Proof.* Consider the string of equalities

$$j(x \wedge y^*) = j(x) \wedge j(y^*) = j(x) \wedge j(x^*) = j(x \wedge x^*) = j(0) = 0.$$

Since every element is below its image under a nucleus, $x \wedge y^* = 0$, which implies $y^* \leq x^*$. A similar argument yields $x^* \leq y^*$, whence the claimed equality. □

Another lemma (which is possibly well known) that we shall need follows.

**Lemma 3.7.** *A nucleus $j\colon M \to M$ on any frame $M$ is one-one iff* $\mathrm{Fix}(j) = M$.

*Proof.* If $\mathrm{Fix}(j) = M$, then for any $a \in M$ we have $a \in \mathrm{Fix}(j)$, so that $a = j(a)$. Therefore if $x$ and $y$ are elements of $M$ with $j(x) = j(y)$, then $x = j(x) = j(y) = y$; showing that $j$ is one-one.

Conversely, if $j$ is one-one then, for any $a \in M$, the equality $j(a) = j(j(a))$ implies $a = j(a)$, so that $a \in \mathrm{Fix}(j)$. Therefore $M \subseteq \mathrm{Fix}(j)$, and hence $\mathrm{Fix}(j) = M$. □

For use in the proof of the result we are aiming for (Theorem 3.8 below), let us recall that, as shown by Banaschewski in [3, Lemma 1], if a coherent map between coherent frames is one-one on compact elements, then it is one-one. Although he shows this in the category **CohFrm**, the argument uses only the fact that the compact elements generate the frames in that category. Exactly the same proof shows that if $h\colon L \to M$ is a coherent map between algebraic frames such that the restriction $h_{|\mathfrak{k}L}$ is one-one, then $h$ is one-one.

In the upcoming result we are going to hypothesize that the algebraic frame in question has the feature that each of its compact elements is a polar. Such frames include regular algebraic frames with the FIP because, as shown by Martínez and Zenk in [24, Theorem 2.4], an algebraic frame is regular precisely when every compact element is complemented (and is therefore a polar).

**Theorem 3.8.** *Suppose that $L$ is an algebraic frame with the FIP and possesses units. If every compact element of $L$ is a polar, then* $\mathrm{Fix}(\kappa) = L$.

*Proof.* By Lemma 3.7, it suffices to show that the nucleus $\kappa\colon L \to L$ is one-one. For this, it suffices to show that the coherent map $\kappa\colon L \to \mathrm{Fix}(\kappa)$ is one-one on compact elements. So let $c$ and $d$ be compact elements of $L$ such that $\kappa(c) = \kappa(d)$. Then $\kappa(c^{\perp\perp}) = \kappa(d^{\perp\perp})$ since each compact element of $L$ is a polar. Since $\kappa$ is a dense nucleus, Lemma 3.6 yields $c^{\perp\perp} = d^{\perp\perp}$, so that (in light of the fact that an element is a polar if and only if it equals the polar of its polar)

$$c = c^{\perp\perp} = d^{\perp\perp} = d.$$

Therefore the coherent map $\kappa\colon L \to \mathrm{Fix}(\kappa)$ is one-one on compact elements, and is therefore one-one. This clearly implies that the nucleus $\kappa\colon L \to L$ is one-one. Therefore $\mathrm{Fix}(\kappa) = L$, by Lemma 3.7. $\square$

We now compare $\kappa$ to a certain well-known nucleus. Let us recall from [24] the *d-nucleus* $d\colon L \to L$ on an algebraic frame $L$ with the FIP that is defined by

$$d(a) = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq a\}.$$

Partially ordering nuclei on a frame pointwise, it is shown in [24] that for any inductive dense nucleus $j$ on $L$, $j \leq d$. This is equivalent to saying $\mathrm{Fix}(d) \subseteq \mathrm{Fix}(j)$. Since $\kappa$ is a dense inductive nucleus, we therefore have that $\kappa \leq d$.

It is actually easy to prove directly (that is, without using the argument indicated in [24, Remark 4.5]) that $dL \subseteq \kappa L$, using the characterization observed in [24] that an element $a$ of $L$ belongs to $dL$ if and only if $c^{\perp\perp} \leq a$ for every $c \in \mathfrak{k}(L)$ with $c \leq a$. Let us record the proof.

**Proposition 3.9.** *If $L$ is an algebraic frame with the FIP and possesses units, then $dL \subseteq \kappa L$.*

*Proof.* Let $x \in dL$, and suppose, by way of contradiction, that $x \notin \kappa L$. Then, since $L$ is algebraic, there is a $c \in \mathfrak{k}(L)$ such that $c \leq \kappa(x)$ but $c \not\leq x$. Since $c$ is compact, Lemma 2.3 tells us that $c \in \mathsf{K}(x)$, so that $c \wedge u \leq x$ for some unit $u$ of $L$. Since $c \wedge u$ is compact and $x$ belongs to $dL$, it follows that $(c \wedge u)^{\perp\perp} \leq x$. But

$$(c \wedge u)^{\perp\perp} = c^{\perp\perp} \wedge u^{\perp\perp} = c^{\perp\perp} \wedge 1 = c^{\perp\perp}.$$

So, $c^{\perp\perp} \leq x$, implying $c \leq x$, a contradiction. □

Recall that a frame $L$ is also a Heyting algebra with the Heyting operation $\to$ given by

$$a \to b = \bigvee\{x \in L \mid x \wedge a \leq b\},$$

and characterized by

$$x \leq a \to b \quad \Longleftrightarrow \quad x \wedge a \leq b.$$

The element $a \to b$ is the largest element whose meet with $a$ is below $b$. We are bringing this up because we want to show that $\kappa L$ is a *fitted* sublocale of $L$ – meaning that it is an intersection of open sublocales of $L$. For this, it suffices to show that the nucleus $\kappa\colon L \to L$ is a join (calculated in the frame of nuclei of $L$) of open nuclei. Throughout, when we talk about joins of nuclei, they will be calculated in the frame of nuclei. In the upcoming proof we shall apply [25, Theorem 2.5], which says (although we are not quoting verbatim):

> *If $L$ is an algebraic frame with the FIP, then a nucleus $j\colon L \to L$ is a join of open nuclei iff $j(c \to a) = j(c) \to j(a)$ for all $c \in \mathfrak{k}(L)$ and $a \in L$.*

Let us recall (from [25, Lemma 1.2], for instance) that if $j\colon L \to L$ is a nucleus on any frame $L$, then $j(x) \to j(y) = x \to j(y)$, for all $x, y \in L$.

**Theorem 3.10.** *If $L$ is an algebraic frame with the FIP and possesses units, then $\kappa\colon L \to L$ is a join of open nuclei. Hence, $\kappa L$ is a fitted sublocale of $L$.*

*Proof.* Let $c \in \mathfrak{k}(L)$ and $a \in L$. As with any $\wedge$-preserving map,

$$\kappa(c \to a) \leq \kappa(c) \to \kappa(a).$$

To reverse the inequality, consider any $d \in \mathfrak{k}(L)$ such that $d \leq \kappa(c) \to \kappa(a)$. Then, by [25, Lemma 1.2], $d \leq c \to \kappa(a)$, which implies $d \wedge c \leq \kappa(a)$. Since $d \wedge c$ is compact, Lemma 2.3 tells us that $d \wedge c \in \mathsf{K}(a)$, so that there is a unit $u \in \mathfrak{u}(L)$ such that $(d \wedge c) \wedge u \leq a$. Consequently, $d \wedge u \leq c \to a$, which implies $d \leq \kappa(c \to a)$. Since $\kappa(c) \to \kappa(a)$ is the join of the compact elements below it, it follows that $\kappa(c) \to \kappa(a) \leq \kappa(c \to a)$, whence we deduce that $\kappa(c \to a) = \kappa(c) \to \kappa(a)$. Therefore, by [25, Theorem 2.5], $\kappa$ is a join of open nuclei. □

As in [4], we say a frame is *Noetherian* if each of its elements is compact. Noetherian frames are algebraic, have the FIP, and possess units because in them every element is compact, and so (among other things) the top element is a unit. We have the following corollary to the foregoing theorem.

**Corollary 3.11.** *If $L$ is a Noetherian frame then $\kappa \colon L \to L$ preserves the Heyting operation. Hence, the frame homomorphism $\kappa \colon L \to \kappa L$ is a Heyting homomorphism.*

*Proof.* In the course of the proof of Theorem 3.10 we showed that for a compact $c$ and arbitrary $a$ in $L$, the equality $\kappa(c \to a) = \kappa(c) \to \kappa(a)$ holds. Since in a Noetherian frame every element is compact, the first part in the assertion in the corollary we are proving follows immediately.

To show the latter part, recall that for any frame $M$ and any sublocale $S$ of $M$, the Heyting operation in $S$ is exactly that in $M$. That is, if $s, t \in S$ and we denote by $\to^S$ the Heyting operation in $S$, then $s \to^S t = s \to^M t$. To avoid possible confusion, let us write $k \colon L \to \kappa L$ for the frame homomorphism $x \mapsto \kappa(x)$. Let $a, b \in L$. Applying Theorem 3.10, keeping in mind that $a \in \mathfrak{k}(L)$ as $L$ is Noetherian, we get the equalities

$$k(a \to^L b) = \kappa(a \to^L b) = \kappa(a) \to^L \kappa(b) = \kappa(a) \to^{\kappa L} \kappa(b) = k(a) \to^{\kappa L} k(b),$$

which shows that the frame homomorphism $\kappa \colon L \to \kappa L$ is a Heyting homomorphism. □

We close this section by describing the prime elements of $\kappa L$. When dealing with spatial frames, it is always of interest to identify its prime elements. Since $\kappa L$ is a sublocale of $L$, the prime elements of $\kappa L$ are precisely those of $L$ that belong to $\kappa L$; that is, $\mathrm{Pr}(\kappa L) = \mathrm{Pr}(L) \cap \kappa L$. We are able describe them in terms of units of $L$.

**Theorem 3.12.** *Let $L$ be an algebraic frame with the FIP and possessing units. Then, the prime elements of $\kappa L$ are precisely those of $L$ that are above* *no unit of $L$. That is,*

$$\mathrm{Pr}(\kappa L) = \{p \in \mathrm{Pr}(L) \mid p \text{ is above no unit of } L\}.$$

*Proof.* Let $p \in \mathrm{Pr}(L)$ be such that there is no unit of $L$ below $p$. We show that $\kappa(p) = p$. Consider any $c \in \mathfrak{k}(L)$ with $c \leq \kappa(p)$. By Lemma 2.3, $c \in \mathsf{K}(p)$, and so $c \wedge u \leq p$ for some $u \in \mathfrak{u}(L)$. Since $p$ is prime, the inequality $c \wedge u \leq p$ implies $c \leq p$ or $u \leq p$. Since the latter is not possible, we must have $c \leq p$. Since $L$ is an algebraic frame, we therefore have $\kappa(p) \leq p$, and hence $\kappa(p) = p$, showing that $p \in \kappa L$. This proves the inclusion $\supseteq$ in the claimed equality.

To reverse it, let $q \in \mathrm{Pr}(\kappa L)$. Then, of course, $q \in \mathrm{Pr}(L)$. We also have $q = \kappa(q)$ since $\kappa L = \mathrm{Fix}(\kappa)$. By Lemma 3.2, $q$ is not above any unit of $L$; else we would have $\kappa(q) = 1$, which is not the case. This establishes the desired inclusion, and hence proves the result. ☐

Recall that, for any element $a$ of a frame $L$, the set

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\}$$

is a sublocale of $L$, and, in fact, the smallest sublocale of $L$ containing $a$. By a *one-point sublocale* of $L$ is meant a sublocale of the form $\mathfrak{b}(p)$ with $p \in \mathrm{Pr}(L)$. Then $\mathfrak{b}(p) = \{p, 1\}$. The spatiality of $\kappa L$ together with the result in Theorem 3.12 yields the following corollary; where the indicated join is calculated in the lattice of sublocales of $L$.

**Corollary 3.13.** *For any algebraic frame $L$ with the FIP and possessing units,*

$$\kappa L = \bigvee\{\mathfrak{b}(p) \mid p \in \mathrm{Pr}(L) \text{ and } p \text{ is above no unit of } L\}.$$

We show, as a corollary to Theorem 3.12, that the minimal primes of $\kappa L$ are precisely the minimal primes of $L$. Recall from [20, Corollary 2.5.1] that if $L$ is an algebraic frame with the FIP, $p \in \mathrm{Min}(L)$, $c \in \mathfrak{k}(L)$, and $c \leq p$, then $c^{\perp} \nleq p$. That is, a minimal prime above a compact element cannot be above the polar of that compact element. Consequently, in an algebraic frame with the FIP and possessing units, a minimal prime cannot be above a unit because it is above 0, the polar of any unit. We therefore have the following result.

**Corollary 3.14.** *Let $L$ be an algebraic frame with the FIP and possessing units. Then* $\mathrm{Min}(\kappa L) = \mathrm{Min}(L)$.

*Proof.* Let $p \in \mathrm{Min}(L)$. Since $p$ is above no unit of $L$ (as we have just argued above), we see from Theorem 3.12 that $\mathrm{Min}(L) \subseteq \mathrm{Pr}(\kappa L)$, and so

$p \in \mathrm{Pr}(\kappa L)$. If $r$ is a prime of $\kappa L$ with $r \leq p$, then $r$ is also a prime of $L$ and is such that $r \leq p$ in $L$. Hence $r = p$, showing that $p \in \mathrm{Min}(\kappa L)$. Therefore $\mathrm{Min}(L) \subseteq \mathrm{Min}(\kappa L)$.

Conversely, suppose that $q \in \mathrm{Min}(\kappa L)$. Then $q \in \mathrm{Pr}(L)$ and, by Theorem 3.12, $q$ is above no unit of $L$. Suppose that $r$ is a prime of $L$ with $r \leq q$. Then $r$ is above no unit of $L$ as $q$ is above no unit of $L$, and so $r$ is a prime of $\kappa L$ below the minimal prime $q$, which implies $r = q$. This shows that $q$ is a minimal prime in $L$, and so $\mathrm{Min}(\kappa L) \subseteq \mathrm{Min}(L)$, whence we have the claimed equality. □

*Remark* 3.15. The results in Theorem 3.12 and Corollary 3.14 bring to mind the following well-known ring-theoretic facts. Let $A$ be a reduced commutative ring with identity and let $qA$ denote its total ring of quotients. As usual, for a ring $R$, $\mathrm{Spec}(R)$ denotes the set of prime ideals of $R$. The ring results we are alluding to are:

(i) The set $\mathrm{Spec}_r(A)$ which is equals to $\{P \in \mathrm{Spec}(A) \mid P \text{ does not contain any non-zerodivisor of } A\}$, is in bijective correspondence with $\mathrm{Spec}(qA)$.

(ii) There is a bijection between the sets $\mathrm{Min}(A)$ and $\mathrm{Min}(qA)$.

In a way, the results in results in Theorem 3.12 and Corollary 3.14 should be expected because, as Banaschewski and Sioen observe in [6], for any ring $A$, the algebraic frames $\kappa\,\mathrm{RId}(A)$ and $\mathrm{RId}(qA)$ are isomorphic.

## 4. On $\kappa$-maps

This section is motivated by what Martínez and Zenk [23] call nuclear typings of frames. In our context, it boils down to characterizing morphisms $h\colon L \to M$ in the category of algebraic frames with the FIP and possessing units, for which there is a morphism $\bar{h}\colon \kappa L \to \kappa M$ in **FIPFrm** such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ \downarrow{\scriptstyle \kappa_L} & & \downarrow{\scriptstyle \kappa_M} \\ \kappa L & \xrightarrow{\bar{h}} & \kappa M \end{array} \tag{4.1}$$

commutes. If such a morphism exists, it is unique. We shall thus denote it by $\kappa(h)$. Following the nomenclature in [23], we shall then say $h$ is $\kappa$-*natural* or that $h$ is a $\kappa$-*map*. It is easy to see (as, indeed, observed in [23]) that $h$ is a $\kappa$-map if and only if, for every pair $a, b \in L$, the equality

$\kappa_L(a) = \kappa_L(b)$ implies $\kappa_M(h(a)) = \kappa_M(h(b))$. In this case, the map $\kappa(h)$ is then the restriction to $\kappa L$ of the composite $\kappa_M \circ h$. That is, $\kappa(h)$ is the homomorphism $(\kappa_M \circ h)|_{\kappa L} \colon \kappa L \to \kappa M$.

To characterize $\kappa$-maps, let us, given $a \in L$, look again at the set $\mathsf{K}(a)$. It is an ideal of $\mathfrak{k}(L)$, which we now use to generate the ideal

$$\widehat{\mathsf{K}}(a) = \{x \in L \mid x \leq c \text{ for some } c \in \mathsf{K}(a)\}$$

of $L$. Observe that

$$\mathsf{K}(a) \subseteq \widehat{\mathsf{K}}(a) \qquad \text{and} \qquad \bigvee \mathsf{K}(a) = \bigvee \widehat{\mathsf{K}}(a);$$

the latter holding because the inequality $\bigvee \widehat{\mathsf{K}}(a) \leq \bigvee \mathsf{K}(a)$ follows from the fact that every element of $\widehat{\mathsf{K}}(a)$ is below some element of $\mathsf{K}(a)$.

When there is more than one frame under consideration and there is a possible danger of confusion, we shall use subscripts to denote the frame whose ideal is being referred to. For instance, if $h \colon L \to M$ is a homomorphism and $a \in L$, there is no ambiguity in writing $\mathsf{K}(a)$, $\widehat{\mathsf{K}}(h(a))$, and so on. However, if, for instance, $a \in \kappa L$ (so that $a$ also belongs to $L$), simply writing $\mathsf{K}(a)$ or $\widehat{\mathsf{K}}(a)$ might lead to ambiguity.

Here is a lemma which serves as a first step towards charactering $\kappa$-maps in terms of the ideals just introduced.

**Lemma 4.1.** *The following are equivalent for any two elements $a, b \in L$.*

(a) $\kappa(a) \leq \kappa(b)$.
(b) $\mathsf{K}(a) \subseteq \mathsf{K}(b)$.
(c) $\widehat{\mathsf{K}}(a) \subseteq \widehat{\mathsf{K}}(b)$.

*Proof.* (a) $\Rightarrow$ (b): Let $c \in \mathsf{K}(a)$. Then $c \leq \kappa(a)$. So, by the hypothesis in (a), $c \leq \kappa(b)$, and hence, by Lemma 2.3, $c \in \mathsf{K}(b)$. Therefore $\mathsf{K}(a) \subseteq \mathsf{K}(b)$.

(b) $\Rightarrow$ (c): This is trivial.

(c) $\Rightarrow$ (a): This follows from the fact that $\bigvee \widehat{\mathsf{K}}(x) = \kappa(x)$ for every $x \in L$. $\square$

An immediate consequence is the following result.

**Corollary 4.2.** *The following are equivalent for any two elements $a, b \in L$.*

(a) $\kappa(a) = \kappa(b)$.
(b) $\mathsf{K}(a) = \mathsf{K}(b)$.
(c) $\widehat{\mathsf{K}}(a) = \widehat{\mathsf{K}}(b)$.

Now here is the characterization of $\kappa$-maps towards which we have been building.

**Theorem 4.3.** *The following are equivalent for a morphism $h\colon L \to M$ in* **FIPFrm** *with $L$ and $M$ possessing units.*

(a) *$h$ is a $\kappa$-map.*
(b) *For any two elements $a, b$ in $L$, the equality $\mathsf{K}(a) = \mathsf{K}(b)$ implies $\mathsf{K}(h(a)) = \mathsf{K}(h(b))$.*
(c) *For any two elements $a, b$ in $L$, the equality $\widehat{\mathsf{K}}(a) = \widehat{\mathsf{K}}(b)$ implies $\widehat{\mathsf{K}}(h(a)) = \widehat{\mathsf{K}}(h(b))$.*

*Proof.* (a) $\Leftrightarrow$ (b): Assume that $h$ is a $\kappa$-map, and let $a, b \in L$ be such that $\mathsf{K}(a) = \mathsf{K}(b)$. Then $\kappa_L(a) = \kappa_L(b)$. Since $h$ is a $\kappa$-map, this implies $\kappa_M(h(a)) = \kappa_M(h(b))$, and so, from Corollary 4.2, we deduce that $\mathsf{K}(h(a)) = \mathsf{K}(h(b))$.

Conversely, assume that the condition in (b) holds. Consider any two elements $a, b \in L$ such that $\kappa_L(a) = \kappa_L(b)$. Then, by Corollary 4.2, $\mathsf{K}(a) = \mathsf{K}(b)$. By hypothesis, this implies $\mathsf{K}(h(a)) = \mathsf{K}(h(b))$, whence

$$\kappa_M(h(a)) = \kappa_M(h(b))$$

by Corollary 4.2 again. It follows therefore that $h$ is a $\kappa$-map.

(a) $\Leftrightarrow$ (c): The equivalence (a) and (c) is shown similarly to the one of (a) and (b). □

We record the following corollary for later use. We commented earlier that if $\gamma\colon L \to L$ is a dense inductive nucleus on $L$ and $L$ possesses units, then $\gamma L$ also possesses units.

**Corollary 4.4.** *For any algebraic frame $L$ with the FIP and possessing units, the frame homomorphism $\kappa_L\colon L \to \kappa L$ is a $\kappa$-map.*

*Proof.* Suppose that $x$ and $y$ are elements of $L$ such that $\mathsf{K}_L(x) = \mathsf{K}_L(y)$. Then, upon taking joins, we have that $\kappa_L(x) = \kappa_L(y)$, and so

$$\mathsf{K}_{\kappa L}(\kappa_L(x)) = \mathsf{K}_{\kappa L}(\kappa_L(y)).$$

Hence, $\kappa_L\colon L \to \kappa L$ is a $\kappa$-map by Theorem 4.3. □

Following the nomenclature in [24], we shall say an algebraic frame $L$ possessing units is *$\kappa$-regular* if $\kappa L$ is regular. Let us observe that the property of being $\kappa$-regular is inherited by quotients induced by $\kappa$-maps. In more detail, we have the following result.

**Proposition 4.5.** *Let $h\colon L \to M$ be a surjective $\kappa$-map. If $L$ is $\kappa$-regular, then so is $M$.*

*Proof.* Since $h$ is a $\kappa$-map, we have the commutative square:

$$\begin{array}{ccc} L & \xrightarrow{\ h\ } & M \\ {\scriptstyle \kappa_L}\downarrow & & \downarrow{\scriptstyle \kappa_M} \\ \kappa L & \xrightarrow{\ \kappa(h)\ } & \kappa M \end{array}$$

Since $\kappa_M \circ h = \kappa(h) \circ \kappa_L$, the surjectivity of the maps $\kappa_M$ and $h$ implies that $\kappa(h)$ is surjective. Since $L$ is $\kappa$-regular, the frame $\kappa L$ is regular. Since every homomorphic image of a regular frame is regular, it follows that $\kappa M$ is regular, that is, $M$ is $\kappa$-regular. □

We end this section with the result (Theorem 4.9 below) which says if $a \in L$ satisfies a certain mild condition, then the frame $\kappa(\uparrow a)$ is isomorphic to the closed quotient of $\kappa L$ induced by the element $\kappa(a)$. We need to clear some ground to be able to prove the result. Let us first agree on some notation. For any frame $K$ and $z \in K$, we write $\mathsf{c}_z\colon K \to \uparrow z$ for the frame homomorphism $x \mapsto z \vee x$. We will use both the notations $\uparrow a$ and $\mathfrak{c}_L(a)$ for the closed quotient associated with $a$. The former can, of course, be ambiguous in some instances, in which case we will prefer the latter. Let $h\colon L \to M$ be any frame homomorphism and let $a \in L$. A direct calculation shows that the function

$$h^a\colon \mathfrak{c}_L(a) \to \mathfrak{c}_M(h(a))$$

which maps exactly as $h$ is a frame homomorphism making the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\ h\ } & M \\ {\scriptstyle \mathsf{c}_a}\downarrow & & \downarrow{\scriptstyle \mathsf{c}_{h(a)}} \\ \mathfrak{c}_L(a) & \xrightarrow{\ h^a\ } & \mathfrak{c}_M(h(a)) \end{array} \tag{4.2}$$

We comment, in passing, that this square is a pushout square in the category **Frm**. A proof of this can be found in Volume 2 of P.T. Johnstone's "Sketches of an Elephant".

Next, we take another look at the definition of the Banaschewski-Sioen nucleus. In Definition 2.1 we set, for any $a \in L$, $\kappa(a)$ to be the join of the set that we denoted by $\mathsf{K}(a)$. We observed that $\mathsf{K}(a)$ is an ideal of $\mathfrak{k}(L)$. Using it, we then defined the ideal of $L$ that we denoted by $\widehat{\mathsf{K}}(a)$. Here we define a new ideal of $L$, which we shall denote by $\mathsf{T}(a)$, in a manner akin to how

$\mathsf{K}(a)$ is defined, and then show that $\kappa(a)$ coincides with the join of $\mathsf{T}(a)$. The advantage of this will be that when computing $\kappa(a)$, we will not be restricted to working only with compact elements – which might be restrictive.

Let us introduce the following terminology and notation of convenience. Let $a \in L$. We say an element $x \in L$ is *tied* to $a$, and write $x \ltimes a$, in case $x \wedge u \leq a$ for some unit $u$ of $L$. That is,

$$x \ltimes a \quad \Longleftrightarrow \quad (\exists u \in \mathfrak{u}(L))(x \wedge u \leq a).$$

In this case we shall say $u$ *witnesses the tiedness* of $x$ to $a$.

We define the subset $\mathsf{T}(a)$ of $L$ and the map $\tau\colon L \to L$ by the equations

$$\mathsf{T}(a) = \{x \in L \mid x \ltimes a\} \qquad \text{and} \qquad \tau(a) = \bigvee \mathsf{T}(a).$$

As before, we shall use subscripts when more than one frame is under discussion to avoid possible confusion.

Observe that, for any $a \in L$:

- $0 \ltimes a$;
- $x \leq y \quad \& \quad y \ltimes a \quad \Longrightarrow x \ltimes a$;
- $x \ltimes a \quad \& \quad y \ltimes a \quad \Longrightarrow (x \vee y) \ltimes a$, by an argument similar to that employed in the proof of Lemma 2.2(a).

A consequence of this is that $\mathsf{T}(a)$ is an ideal of $L$ with $\mathsf{K}(a) \subseteq \mathsf{T}(a)$. In fact,

$$\mathsf{K}(a) = \mathsf{T}(a) \cap \mathfrak{k}(L),$$

as a routine calculation shows.

**Lemma 4.6.** *Let $L$ be an algebraic frame with the FIP and possessing units. Then for any $a \in L$, $\kappa(a) = \tau(a)$.*

*Proof.* Since $\mathsf{K}(a) \subseteq \mathsf{T}(a)$, we immediately have the inequality $\kappa(a) \leq \tau(a)$. To reverse it, consider any $x \in L$ with $x \ltimes a$. Pick any $u \in \mathfrak{u}(L)$ that witnesses this, so that $x \wedge u \leq a$. Let

$$X = \{c \in \mathfrak{k}(L) \mid c \leq x\},$$

so that $x = \bigvee X$ since $L$ is an algebraic frame. Each element of $X$ is tied to $a$, and thus (being compact) belongs to $\mathsf{K}(a)$, and is therefore below $\kappa(a)$. Consequently, $x \leq \kappa(a)$. Since $\tau(a)$ is the join of the elements of $L$ that are tied to $a$, it follows that $\tau(a) \leq \kappa(a)$, whence we obtain the asserted equality. □

We mentioned above that in our target result we will impose some condition on an element $a \in L$. We will hypothesize that $a$ has the feature that:

- The homomorphism $\mathsf{c}_a\colon L \to {\uparrow}a$ sends units to units; and
- $\mathfrak{u}({\uparrow}a) = \{a \vee u \mid u \in \mathfrak{u}(L)\}$.

In such a case we shall say ${\uparrow}a$ is *favored by units*. Here is an easy example of an $L$ with a closed quotient favored by units.

**Example 4.7.** Let $L$ be the four-element Boolean algebra $L = \{0, a, a', 1\}$. Then $\mathfrak{u}(L) = \mathfrak{u}({\uparrow}a) = \{1\}$, so that $\mathsf{c}_a$ sends units to units, and

$$\mathfrak{u}({\uparrow}a) = \{a \vee u \mid u \in \mathfrak{u}(L)\}.$$

In fact, every non-trivial closed quotient of $L$ is favored by units.

The following lemma shows that if $a$ is favored by units, then the maps $\kappa_L$ and $\kappa_{\uparrow a}$ agree on the elements of ${\uparrow}a$.

**Lemma 4.8.** *Let $L$ be an algebraic frame with the FIP and possessing units, and let $a \in L$. If ${\uparrow}a$ is favored by units, then for any $z \in {\uparrow}a$, $\kappa_L(z) = \kappa_{\uparrow a}(z)$.*

*Proof.* By Lemma 4.6,

$$\kappa_{\uparrow a}(z) = \tau_{\uparrow a}(z) = \bigvee\{t \in {\uparrow}a \mid t \wedge v \leq z \text{ for some } v \in \mathfrak{u}({\uparrow}a)\}.$$

Consider any $t \in {\uparrow}a$ which is tied to $z$ in the frame ${\uparrow}a$. Pick $v \in \mathfrak{u}({\uparrow}a)$ such that $t \wedge v \leq z$. Since ${\uparrow}a$ is favored by units, there is a $u \in \mathfrak{u}(L)$ such that $v = a \vee u$. Thus, $u$ is a unit of $L$ such that

$$t \wedge u \leq t \wedge (a \vee u) = t \wedge v \leq z;$$

showing that $t$ is tied to $z$ in the frame $L$, whence $t \leq \tau_L(z) = \kappa_L(z)$, by Lemma 4.6. Since $\kappa_{\uparrow a}(z)$ is the join of all elements of ${\uparrow}a$ that are tied to $z$ in this frame, it follows that $\kappa_{\uparrow a}(z) \leq \kappa_L(z)$.

To reverse the inequality, consider any $c \in \mathsf{K}_L(z)$, and find $u \in \mathfrak{u}(L)$ such that $c \wedge u \leq z$. Then, for the frame homomorphism $\mathsf{c}_a\colon L \to {\uparrow}a$, we have

$$\begin{aligned} \mathsf{c}_a(c \wedge u) \leq \mathsf{c}_a(z) &\implies a \vee (c \wedge u) \leq a \vee z \\ &\implies (a \vee c) \wedge (a \vee u) \leq a \vee z \\ &\implies (a \vee c) \wedge (a \vee u) \leq z \qquad \text{since } a \leq z. \end{aligned}$$

Since ${\uparrow}a$ is favored by units, $a \vee u$ is a unit in the frame ${\uparrow}a$. Since $c$ is a compact element in $L$, $a \vee c$ is a compact element in ${\uparrow}a$. Thus, $a \vee c$ is a compact element of ${\uparrow}a$ which meets some unit of ${\uparrow}a$ below $z$, which says $a \vee c \in \mathsf{K}_{\uparrow a}(z)$. Since $c \leq a \vee c$, we therefore have that every element of $\mathsf{K}_L(z)$ is below some element of $\mathsf{K}_{\uparrow a}(z)$, and so, upon taking joins,

$$\kappa_L(z) = \bigvee \mathsf{K}_L(z) \leq \bigvee \mathsf{K}_{\uparrow a}(z) = \kappa_{\uparrow a}(z).$$

Therefore $\kappa_L(z) = \kappa_{\uparrow a}(z)$. $\square$

Here is the result we have been aiming for.

**Theorem 4.9.** *Let $L$ be an algebraic frame with the FIP and possessing units, and let $a \in L$ be such that $\mathfrak{c}_L(a)$ is favored by units. Then*

$$\kappa(\mathfrak{c}_L(a)) \cong \mathfrak{c}_{\kappa L}(\kappa_L(a)).$$

*Proof.* Consider the frame homomorphism $\kappa_L^a \colon \mathfrak{c}_L(a) \to \mathfrak{c}_{\kappa L}(\kappa_L(a))$, emanating from the frame homomorphism $\kappa_L \colon L \to \kappa L$ as per Diagram (4.2). Define the map

$$g \colon \kappa(\mathfrak{c}_L(a)) \to \mathfrak{c}_{\kappa L}(\kappa_L(a))$$

to be the restriction of $\kappa_L^a$ to the subset $\kappa(\mathfrak{c}_L(a))$ of $\mathfrak{c}_L(a)$. This means that, by Lemma 4.8, for any $t \in \mathfrak{c}_L(a)$, $g(t) = \kappa_L(t) = \kappa_{\mathfrak{c}_L(a)}(t)$. We show that $g$ is a frame isomorphism. The bottom element of $\kappa(\mathfrak{c}_L(a))$ is $\kappa_L(a)$, and $g(\kappa_L(a)) = \kappa_L(\kappa_L(a)) = \kappa_L(a)$, which is the bottom element of $\mathfrak{c}_{\kappa L}(\kappa_L(a))$. Therefore $g$ sends the bottom of its domain to the bottom of its codomain. It is clear that $g$ sends the top to the top. Since meets in a subocale are exactly those in the containing frame, it is also easy to see that $g$ preserves binary meets. Regarding preservation of joins, let $S \subseteq \kappa(\mathfrak{c}_L(a))$. For any sublocale $T$ of $L$ involved in the rest of the proof, we shall write $\bigvee^T$ for the join in that sublocale. An unadorned $\bigvee$ will denote the join in $L$. Observe that each element of $S$ belongs to $\mathfrak{c}_L(a)$ because $\kappa(\mathfrak{c}_L(a) \subseteq \mathfrak{c}_L(a)$. Hence $g(s) = \kappa_L(s)$ for each $s \in S$ by Lemma 4.8. Now,

$$\begin{aligned}
g\Big(\bigvee\nolimits^{\kappa(\mathfrak{c}_L(a))} S\Big) &= g\Big(\kappa_{\mathfrak{c}_L(a)}\big(\bigvee S\big)\Big) \\
&= \kappa_L\Big(\kappa_{\mathfrak{c}_L(a)}\big(\bigvee S\big)\Big) && \text{by how } g \text{ is defined} \\
&= \kappa_L\Big(\kappa_L\big(\bigvee S\big)\Big) && \text{by Lemma 4.8} \\
&= \kappa_L\big(\bigvee S\big) \\
&= \bigvee\nolimits^{\kappa L}\{\kappa_L(s) \mid s \in S\} \\
&= \bigvee\nolimits^{\mathfrak{c}_{\kappa L}(\kappa_L(a))}\{\kappa_L(s) \mid s \in S\} && \textstyle\bigvee{\uparrow}\kappa_L(a) \text{ are those in } \kappa L \\
&= \bigvee\nolimits^{\mathfrak{c}_{\kappa L}(\kappa_L(a))}\{g(s) \mid s \in S\},
\end{aligned}$$

which shows that $g$ preserves joins. Therefore $g$ is a frame homomorphism. To show that $g$ is surjective, let $y \in \mathfrak{c}_{\kappa L}(\kappa_L(a))$. Then $y \in \kappa L$ (so that

$y \in \mathrm{Fix}(\kappa_L)$, hence $y = \kappa_L(y))$ and $\kappa_L(a) \le y$. The latter implies $a \le y$, so that $y \in \mathfrak{c}_L(a)$. Since $g(y) = \kappa_L(y) = y$, it follows that $g$ is surjective. Finally, we show that $g$ is injective. Let $u, v \in \kappa(\mathfrak{c}_L(a))$ be such that $g(u) = g(v)$. Then

$$u \in \mathfrak{c}_L(a), \quad v \in \mathfrak{c}_L(a), \quad u = \kappa_{\mathfrak{c}_L(a)}(u), \quad v = \kappa_{\mathfrak{c}_L(a)}(v).$$

By Lemma 4.8, this implies that $u = \kappa_L(u)$ and $v = \kappa_L(v)$. Consequently,

$$\begin{aligned} g(u) = g(v) \quad &\Longrightarrow \quad \kappa_L(\kappa_L(u)) = \kappa_L(\kappa_L(v)) \\ &\Longrightarrow \quad \kappa_L(u) = \kappa_L(v) \\ &\Longrightarrow \quad u = v. \end{aligned}$$

In all then, $g$ is an isomorphism of frames, as desired. □

## 5. A nucleus derived from $\mathsf{T}$

In the style of Simmons [28], in this section we define a nucleus on $\mathfrak{J}L$, the frame of ideals of $L$, utilizing the operator $\mathsf{T}(-)$. Throughout the section, $L$ denotes an algebraic frame with the FIP and possessing units. We shall however sometimes mention this in the statements of lemmas, theorems, etc. Let us recall that, for any $a \in L$,

$$\mathsf{T}(a) = \{x \in L \mid x \wedge u \le a \text{ for some } u \in \mathfrak{u}(L)\}.$$

We observed earlier that $\mathsf{T}(a)$ is an ideal of $L$. The following lemma is going to be useful.

**Lemma 5.1.** *Let $L$ be an algebraic frame with the FIP and possessing units.*
- (a) *For any $a \in L$, $a \in \mathsf{T}(a)$.*
- (b) $\mathsf{T}(0) = \{0\}$ *and* $\mathsf{T}(1) = L$.
- (c) *If $a \le b$, then $\mathsf{T}(a) \subseteq \mathsf{T}(b)$.*
- (d) *For any $a, b \in L$, $\mathsf{T}(a \wedge b) = \mathsf{T}(a) \cap \mathsf{T}(b)$.*

*Proof.* (a) Let $u$ be a unit in $L$. Since $a \wedge u \le a$, it follows that $a \in \mathsf{T}(a)$.

(b) Let $x \in \mathsf{T}(0)$. Then there exists a unit $u$ in $L$ such that $x \wedge u \le 0$. Since $u$ is a dense element, this implies $x = 0$, and so $\mathsf{T}(0) = \{0\}$. That $\mathsf{T}(1) = L$ follows from the fact that $\mathsf{T}(1)$ is an ideal of $L$, and $1 \in \mathsf{T}(1)$ by part (a).

(c) If $x \in \mathsf{T}(a)$ and $u$ is a unit of $L$ witnessing this, then $u$ also witnesses that $x \in \mathsf{T}(b)$. Therefore $\mathsf{T}(a) \subseteq \mathsf{T}(b)$.

(d) The containment $\mathsf{T}(a \wedge b) \subseteq \mathsf{T}(a) \cap \mathsf{T}(b)$ follows from (c). For the reverse containment, let $x \in \mathsf{T}(a) \cap \mathsf{T}(b)$. Find $u, v \in \mathfrak{u}(L)$ such that $x \wedge u \le a$ and $x \wedge v \le b$. Then $u \wedge v$ is a unit of $L$ such that

$$x \wedge (u \wedge v) = (x \wedge u) \wedge (x \wedge v) \le a \wedge b,$$

showing that $x \in \mathsf{T}(a \wedge b)$. Therefore $\mathsf{T}(a \wedge b) = \mathsf{T}(a) \cap \mathsf{T}(b)$. $\square$

In preparation for the upcoming definition, note that (in light of item (c) in Lemma 5.1) if $I$ is an ideal of $L$, then the set $\{\mathsf{T}(x) \mid x \in I\}$ is directed, hence its union is an ideal of $L$.

**Definition 5.2.** We define a map $\vartheta\colon \mathfrak{J}L \to \mathfrak{J}L$ by

$$\vartheta(I) = \bigcup\{\mathsf{T}(x) \mid x \in I\},$$

for each $I \in \mathfrak{J}L$.

In the "tied to" notation, we see easily that

$$\vartheta(I) = \{x \in L \mid (\exists i \in I)(x \ltimes i)\}.$$

Our first aim is to show that $\vartheta$ is a nucleus on $\mathfrak{J}L$. We show that, in fact, $\vartheta$ is a dense inductive nucleus on $\mathfrak{J}L$.

Since binary meet and directed join in $\mathfrak{J}L$ are, respectively, intersection and union, we shall at times write them as so instead of $\wedge$ and $\bigvee$.

**Theorem 5.3.** *The map $\vartheta\colon \mathfrak{J}L \to \mathfrak{J}L$ is a dense inductive nucleus for any algebraic frame $L$ with the FIP and possessing units.*

*Proof.* Let $I \in \mathfrak{J}L$. Since every element of $L$ is tied to itself (as $\mathfrak{u}(L) \neq \emptyset$), it follows immediately that $I \subseteq \vartheta(I)$. Observe that, as follows easily from the definition, $\vartheta$ preserves order. Thus, if $I, J \in \mathfrak{J}L$, then $\vartheta(I \cap J) \subseteq \vartheta(I) \cap \vartheta(J)$. Now, let $x \in \vartheta(I) \cap \vartheta(J)$. Pick $i \in I$ and $j \in J$ such that $x \in \mathsf{T}(i)$ and $x \in \mathsf{T}(j)$. Then $x \in \mathsf{T}(i) \cap \mathsf{T}(j)$, so that $x \in \mathsf{T}(i \wedge j)$ by Lemma 5.1(d). Since $i \wedge j$ is an element of $I \cap J$, it follows that $x \in \vartheta(I \cap J)$. Consequently, $\vartheta(I \cap J) = \vartheta(I) \cap \vartheta(J)$, and so $\vartheta$ preserves binary meets.

Next, we show that $\vartheta(\vartheta(I)) \subseteq \vartheta(I)$, whence it will follows that $\vartheta$ is idempotent as it is order preserving. So, let $x \in \vartheta(\vartheta(I))$. Then there exists some $w \in \vartheta(I)$ such that $x$ is tied to $w$. Pick $u \in \mathfrak{u}(L)$ such that $x \wedge u \leq w$. Since $w \in \vartheta(I)$, there is an $i \in I$ such that $w$ is tied to $i$. Pick $v \in \mathfrak{u}(L)$ such that $w \wedge v \leq i$. Now, $u \wedge v$ is a unit of $L$ such that

$$x \wedge (u \wedge v) = (x \wedge u) \wedge v \leq w \wedge v \leq i,$$

showing that $w$ is tied to $i$, so that $w \in \vartheta(I)$, as desired. Therefore $\vartheta(\vartheta(I)) \subseteq \vartheta(I)$, and so $\vartheta(\vartheta(I)) = \vartheta(I)$. In all then, $\vartheta$ is a nucleus on $\mathfrak{J}L$. In light of Lemma 5.1(b), it is clear that $\vartheta(\{0\}) = \{0\}$, and so $\vartheta$ is a dense nucleus.

Finally, we show that it is inductive. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a directed family of ideals of $L$. Since $\vartheta$ is order-preserving,

$$\bigcup\{\vartheta(I_\lambda) \mid \lambda \in \Lambda\} \subseteq \vartheta\left(\bigcup_{\lambda\in\Lambda} I_\lambda\right).$$

On the other hand, let $z \in \vartheta\left(\bigcup_{\lambda\in\Lambda} I_\lambda\right)$. Then there is a $t \in \bigcup_{\lambda\in\Lambda} I_\lambda$ such that $z$ is tied to $t$. There is an index $\lambda_0 \in \Lambda$ such that $t \in I_{\lambda_0}$, and so $z \in \vartheta(I_{\lambda_0})$. From this we deduce that

$$\vartheta\left(\bigcup_{\lambda\in\Lambda} I_\lambda\right) \subseteq \bigcup\{\vartheta(I_\lambda) \mid \lambda \in \Lambda\}.$$

It follows therefore that $\vartheta$ preserves directed joins, which is to say it is inductive. □

By applying the result of Martìnez and Zenk [24] that we cited earlier about $\mathrm{Fix}(k)$, for an inductive nucleus $k$, we deduce that $\mathrm{Fix}(\vartheta)$ is an algebraic frame with the FIP because $\mathfrak{J}A$ is a coherent frame for any bounded distributive lattice $A$. By [1, Lemma 2], $\mathrm{Fix}(\vartheta)$ is compact. In all, then, we have the following corollary.

**Corollary 5.4.** $\mathrm{Fix}(\vartheta)$ *is a coherent frame.*

We are now going to characterize the elements of the frame $\mathrm{Fix}(\vartheta)$, that is, the ideals of $L$ that are fixed by $\vartheta$. Recall that, for any bounded distributive lattice $A$, the set of compact elements of $\mathfrak{J}A$ is

$$\mathfrak{k}(\mathfrak{J}A) = \{{\downarrow}a \mid a \in A\}.$$

It is thus of benefit to know how the principal ideals of $L$ are mapped by $\vartheta$. Let $x, a \in L$. The calculation

$$\begin{aligned} x \in \vartheta({\downarrow}a) \quad &\Longleftrightarrow \quad x \ltimes t \text{ for some } t \leq a \\ &\Longleftrightarrow \quad x \ltimes a \\ &\Longleftrightarrow \quad x \in \mathsf{T}(a) \end{aligned}$$

shows that

$$\vartheta({\downarrow}a) = \mathsf{T}(a).$$

This, of course, also follows from the definition of $\vartheta$ and Lemma 5.1(c). Emanating from this is that, for any $a \in L$,

$$\vartheta(\mathsf{T}(a)) = \vartheta(\vartheta({\downarrow}a)) = \vartheta({\downarrow}a) = \mathsf{T}(a).$$

Let us call an ideal $J$ of $L$ a $\mathsf{T}$-*ideal* if it is of the form $\mathsf{T}(x)$ for some $x \in L$. Thus, every $\mathsf{T}$-ideal of $L$ is fixed by $\vartheta$. It is on the basis of this that we are able to characterize the fixed elements of $\vartheta$, in the following theorem.

**Theorem 5.5.** *An ideal of $L$ is fixed by $\vartheta$ iff it is a directed union of $\mathsf{T}$-ideals.*

*Proof.* Suppose that $I$ is an ideal of $\mathfrak{J}L$ with $I = \bigcup_\alpha \mathsf{T}(a_\alpha)$, where the set $\{\mathsf{T}(a_\alpha)\}$ is directed. Since $\vartheta$ preserves directed unions (as it is an inductive nucleus),

$$\vartheta(I) = \vartheta\Big(\bigcup_\alpha \mathsf{T}(a_\alpha)\Big) = \bigcup_\alpha \vartheta(\mathsf{T}(a_\alpha)) = \bigcup_\alpha \mathsf{T}(a_\alpha) = I,$$

showing that $I \in \mathrm{Fix}(\vartheta)$.

Conversely, suppose that $J = \vartheta(J)$. The set $\{\downarrow\! a \mid a \in J\}$ is directed, and, by Lemma 5.1(c), the set $\{\mathsf{T}(a) \mid a \in J\}$ is also directed. Since $J = \bigcup_{a\in J}\downarrow\! a$ (a directed union), we have

$$J = \vartheta(J) = \vartheta\Big(\bigcup_{a\in J} \downarrow\! a\Big) = \bigcup_{a\in J} \vartheta(\downarrow\! a) = \bigcup_{a\in J} \mathsf{T}(a),$$

which shows that $J$ is a directed union of $\mathsf{T}$-ideals. ☐

Recall that an ideal $I$ of a bounded distributive lattice $A$ is called *prime* if $I \neq L$ and for any $x, y \in A$, the containment $x \wedge y \in I$ implies $x \in I$ or $y \in I$. In [17, Lemma II.3.4], Johnstone shows that the prime elements of $\mathfrak{J}A$ are precisely the prime ideals of $A$. Since $\mathrm{Fix}(\vartheta)$ is a sublocale of $\mathfrak{J}L$, the prime elements of $\mathrm{Fix}(\vartheta)$ are the prime ideals of $L$ that are fixed by $\vartheta$.

In light of Theorem 5.5, the prime elements of $\mathrm{Fix}(\vartheta)$ are precisely the prime ideals of $L$ that are directed unions of $\mathsf{T}$-ideals. We show that among them are the minimal prime ideals of $L$. Of course, as in rings, a *minimal prime ideal* of a lattice is one that is a minimal element in the poset (ordered by inclusion) of prime ideals of the lattice. We draw the attention of the reader to the fact that, by an argument similar to that on page 169 of Gratzer's book [13],

> *A prime ideal of a frame is minimal prime iff it contains no dense element of the frame.*

Now, here is the result that identifies some prime elements of $\mathrm{Fix}(\vartheta)$.

**Proposition 5.6.** *Every minimal prime ideal of $L$ is a prime element of* $\mathrm{Fix}(\vartheta)$.

*Proof.* Let $P$ be a minimal prime ideal of $L$. We show that $\vartheta(P) \subseteq P$, whence it will follow that $\vartheta(P) = P$, as required. Let $x \in \vartheta(P)$. Pick $u \in \mathfrak{u}(L)$ and $p \in P$ such that $x \wedge u \leq p$. Then $x \wedge u \in P$, and since $P$ is a prime ideal, $x \in P$ or $u \in P$. The latter is not possible because $u$ is a dense element and minimal prime ideals contain no dense elements. Therefore $x \in P$, showing that $\vartheta(P) \subseteq P$, and hence $\vartheta(P) = P$. Therefore $P \in \mathrm{Pr}(\mathrm{Fix}(\vartheta))$, as claimed. □

A remark emanating from the proof of this proposition is in order.

*Remark* 5.7. Recall that an element $r$ of a ring $R$ is said to be prime to an ideal $I$ of $R$ if, for any $x \in R$, $rx \in I$ implies $x \in I$. We may extend this usage to lattices, and say an element $a$ of a lattice $A$ is *prime to* an ideal $J$ of $A$ if, for any $y \in A$, the containment $a \wedge y \in J$ implies $y \in J$. Now, exactly as in the proof of Proposition 5.6, we see that if $J$ is an ideal of $L$ such that every unit of $L$ is prime to $J$, then $J \in \mathrm{Fix}(\vartheta)$.

As in [28], given a nucleus $j$ on a frame $L$, we write $\nabla(j)$ for the filter of elements that $j$ sends to the top. That is

$$\nabla(j) = \{a \in L \mid j(a) = 1\}.$$

Of course, a nucleus $j$ is codense if and only if $\nabla(j) = \{1\}$.

**Proposition 5.8.** *For any $L$, $\nabla(\vartheta) = \{J \in \mathfrak{J}L \mid J$ contains a unit of $L\}$. That is, for every $J \in \mathfrak{J}L$,*

$$\vartheta(J) = L \iff J \text{ contains a unit of } L.$$

*Proof.* Suppose that $J$ is an ideal of $L$ with $\vartheta(J) = L$. Then $L = \bigcup_{a \in J} \mathsf{T}(a)$, which implies that there exists an element $j \in J$ such that $1 \in \mathsf{T}(j)$. Therefore there exists some $u \in \mathfrak{u}(L)$ such that $1 \wedge u \leq j$, which implies $u \leq j$, and so $u \in J$, showing that $J$ contains a unit of $L$.

Conversely, suppose that $I$ is an ideal of $L$ containing a unit, say $u$, of $L$. Since $1 \wedge u = u$, it follows that 1 is tied to $u$, so that $1 \in \vartheta(I)$, implying that $\vartheta(I) = L$ because $\vartheta(I)$ is an ideal of $L$. □

We saw in Proposition 5.6 that, for any $L$, every minimal prime ideal of $L$ is a prime element of $\mathrm{Fix}(\vartheta)$. In [10], we defined an algebraic frame $L$ to be *zipped* if every dense element of $L$ is above some dense compact element of $L$. The choice of name was based on the fact that these algebraic frames are generalizations of what are called zip-rings.

Combining the results in Proposition 5.6 and Proposition 5.8, we are able to identify the prime elements of $\mathrm{Fix}(\vartheta)$ if $L$ is a zipped. Observe that if $L$

is zipped, then it possesses a unit because $x \vee x^{\perp}$ is a dense element for any $x \in L$.

**Corollary 5.9.** *If $L$ is a zipped algebraic frame, then the primes of* $\mathrm{Fix}(\vartheta)$ *are precisely the minimal prime ideals of $L$.*

*Proof.* We need to show that every prime ideal of $L$ which belongs to $\mathrm{Fix}(\vartheta)$ is a minimal prime ideal of $L$. So, let $P$ be a prime ideal of $L$ such that $\vartheta(P) = P$. Since $P \neq L$, Proposition 5.8 tells us that $P$ does not contain a unit of $L$. Since $L$ is zipped, $P$ cannot contain a dense element of $L$, because if it did, then any unit below that dense element would also belong to $P$ (as $P$ is an ideal), and so, by Proposition 5.8, we would have $P = L$. It follows therefore that the prime elements of $\mathrm{Fix}(\vartheta)$ are exactly the minimal prime ideals of $L$. □

For zipped algebraic frames we show that the Banaschewski-Sioen nucleus on $\mathfrak{J}L$ coincides with $\vartheta$. To do this we first take cognizance of some facts; including some we believe are known.

**Lemma 5.10.** *Let $L$ be an arbitrary frame and denote pseudocomplementation in $L$ by $(-)^*$. Then, for any $a \in L$, the pseudocomplement of $\downarrow a$ in $\mathfrak{J}L$ is given by $(\downarrow a)^{\perp} = \downarrow(a^*)$.*

*Proof.* Since $\downarrow a \wedge \downarrow(a^*) = \downarrow(a \wedge a^*) = \{0\}$, we need only show that if $H \cap \downarrow a = \{0\}$ for some $H \in \mathfrak{J}L$, then $H \subseteq \downarrow(a^*)$. For any $y \in H$, $y \wedge a$ belongs both to $H$ and $\downarrow a$, and so $y \wedge a = 0$, which implies $y \leq a^*$, whence $H \subseteq \downarrow(a^*)$. □

As the referee has remarked, the set $\mathrm{Idl}(M)$, which consists of all ideals of a distributive lattice $M$, is recognized as a pseudocomplemented distributive lattice. For any $I \in \mathrm{Idl}(M)$, the pseudocomplemented $I^{\perp}$ is defined as: $I^{\perp} = \{a \in M \mid a \wedge i = 0, \text{ for all } i \in I\}$. This supports the correctness of the foregoing lemma. Since the compact elements of $\mathfrak{J}L$ are precisely the principal ideals of $L$, we deduce the following from the foregoing lemma.

**Proposition 5.11.** *For any frame $L$,*

$$\mathfrak{u}(\mathfrak{J}L) = \{\downarrow d \mid d \text{ is a dense element of } L\}.$$

Next, we observe that if $k$ and $\ell$ are inductive nuclei on an algebraic frame $L$ such that $k(c) \leq \ell(c)$ for every $c \in \mathfrak{k}(L)$, then $k \leq \ell$. Indeed, let $a \in L$, and put $C = \{c \in \mathfrak{k}(L) \mid c \leq a\}$. Note that $C$ is a directed set. Since $a = \bigvee C$ as

$L$ is algebraic, and since $k$ and $\ell$ preserve directed joins, we have

$$\begin{aligned} k(a) = k\big(\bigvee C\big) &= \bigvee\{k(c) \mid c \in C\} \leq \bigvee\{\ell(c) \mid c \in C\} \\ &= \ell\big(\bigvee C\big) = \ell(a), \end{aligned}$$

showing that $k \leq \ell$ since $a$ is an arbitrary element of $L$. Hence,

> *if $k$ and $\ell$ are inductive nuclei on an algebraic frame $L$ and they agree on the compact elements of $L$, then they are equal.*

**Theorem 5.12.** *For any zipped algebraic frame $L$, the Banaschewski-Sioen nucleus on $\mathfrak{J}L$ coincides with $\vartheta$.*

*Proof.* Let us simply write $\kappa\colon \mathfrak{J}L \to \mathfrak{J}L$ for the Banaschewski-Sioen nucleus on $\mathfrak{J}L$, and write $\mathfrak{d}(L)$ for the set of dense elements of $L$. By what we have observed above, to prove that $\kappa = \vartheta$, we need only compare these nuclei at the compact elements of their domain. Let then $a$ be an element of $L$, and consider the compact element ${\downarrow}a$ of $\mathfrak{J}L$. By the definition of $\kappa$, and taking into account the result in Lemma 5.10,

$$\begin{aligned} \kappa({\downarrow}a) &= \bigvee\{{\downarrow}c \mid c \in L \text{ and } {\downarrow}c \wedge {\downarrow}d \subseteq {\downarrow}a \text{ for some } d \in \mathfrak{d}(L)\} \\ &= \bigvee\{{\downarrow}c \mid c \in L \text{ and } {\downarrow}(c \wedge d) \subseteq {\downarrow}a \text{ for some } d \in \mathfrak{d}(L)\} \\ &= \bigvee\{{\downarrow}c \mid c \in L \text{ and } c \wedge d \leq a \text{ for some } d \in \mathfrak{d}(L)\}. \end{aligned}$$

Now consider the sets

$$\{c \in L \mid c \wedge d \leq a \text{ for some } d \in \mathfrak{d}(L)\}$$

and

$$\{c \in L \mid c \wedge u \leq a \text{ for some } u \in \mathfrak{u}(L)\}.$$

Since units are dense elements, the set on the right is clearly contained in the one on the left. Let $c$ be in the set on the left, and pick a dense element $d$ such that $c \wedge d \leq a$. Since $L$ is zipped, there is a unit $u \leq d$. Then $c \wedge u \leq a$, showing that the set on the left is also contained in the one on the right. Hence they are equal, and so

$$\begin{aligned} \kappa({\downarrow}a) &= \bigvee\{{\downarrow}c \mid c \in L \text{ and } c \wedge u \leq a \text{ for some } u \in \mathfrak{u}(L)\} \\ &= \bigvee\{{\downarrow}c \mid c \in \mathsf{T}(a)\} \\ &\subseteq \mathsf{T}(a) \qquad \text{since } \mathsf{T}(a) \text{ is an ideal of } L. \end{aligned}$$

Note that $\mathsf{T}(a) \subseteq \bigvee\{\downarrow c \mid c \in \mathsf{T}(a)\}$, as a consequence of which $\kappa(\downarrow a) = \mathsf{T}(a) = \vartheta(\downarrow a)$. It follows that $\kappa$ and $\vartheta$ agree on compact elements of $\mathfrak{J}L$, and so (each being an inductive nucleus) they are equal. □

## 6. More on the sets $\mathsf{K}(a)$, $\widehat{\mathsf{K}}(a)$, and $\mathsf{T}(a)$

We commented above (just prior to Lemma 4.6) that, for any $a \in L$, $\mathsf{K}(a) = \mathsf{T}(a) \cap \mathfrak{k}(L)$. We actually have more. We list in the following proposition the containments and equalities that exist among these sets.

**Proposition 6.1.** *Let $L$ be an algebraic frame with the FIP and possessing units.*

(a) *For any $a \in L$,*

$$\mathsf{K}(a) \subseteq \widehat{\mathsf{K}}(a) \subseteq \mathsf{T}(a) \qquad \textit{and} \qquad \mathsf{K}(a) = \widehat{\mathsf{K}}(a) \cap \mathfrak{k}(L) = \mathsf{T}(a) \cap \mathfrak{k}(L).$$

(b) *If $L$ is Noetherian, then $\mathsf{K}(a) = \widehat{\mathsf{K}}(a) = \mathsf{T}(a)$ for every $a \in L$.*
(c) *$L$ is Noetherian iff $\mathsf{K}(a) = \mathsf{T}(a)$ for every $a \in L$.*

*Proof.* (a) The containments follow easily from the definitions. Since $\mathsf{K}(a) = \mathsf{T}(a) \cap \mathfrak{k}(L)$, the two asserted equalities in the statement of (a) follow upon intersecting every set in the containments with $\mathfrak{k}(L)$.

(b) This holds from the equalities in (a) because if $L$ is Noetherian, then $\mathfrak{k}(L) = L$.

(c) The "only if" part is part of (b). Conversely, suppose $\mathsf{K}(a) = \mathsf{T}(a)$ for every $a \in L$. Let $x \in L$. Since $x \in \mathsf{T}(x)$, as noted in Lemma 5.1, we then have $x \in \mathsf{K}(x)$. But every element of $\mathsf{K}(x)$ is compact, therefore $x$ is compact, and so we deduce that $L$ is Noetherian. □

Consider $\mathsf{K}$ and $\widehat{\mathsf{K}}$ as functions mapping out of $L$ into the frame of ideals of $\mathfrak{k}(L)$ and the frame of ideals of $L$, respectively. We show that the map $\widehat{\mathsf{K}}\colon L \to \mathfrak{J}L$ is a zero-preserving meet-semilattice homomorphism, that it preserves directed joins, that it preserves the top if $L$ is coherent, and that it is a Heyting homomorphism if $L$ is Noetherian. We separate the results because the latter requires some background.

**Theorem 6.2.** *Let $L$ be an algebraic frame with the FIP and possessing units. Then the mappings $\widehat{\mathsf{K}}\colon L \to \mathfrak{J}L$ and $\mathsf{K}\colon L \to \mathfrak{J}(\mathfrak{k}L)$ have the following properties:*

(a) *They preserve the bottom, binary meets, and directed joins.*
(b) *They preserve the top if $L$ is a coherent frame.*

*Proof.* We prove the result for the map $\widehat{\mathsf{K}}$, and remark that minor modification of the proofs we shall present shows that the result holds for the map $\mathsf{K}\colon L \to \mathfrak{J}(\mathfrak{k}L)$. (a) Let $x \in \widehat{\mathsf{K}}(0)$. Pick $c \in \mathfrak{k}(L)$ and $u \in \mathfrak{u}(L)$ such that $x \leq c$ and $c \wedge u \leq 0$. Since $u$ is a dense element in $L$, this implies $c = 0$, and hence $x = 0$. Therefore $\widehat{\mathsf{K}}(0) = \{0\}$, showing that $\widehat{\mathsf{K}}$ sends the bottom of $L$ to the bottom of $\mathfrak{J}L$.

To show that $\widehat{\mathsf{K}}$ preserves binary meets, we first note that, as follows immediately from the definition, if $x \leq y$ in $L$ then $\widehat{\mathsf{K}}(x) \subseteq \widehat{\mathsf{K}}(y)$, so that $\widehat{\mathsf{K}}$ is order-preserving. Now let $a, b \in L$. Then $\widehat{\mathsf{K}}(a \wedge b) \subseteq \widehat{\mathsf{K}}(a) \cap \widehat{\mathsf{K}}(b)$. To reserve the inclusion, let $x \in \widehat{\mathsf{K}}(a) \cap \widehat{\mathsf{K}}(b)$. Pick $c, d \in \mathfrak{k}(L)$ and $u, v \in \mathfrak{u}(L)$ such that

$$x \leq c, \quad c \wedge u \leq a \qquad \& \qquad x \leq d, \quad d \wedge v \leq b.$$

Then,

$$\begin{aligned}(c \wedge d) \wedge (u \wedge v) &= (c \wedge d \wedge u) \wedge (c \wedge d \wedge v) \\ &\leq (c \wedge u) \wedge (d \wedge v) \\ &\leq a \wedge b.\end{aligned}$$

Since $c \wedge d \in \mathfrak{k}(L)$, $u \wedge v \in \mathfrak{u}(L)$, and $x \leq c \wedge d$, it follows that $x \in \widehat{\mathsf{K}}(a \wedge b)$, and thus $\widehat{\mathsf{K}}(a) \cap \widehat{\mathsf{K}}(b) \subseteq \widehat{\mathsf{K}}(a \wedge b)$. Therefore $\widehat{\mathsf{K}}$ preserves binary meets.

Finally, let $D$ be a directed subset of $L$. Observe that, since $\widehat{\mathsf{K}}$ is order-preserving, the collection $\{\widehat{\mathsf{K}}(d) \mid d \in D\}$ of ideals of $L$ is directed, and so its join in $\mathfrak{J}L$ equals $\bigcup_{d\in D}\widehat{\mathsf{K}}(d)$. Again, since $\widehat{\mathsf{K}}$ is order-preserving, we immediately have

$$\bigcup_{d\in D}\widehat{\mathsf{K}}(d) \subseteq \widehat{\mathsf{K}}\big(\bigvee D\big).$$

To see the reverse inclusion, let $x \in \widehat{\mathsf{K}}(\bigvee D)$. Find $c \in \mathfrak{k}(L)$ and $u \in \mathfrak{u}(L)$ such that $x \leq c$ and $c \wedge u \leq \bigvee D$. Since $c \wedge u$ is compact and $\bigvee D$ is a directed join, there is an element $d_0 \in D$ such that $c \wedge u \leq d_0$. Therefore

$$x \in \widehat{\mathsf{K}}(d_0) \subseteq \bigcup_{d\in D}\widehat{\mathsf{K}}(d),$$

proving the desired inclusion. We therefore have the claimed equality.

(b) If $L$ is coherent, then $1 \in \mathfrak{u}(L)$. So, from the inequality $1 \wedge 1 \leq 1$, we have that $1 \in \widehat{\mathsf{K}}(1)$, whence $\widehat{\mathsf{K}}(1) = L$. Since the top of $\mathfrak{J}L$ is $L$, this proves the result.

□

We claimed above that the map $\widehat{\mathsf{K}}\colon L \to \mathfrak{J}L$ is a Heyting homomorphism if $L$ is Noetherian. To prove this, we first recall from [7, Lemma 2.5] that, for any bounded distributive lattice $A$, the Heyting operation in $\mathfrak{J}A$ is given by

$$I \to J = \{a \in A \mid {\downarrow}a \cap I \subseteq J\}.$$

An examination of the proof in [7] shows that the same holds even if $A$ is a meet-semilattice.

Before we show that if $L$ is Noetherian, then $\widehat{\mathsf{K}}\colon L \to \mathfrak{J}L$ preserves the Heyting implication, let us first give an example to show that this map is not necessarily an isomorphism of frames even if $L$ is Noetherian. Of course, if $\widehat{\mathsf{K}}\colon L \to \mathfrak{J}L$ were an isomorphism of frames, the result would be trivial.

**Example 6.3.** Consider the three-element chain

$$\mathbf{3} = \{0, m, 1\}$$

with $0 < m < 1$. Then $\widehat{\mathsf{K}}(m) = \mathbf{3} = \widehat{\mathsf{K}}(1)$, showing that $\widehat{\mathsf{K}}\colon \mathbf{3} \to \mathfrak{J}\mathbf{3}$ is not one-one. It is also not codense.

For use below, observe that if $L$ is an algebraic frame with the FIP and possessing units, then $c \in \mathsf{K}(c)$ for every compact element $c$. This is so because, for any unit $u$ of $L$, $c \wedge u \leq c$. We saw in Proposition 6.1 that if $L$ is Noetherian, then $\widehat{\mathsf{K}}(-) = \mathsf{K}(-)$. We shall thus state the theorem using $\mathsf{K}$ instead of $\widehat{\mathsf{K}}$.

**Theorem 6.4.** *If $L$ is Noetherian, then $\mathsf{K}\colon L \to \mathfrak{J}L$ preserves the Heyting implication.*

*Proof.* Let $a, b \in L$. Then, in view of Theorem 6.2,

$$\mathsf{K}(a \to b) \wedge \mathsf{K}(a) = \mathsf{K}(a) \cap \mathsf{K}(a \to b) = \mathsf{K}(a \wedge (a \to b)) = \mathsf{K}(a \wedge b) \leq \mathsf{K}(b),$$

and so $\mathsf{K}(a \to b) \subseteq \mathsf{K}(a) \to \mathsf{K}(b)$. For the opposite inequality, let $c \in \mathsf{K}(a) \to \mathsf{K}(b)$. Then ${\downarrow}c \cap \mathsf{K}(a) \subseteq \mathsf{K}(b)$. Since $a$ is compact, $a \in \mathsf{K}(a)$, hence $c \wedge a \in {\downarrow}c \cap \mathsf{K}(a)$, and therefore $c \wedge a \in \mathsf{K}(b)$. There is therefore a unit $u \in \mathfrak{u}(L)$ such that $c \wedge a \wedge u \leq b$. This implies $c \wedge u \leq a \to b$ hence $c \in \mathsf{K}(a \to b)$, and so $\mathsf{K}(a) \to \mathsf{K}(b) \subseteq \mathsf{K}(a \to b)$. In all, then, $\mathsf{K}(a \to b) = \mathsf{K}(a) \to \mathsf{K}(b)$, as required. ☐

It is clear from the presented argument that the containment

$$\mathsf{K}(a \to b) \subseteq \mathsf{K}(a) \to \mathsf{K}(b)$$

holds even if $L$ is not Noetherian.

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# THE BANASCHEWSKI–SIOEN NUCLEUS ON AN ALGEBRAIC FRAME

S. BLOSE AND T. DUBE

# هسته‌ی باناچفسکی–سیون بر روی یک قاب جبری

سیفامندلا بلوس[۱] و تمبا دوب[۲]

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در ساخت فشرده‌سازی استون–چخ با استفاده‌ی صرف از ایده‌آل‌های حلقه (و در مقابل $\ell$-ایده‌آل‌ها)، باناچفسکی و سیون در [۶] هسته‌ای خاص را بر روی قاب کوهرنت $\mathrm{RId}(A)$، از ایده‌آل‌های رادیکال یک حلقه‌ی جابه‌جایی $A$ را تعریف کردند. در این مقاله، ما این هسته را به هر قاب جبری‌ای که دارای یک عنصر فشرده‌ی چگال می‌باشد و در آن اشتراک هر دو عنصر فشرده نیز فشرده است، تعمیم می‌دهیم. بدیهی است که هر قاب جبری با این ویژگی‌ها الزاماً کوهرنت نیست؛ از این‌رو، این تعمیم حقیقتاً یک تعمیم اصیل است. سپس برخی از ویژگی‌ها و خواص این هسته را که در مقاله‌ی [۶] بررسی نشده‌اند، مطالعه می‌کنیم.

کلمات کلیدی: قاب جبری، حلقه‌ی جابه‌جایی، هسته، زیرلوکال.