

ALGEBRAIC CONDITIONS WITH HOMODERIVATIONS UNDER WHICH NEAR RINGS ARE RINGS

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ABSTRACT. An additive mapping $\Gamma : \mathfrak{N} \rightarrow \mathfrak{N}$ is called a homoderivation on a near ring $\mathfrak{N}$ if $\Gamma(uv) = \Gamma(u)\Gamma(v) + \Gamma(u)v + u\Gamma(v)$ for all $u, v \in \mathfrak{N}$. The aim of the present paper is to explore the following algebraic identities: (i) $\Gamma([j, u]) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, (ii) $\Gamma([j, u]) + [j, u] = (j \circ u)$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, (iii) $\Gamma(j \circ u) + (j \circ u) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, (iv) $\Gamma([j, n]) = 0$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$, (v) $\Gamma(jn) + jn = [j, n]$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$, (vi) $\Gamma(jn) + jn = (j \circ n)$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$, where $\mathfrak{J}$ is a nonzero Jordan ideal of $\mathfrak{N}$ and $\mathfrak{U}$, a nonzero semigroup ideal of $\mathfrak{N}$. In addition, we present examples to support the hypotheses of our results.

1. INTRODUCTION

A left near ring $\mathfrak{N}$ is a group that is additive and has a binary operation $\cdot$ such that (i) $(\mathfrak{N}, \cdot)$ is a semigroup (ii) $z \cdot (x + y) = z \cdot x + z \cdot y$ for all $x, y, z \in \mathfrak{N}$. If the right distributive law holds instead of (ii), then $\mathfrak{N}$ is called a right near ring. Remember that a near ring $\mathfrak{N}$ is said to be zero-symmetric if $0x = 0$ for all $x \in \mathfrak{N}$ (left distributive law gives that $x0 = 0$). Throughout the paper, $\mathfrak{N}$ is used to represent a zero-symmetric left near ring and $\mathfrak{Z}(\mathfrak{N}) = \{x \in \mathfrak{N} \mid xy = yx \text{ for all } y \in \mathfrak{N}\}$ is the multiplicative center of $\mathfrak{N}$. For every $x, y \in \mathfrak{N}$, $(x \circ y)$ and $[x, y]$ denote the well known Jordan product $xy + yx$ and Lie product $xy - yx$ respectively. A near ring $\mathfrak{N}$ is said to be prime if for every $x, y \in \mathfrak{N}$, $x\mathfrak{N}y = \{0\}$ indicates that either $x = 0$ or $y = 0$. A non-void subset $\mathfrak{U}$ of a near ring $\mathfrak{N}$ is known as right semigroup (resp. left semigroup) ideal of $\mathfrak{N}$ if $\mathfrak{U}\mathfrak{N} \subseteq \mathfrak{U}$ (resp. $\mathfrak{N}\mathfrak{U} \subseteq \mathfrak{U}$). Generally, $\mathfrak{U}$ is called a semigroup ideal of $\mathfrak{N}$ if $\mathfrak{U}$ is right and left semigroup ideal of $\mathfrak{N}$. An additive subgroup $\mathfrak{J}$ of a near ring $\mathfrak{N}$ is said to be a Jordan ideal of $\mathfrak{N}$ if $j \circ n \in \mathfrak{J}$ and $n \circ j \in \mathfrak{J}$ for all $j \in \mathfrak{J}$ and $n \in \mathfrak{N}$. For $\mathfrak{S} \subseteq \mathfrak{N}$, a mapping $\xi : \mathfrak{N} \rightarrow \mathfrak{N}$ is said to be zero power valued on $\mathfrak{S}$ if for every $u \in \mathfrak{S}$, there exists a positive integer $k(u) > 1$ such that $\xi^{k(u)}(u) = 0$. A mapping $\xi : \mathfrak{N} \rightarrow \mathfrak{N}$ preserves $\mathfrak{S}$ if $\xi(\mathfrak{S}) \subseteq \mathfrak{S}$.

MSC(2020): Primary: 16N60; Secondary: 16W25, 16Y30.

Keywords: Homoderivation; Prime near ring; Jordan ideal; Semigroup ideal.

Received: 13 May 2024, Accepted: 4 March 2025.

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Bell and Mason [7] established the convention of derivation in near rings. A derivation on $\mathfrak{N}$ is defined as an additive mapping $d : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathfrak{N}$ or alternatively, as stated in [19], $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathfrak{N}$. Numerous results on the commutativity of prime and semiprime rings and near rings with specific constraints on derivations, generalized derivations, semiderivations and (σ, τ) -derivations have been examined by a large number of authors (for reference see [1], [2], [3], [4], [9], [10], [12], [14], [15], [17] and [18]).

El Sofy [13] established the concept of homoderivations in rings as follows: An additive mapping $\Gamma : \mathfrak{R} \rightarrow \mathfrak{R}$ is called a homoderivation on a ring $\mathfrak{R}$ if $\Gamma(uv) = \Gamma(u)\Gamma(v) + \Gamma(u)v + u\Gamma(v)$ for all $u, v \in \mathfrak{R}$. Also, El Sofy [13] proved the following result: Let $\mathfrak{R}$ be a prime ring and $\mathfrak{U}$, a nonzero two sided ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a homoderivation Γ such that $\Gamma([u, v]) = \pm[u, v]$ for all $u, v \in \mathfrak{U}$, then $\mathfrak{R}$ is commutative. Further many authors investigated the commutativity on different subsets of prime ring and $*$ -prime ring admitting a homoderivation.

In [16], the authors proved the following result: Let $\mathfrak{R}$ be a prime ring and $\mathfrak{U}$, a nonzero ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a homoderivation Γ such that $\Gamma([u, v]) = 0$ for all $u, v \in \mathfrak{U}$, then $\mathfrak{R}$ is commutative. In [5] A. Al-Kenani et al. extended these results for nonzero $*$ ideal $\mathfrak{U}$ of a $*$ -prime ring via homoderivation Γ which commutes with $*$ and satisfies one of the following assertions: (i) $\Gamma([u, v]) = 0$, (ii) $\Gamma(u \circ v) = 0$, (iii) $\Gamma([u, v]) = [u, v]$ and (iv) $\Gamma(u \circ v) = (u \circ v)$ for all $u, v \in \mathfrak{U}$. Recently, Boua [8] defined the notion of homoderivations in near rings and proved the commutativity of prime near rings by considering the appropriate constraints on homoderivations. Also, Boua [8] investigated the following algebraic identities: (i) $\Gamma(u \circ v) = 0$, (ii) $\Gamma(u \circ v) = (u \circ v)$, (iii) $\Gamma([u, v]) = (u \circ v)$ for all $u, v \in \mathfrak{N}$.

In this line of investigation, it is interesting to develop these results on some appropriate subsets of near rings.

2. SOME PRELIMINARY RESULTS

We will require the following lemmas in order to establish the proofs of our primary conclusions.

Lemma 2.1. [6, Lemma 1.3 and Lemma 1.4] Let $\mathfrak{N}$ be a prime near ring and $\mathfrak{U}$, a nonzero semigroup ideal of $\mathfrak{N}$.

- (i) If $u, v \in \mathfrak{N}$ and $u\mathfrak{U}v = \{0\}$, then $u = 0$ or $v = 0$.
- (ii) If $u \in \mathfrak{N}$ and $u\mathfrak{U} = \{0\}$ or $\mathfrak{U}u = \{0\}$, then $u = 0$.

Lemma 2.2. [6, Lemma 1.5] Let $\mathfrak{N}$ be a prime near ring and $\mathfrak{U}$, a nonzero left (or right) semigroup ideal contained in $\mathfrak{Z}(\mathfrak{N})$, then $\mathfrak{N}$ is a commutative ring.

Lemma 2.3. [11, Lemma 1] Let $\mathfrak{N}$ be a 2-torsion free prime near ring and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$ such that $\mathfrak{J} \subseteq \mathfrak{Z}(\mathfrak{N})$, then $\mathfrak{N}$ is a commutative ring.

Lemma 2.4. [8, Theorem 4.9] Let $\mathfrak{N}$ be a 2-torsion free prime near ring and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If $\mathfrak{N}$ admits a nonzero homoderivation Γ such that $\Gamma(\mathfrak{J}) = \{0\}$, then $\mathfrak{J}$ is commutative.

Lemma 2.5. [8, Lemma 2.4] Let $\mathfrak{N}$ be a 2-torsion free prime near ring admitting a homoderivation Γ such that $\Gamma^2(\mathfrak{N}) = \{0\}$, then $\Gamma = 0$.

Lemma 2.6. [8, Lemma 2.5] Let $\mathfrak{N}$ be a near ring and Γ , a homoderivation on $\mathfrak{N}$, then

$$\Gamma(uv)(\Gamma(w) + w) = \Gamma(u)\Gamma(v)(\Gamma(w) + w) + \Gamma(u)v(\Gamma(w) + w) + u\Gamma(v)(\Gamma(w) + w)$$

for all $u, v, w \in \mathfrak{N}$.

3. COMMUTATIVITY OF PRIME NEAR RINGS ADMITTING HOMODERIVATION WITH JORDAN IDEAL

Inspired by the results appeared in [2], [3], [8], [9], [10], [11] and [18], the aim of the study is to extend these results in the combining context of Jordan ideal and semigroup ideal of a prime near ring admitting a homoderivation.

Theorem 3.1. *Let $\mathfrak{N}$ be a 2-torsion free prime near ring, $\mathfrak{U}$ a nonzero semigroup ideal of $\mathfrak{N}$ and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If Γ is a nonzero homoderivation on $\mathfrak{N}$ which preserves $\mathfrak{U}$ and is zero power valued on $\mathfrak{N}$ satisfying $\Gamma([j, u]) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, then $\mathfrak{N}$ is a commutative ring.*

Proof. Suppose that

$$\Gamma([j, u]) = [j, u], \quad \forall j \in \mathfrak{J}, u \in \mathfrak{U}. \quad (3.1)$$

Replacing u by ju in (3.1) with the fact that $[j, ju] = j[j, u]$, we find that

$$\begin{aligned} j[j, u] &= \Gamma(j[j, u]) \\ &= \Gamma(j)\Gamma([j, u]) + \Gamma(j)[j, u] + j\Gamma([j, u]) \\ &= \Gamma(j)[j, u] + \Gamma(j)[j, u] + j[j, u], \end{aligned}$$

which reduces to $2\Gamma(j)[j, u] = 0$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$. Since $\mathfrak{N}$ is 2-torsion free, so we have $\Gamma(j)[j, u] = 0$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$. The last relation yields that

$$\Gamma(j)ju = \Gamma(j)uj, \quad \forall j \in \mathfrak{J}, u \in \mathfrak{U}. \quad (3.2)$$

Substituting un for u in (3.2), where $n \in \mathfrak{N}$ and using (3.2) again, we obtain $\Gamma(j)\mathfrak{U}[j, n] = \{0\}$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. In view of Lemma 2.1(i), we get

$$\Gamma(j) = 0 \quad \text{or} \quad j \in \mathfrak{Z}(\mathfrak{N}), \quad \forall j \in \mathfrak{J}. \quad (3.3)$$

Suppose that there exists an element $j_0 \in \mathfrak{J}$ such that $\Gamma(j_0) = 0$, then by our hypothesis $[j_0, u] = \Gamma(j_0u) - \Gamma(uj_0) = j_0\Gamma(u) - \Gamma(u)j_0 = [j_0, \Gamma(u)]$ for all $u \in \mathfrak{U}$. By the recurrence, we deduce that

$$[j_0, \Gamma^l(u)] = [j_0, u], \quad \forall u \in \mathfrak{U}, l \in \mathbb{N}^*. \quad (3.4)$$

Since Γ is zero power valued homoderivation on $\mathfrak{N}$, then there exists an integer $l(u) > 1$ such that $\Gamma^{l(u)}(u) = 0$. Putting $l(u)$ in place of l in (3.4), we get $[j_0, u] = 0$ for all $u \in \mathfrak{U}$. Replacing u by um for $m \in \mathfrak{N}$ in the preceding relation and using it again, we arrive at $\mathfrak{U}[j_0, m] = \{0\}$ for all $m \in \mathfrak{N}$. Invoking Lemma 2.1(ii), we get $j_0 \in \mathfrak{Z}(\mathfrak{N})$, therefore (3.3) gives that $\mathfrak{J} \subseteq \mathfrak{Z}(\mathfrak{N})$ and hence by Lemma 2.3, $\mathfrak{N}$ is a commutative ring. $\square$

Theorem 3.2. *Let $\mathfrak{N}$ be a 2-torsion free prime near ring, $\mathfrak{U}$ a nonzero semi-group ideal of $\mathfrak{N}$ and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If Γ is a nonzero homoderivation on $\mathfrak{N}$ satisfying $\Gamma([j, u]) + [j, u] = (j \circ u)$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, then $\mathfrak{J}$ is commutative.*

Proof. By the hypothesis

$$\Gamma([j, u]) + [j, u] = (j \circ u), \quad \forall j \in \mathfrak{J}, u \in \mathfrak{U}. \quad (3.5)$$

Substituting ju instead of u in (3.5) and using the definition of Γ , we obtain

$$\begin{aligned} j(j \circ u) &= \Gamma(j[j, u]) + j[j, u], \\ &= \Gamma(j)\Gamma([j, u]) + \Gamma(j)[j, u] + j\Gamma([j, u]) + j[j, u], \\ &= \Gamma(j)(\Gamma([j, u]) + [j, u]) + j(\Gamma([j, u]) + [j, u]). \end{aligned}$$

In view of (3.5), the above expression reduces to $\Gamma(j)(j \circ u) = 0$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, which can be written as $\Gamma(j)ju = -\Gamma(j)uj$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$. Replacing u by un in the last expression and using it again, we find that

$$\Gamma(j)jun = ((-\Gamma(j)uj)n) = ((\Gamma(j)u)(-j))n = ((\Gamma(j)un)(-j))$$

for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, $n \in \mathfrak{N}$. The preceding expression yields that $\Gamma(j)\mathfrak{U}[-j, n] = \{0\}$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. Since $\mathfrak{J}$ is an additive subgroup of $\mathfrak{N}$, then replacing j by $-j$ in the previous expression, we deduce that $\Gamma(-j)\mathfrak{U}[j, n] = \{0\}$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. An application of Lemma 2.1(i) gives that $\Gamma(-j) = 0$ or $j \in \mathfrak{Z}(\mathfrak{N})$ for all $j \in \mathfrak{J}$ which means that

$$\Gamma(j) = 0 \quad \text{or} \quad j \in \mathfrak{Z}(\mathfrak{N}), \quad \forall j \in \mathfrak{J}. \quad (3.6)$$

Suppose that there exists $j_0 \in \mathfrak{J}$ such that $j_0 \in \mathfrak{Z}(\mathfrak{N})$, then (3.5) gives that $2j_0u = 0$ for all $u \in \mathfrak{U}$. By 2-torsion freeness of $\mathfrak{N}$, we get $j_0u = 0$ for all $u \in \mathfrak{U}$. Applying Lemma 2.1(ii), we get $j_0 = 0$, a contraction. Therefore, (3.6) implies that $\Gamma(j) = 0$ for all $j \in \mathfrak{J}$ and hence $\mathfrak{J}$ is commutative by Lemma 2.4. $\square$

Theorem 3.3. *Let $\mathfrak{N}$ be a 2-torsion free prime near ring, $\mathfrak{U}$ a nonzero semi-group ideal of $\mathfrak{N}$ and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If Γ is a nonzero homoderivation on $\mathfrak{N}$ which preserves $\mathfrak{U}$ and is zero power valued on $\mathfrak{N}$ satisfying $\Gamma(j \circ u) + (j \circ u) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, then $\mathfrak{J}$ is commutative.*

Proof. By the assumption

$$\Gamma(j \circ u) + (j \circ u) = [j, u], \quad \forall j \in \mathfrak{J}, \quad u \in \mathfrak{U}. \quad (3.7)$$

Replacing u by ju in (3.7) and noting that $(j \circ ju) = j(j \circ u)$, $[j, ju] = j[j, u]$, we get

$$\begin{aligned} j[j, u] &= \Gamma(j(j \circ u)) + j(j \circ u), \\ &= \Gamma(j)\Gamma(j \circ u) + \Gamma(j)(j \circ u) + j\Gamma(j \circ u) + j(j \circ u), \\ &= \Gamma(j)(\Gamma(j \circ u) + (j \circ u)) + j(\Gamma(j \circ u) + (j \circ u)), \\ &= \Gamma(j)[j, u] + j[j, u], \quad \forall j \in \mathfrak{J}, \quad u \in \mathfrak{U}. \end{aligned}$$

The above expression implies that $\Gamma(j)ju = \Gamma(j)uj$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$. Substituting un for u in the preceding relation and using it again, we arrive at $\Gamma(j)u[j, n] = 0$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$, $n \in \mathfrak{N}$. In view of Lemma 2.1(i), we obtain

$$\Gamma(j) = 0 \quad \text{or} \quad j \in \mathfrak{Z}(\mathfrak{N}), \quad \forall j \in \mathfrak{J}. \quad (3.8)$$

Assume that there exists $j_0 \in \mathfrak{J}$ such that $j_0 \in \mathfrak{Z}(\mathfrak{N})$, then the hypothesis yields that $\Gamma(2j_0u) + 2j_0u = 0$ for all $u \in \mathfrak{U}$. Therefore, by recurrence, we have $\Gamma^l(2j_0u) + (-1)^{l+1}2j_0u = 0$ for all $u \in \mathfrak{U}$, $l \in \mathbb{N}^*$. Since Γ is zero power valued on $\mathfrak{N}$, then there exists an integer $l(2j_0u) > 1$ such that $\Gamma^{l(2j_0u)}(2j_0u) = 0$. Putting $l(2j_0u)$ in place of l in the last expression, we obtain $2j_0u = 0$ for all $u \in \mathfrak{U}$. The 2-torsion freeness of $\mathfrak{N}$ implies that $j_0u = 0$ for all $u \in \mathfrak{U}$.

and hence Lemma 2.1(ii) forces that $j_0 = 0$, a contradiction. Therefore, (3.8) gives that $\Gamma(j) = 0$ for all $j \in \mathfrak{J}$ and hence $\mathfrak{J}$ is commutative by Lemma 2.4. $\square$

The following example demonstrates that the primeness hypothesis of $\mathfrak{N}$ in Theorems 3.1, 3.2 and 3.3 is superfluous.

Example 3.4. Let $\mathfrak{S}$ be a 2-torsion free, non-abelian, zero-symmetric left near ring. Define $\mathfrak{N}, \mathfrak{U}, \mathfrak{J}$ and $\Gamma : \mathfrak{N} \rightarrow \mathfrak{N}$ by

$$\mathfrak{N} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, w \in \mathfrak{S} \right\}, \quad \mathfrak{U} = \left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u \in \mathfrak{S} \right\}$$

$$\mathfrak{J} = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \mid v, w \in \mathfrak{S} \right\}, \quad \Gamma \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

. It is easy to check that $\mathfrak{N}$ is a non-prime, non abelian left near ring w.r.t. addition and multiplication of matrix, $\mathfrak{J}$ a nonzero Jordan ideal of $\mathfrak{N}$, $\mathfrak{U}$ a nonzero semigroup ideal of $\mathfrak{N}$ and Γ is a nonzero homoderivation which is zero power valued on $\mathfrak{N}$ satisfying the following assertions:

- (i) $\Gamma([j, u]) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$,
- (ii) $\Gamma([j, u]) + [j, u] = (j \circ u)$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$
- (iii) $\Gamma(j \circ u) + (j \circ u) = [j, u]$ for all $j \in \mathfrak{J}$, $u \in \mathfrak{U}$.

However, $\mathfrak{N}$ is not commutative.

Theorem 3.5. *Let $\mathfrak{N}$ be 2-torsion free prime near ring and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If Γ is a nonzero homoderivation on $\mathfrak{N}$ satisfying $\Gamma([j, n]) = 0$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$, then $\mathfrak{N}$ is a commutative ring.*

Proof. By hypothesis

$$\Gamma([j, n]) = 0, \quad \forall j \in \mathfrak{J}, n \in \mathfrak{N}. \quad (3.9)$$

Replacing n by jn in (3.9) and applying the definition of Γ , we get

$$\begin{aligned} 0 &= \Gamma(j[j, n]) \\ &= \Gamma(j)\Gamma([j, n]) + \Gamma(j)[j, n] + j\Gamma([j, n]), \quad \forall j \in \mathfrak{J}, n \in \mathfrak{N}, \end{aligned}$$

which reduces to $\Gamma(j)[j, n] = 0$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. The last expression implies that $\Gamma(j)jn = \Gamma(j)nj$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. Substituting nm for n , where $m \in \mathfrak{N}$ in the preceding relation and using it again, we arrive at

$\Gamma(j)\mathfrak{N}[j, m] = \{0\}$ for all $j \in \mathfrak{J}$, $m \in \mathfrak{N}$. Since $\mathfrak{N}$ is prime near ring, we obtain

$$\Gamma(j) = 0 \quad \text{or} \quad j \in \mathfrak{Z}(\mathfrak{N}), \quad \forall j \in \mathfrak{J}. \quad (3.10)$$

Suppose there exists an element $j_0 \in \mathfrak{J}$ such that $\Gamma(j_0) = 0$, then by (3.9), we have $\Gamma(j_0 n) = \Gamma(n j_0)$ for all $n \in \mathfrak{N}$ which gives that $j_0 \Gamma(n) = \Gamma(n) j_0$ for all $n \in \mathfrak{N}$. The last expression can be written as

$$(\Gamma(j_0) + j_0)\Gamma(n) = \Gamma(n)(\Gamma(j_0) + j_0)$$

for all $n \in \mathfrak{N}$. Replacing n by $\Gamma(n)w$ in the previous relation and using the definition of Γ together with Lemma 2.6, we find that

$$\begin{aligned} j_0 \Gamma^2(n) \Gamma(w) + j_0 \Gamma^2(n) w + j_0 \Gamma(n) \Gamma(w) \\ = \Gamma^2(n) \Gamma(w) j_0 + \Gamma^2(n) w j_0 + \Gamma(n) \Gamma(w) j_0. \end{aligned}$$

Using the relation that $j_0 \Gamma(n) = \Gamma(n) j_0$ for all $n \in \mathfrak{N}$ in the last expression, we deduce that $j_0 \Gamma^2(n) w = \Gamma^2(n) w j_0$ for all $n, w \in \mathfrak{N}$. Substituting $w m$ in place of w in the preceding relation and using it again, we get $\Gamma^2(n) \mathfrak{N}[j_0, m] = \{0\}$ for all $m, n \in \mathfrak{N}$. By primeness of $\mathfrak{N}$ with Lemma 2.5, we get $j_0 \in \mathfrak{Z}(\mathfrak{N})$. Therefore, (3.10) implies that $j \in \mathfrak{Z}(\mathfrak{N})$ for all $j \in \mathfrak{J}$ and hence $\mathfrak{N}$ is a commutative ring by Lemma 2.3. $\square$

Theorem 3.6. *Let $\mathfrak{N}$ be a 2-torsion free prime near ring and $\mathfrak{J}$, a nonzero Jordan ideal of $\mathfrak{N}$. If Γ is a nonzero homoderivation on $\mathfrak{N}$ satisfying any one of the following assertions:*

- (i) $\Gamma(jn) + jn = [j, n]$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$,
- (ii) $\Gamma(jn) + jn = (j \circ n)$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$,

then $\mathfrak{N}$ is a commutative ring.

Proof. (i) Suppose that

$$\Gamma(jn) + jn = [j, n], \quad \forall j \in \mathfrak{J}, \quad n \in \mathfrak{N}. \quad (3.11)$$

Replacing n by jn and applying the definition of Γ , we get

$$\begin{aligned} j[j, n] &= \Gamma(jjn) + jjn, \\ &= \Gamma(j)\Gamma(jn) + \Gamma(j)jn + j\Gamma(jn) + jjn, \\ &= \Gamma(j)(\Gamma(jn) + jn) + j(\Gamma(jn) + jn), \quad \forall j \in \mathfrak{J}, \quad n \in \mathfrak{N}. \end{aligned}$$

Using (3.11), the above expression reduces to $\Gamma(j)[j, n] = 0$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$ which means that $\Gamma(j)jn = \Gamma(j)nj$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$. Replacing n by nm in the preceding relation and using it again, we arrive at

$$\Gamma(j)\mathfrak{N}[j, m] = \{0\}$$

for $j \in \mathfrak{J}$, $m \in \mathfrak{N}$. The primeness of $\mathfrak{N}$ yields that

$$\Gamma(j) = 0 \quad \text{or} \quad j \in \mathfrak{Z}(\mathfrak{N}), \quad \forall j \in \mathfrak{J}. \quad (3.12)$$

Suppose that there exists an element $j_0 \in \mathfrak{J}$ such that $\Gamma(j_0) = 0$, then returning to (3.11), we find that $\Gamma(j_0 n) + j_0 n = [j_0, n]$ for all $n \in \mathfrak{N}$. Applying the definition of Γ and using the fact that $\Gamma(j_0) = 0$, we obtain $j_0(\Gamma(n) + n) = [j_0, n]$. Particularly, for $n = j_0$, the last expression gives that $j_0^2 = 0$. Now substituting $j_0 n$ instead of n in the previous expression with the fact that $j_0^2 = 0$, we arrive at $-j_0 n j_0 = 0$ for all $n \in \mathfrak{N}$. This implies that $j_0 n j_0 = 0$ for all $n \in \mathfrak{N}$ and primeness of $\mathfrak{N}$ yields that $j_0 = 0$, a contradiction. Hence, (3.12) forces that $j \in \mathfrak{Z}(\mathfrak{N})$ and an application of Lemma 2.3 concludes that $\mathfrak{N}$ is a commutative ring.

Arguing in the similar manner as we have done in (i), we can obtain the result. $\square$

The following example shows that the primeness hypothesis of $\mathfrak{N}$ in Theorems 3.5 and 3.6 cannot be omitted.

Example 3.7. Let $\mathfrak{S}$ be a 2-torsion free, non-abelian, zero-symmetric left near ring. Define $\mathfrak{N}, \mathfrak{J}$ and $\Gamma : \mathfrak{N} \rightarrow \mathfrak{N}$ by

$$\mathfrak{N} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \mid u, v, w \in \mathfrak{S} \right\}, \quad \mathfrak{J} = \left\{ \begin{pmatrix} 0 & w & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid w \in \mathfrak{S} \right\},$$

$$\Gamma \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that $\mathfrak{N}$ is a non-prime, non abelian left near ring w.r.t. addition and multiplication of matrix, $\mathfrak{J}$ a nonzero Jordan ideal of $\mathfrak{N}$ and Γ is a nonzero homoderivation on $\mathfrak{N}$ satisfying the following assertions:

- (i) $\Gamma([j, n]) = 0$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$,
- (ii) $\Gamma(jn) + jn = [j, n]$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$,
- (iii) $\Gamma(jn) + jn = (j \circ n)$ for all $j \in \mathfrak{J}$, $n \in \mathfrak{N}$.

However, $\mathfrak{N}$ is not commutative.

4. CONCLUSIONS

Finally, we conclude our paper by posing two important queries for future work: (i) Can we prove these results replacing homoderivation by generalized

homoderivation or generalized semi-homoderivation or multiplicative generalized semi-homoderivation? (ii) Do the results obtained in the different theorems remain true for semiprime near rings which is another interesting work for the future?

Acknowledgement

The authors wish to thank the editor and referees for their valuable comments and suggestions.

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ALGEBRAIC CONDITIONS WITH HOMODERIVATIONS UNDER WHICH
NEAR RINGS ARE RINGS

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شرایط جبری با همومشتق‌گیری‌ها که تحت آن‌ها شبه حلقه‌ها، حلقه هستند

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نگاشت جمعی $\Gamma : \mathfrak{N} \rightarrow \mathfrak{N}$ را روی یک شبه حلقه $\mathfrak{N}$ ، همومشتق‌گیری می‌نامیم هرگاه برای هر $u, v \in \mathfrak{N}$ داشته باشیم $\Gamma(uv) = \Gamma(u)\Gamma(v) + \Gamma(u)v + u\Gamma(v)$. هدف این مقاله، بررسی همانی‌های جبری زیر است: (i) $\Gamma([j, u]) = [j, u]$ برای هر $j \in \mathfrak{J}$ و $u \in \mathfrak{L}$ ؛ (ii)

$$\Gamma([j, u]) + [j, u] = (j \circ u)$$

برای هر $j \in \mathfrak{J}$ و $u \in \mathfrak{L}$ ؛ (iii) $\Gamma(j \circ u) + (j \circ u) = [j, u]$ برای هر $j \in \mathfrak{J}$ و $u \in \mathfrak{L}$ ؛ (iv) $\Gamma([j, n]) = 0$ برای هر $j \in \mathfrak{J}$ و $n \in \mathfrak{N}$ ؛ (v) $\Gamma(jn) + jn = [j, n]$ برای هر $j \in \mathfrak{J}$ و $n \in \mathfrak{N}$ ؛ (vi) $\Gamma(jn) + jn = (j \circ n)$ برای هر $j \in \mathfrak{J}$ و $n \in \mathfrak{N}$ ، که در آن $\mathfrak{J}$ یک ایده‌آل جردن ناصفر از $\mathfrak{N}$ و $\mathfrak{L}$ یک ایده‌آل نیم‌گروهی ناصفر از $\mathfrak{N}$ است. به علاوه، برای تأیید فرضیات نتایج به‌دست‌آمده، مثال‌هایی ارائه می‌کنیم.

کلمات کلیدی: همومشتق‌گیری، شبه حلقه اول، ایده‌آل جردن، ایده‌آل نیم‌گروهی.