

APPLICATIONS OF ROUGH SET THEORY TO POWER MAPPINGS AND THEIR RELATED SPACES

S. M. Musavi Ahmadabadi, S. M. Anvariye^{*}h and S. Mirvakili

ABSTRACT. In this paper, we first introduce the concept of a power mapping and demonstrate the existence of a natural one-to-one correspondence between all relations from A to B and all power mappings from A to $\mathcal{P}(B)$. Then, we define several binary operations on a power mapping and investigate their fundamental properties. Next, we investigate the power space $\langle U, W \mid K = \overline{S} \rangle$ and prove that it admits a unique basis. Finally, we define several important subsets of a power mapping, prove fundamental results and provide a variety of examples.

1. INTRODUCTION

The development of rough set theory has expanded significantly over the years covering a variety of disciplines and applications. In 2007, Pawlak and Skowron provided a comprehensive overview of the rudiments of rough sets, summarizing both theoretical advancements and practical implementations of this versatile mathematical tool [17].

Past studies made a great contribution to the theory of rough sets and their applications to various algebraic structures. Notable works include Kuroki's investigation (1997) of rough ideals in semigroups [13], and Mordeson's application of rough set theory to (fuzzy) ideal theory [14]. Davvaz has extensively examined roughness in rings [6], roughness based on fuzzy ideals [5], algebraic T-rough sets [4] and approximations in hyperrings [3].

Further contributions by Kazanci and Davvaz (2008) delved into the structure of rough prime and primary ideals, and their fuzzy counterparts in commutative rings [12]. Estaji et al. (2011) discussed rough set and fuzzy sublattice [9], while Estaji, Hooshmandasl, and Davvaz (2012) applied rough set theory to lattice theory [8].

In the same year, Hosseini and colleagues (2012) explored T-rough ideals and T-rough fuzzy ideals in semigroups [10], as well as results on T-rough prime and primary ideals and T-rough fuzzy prime and primary ideals in commutative rings [11]. Bagirmaz's (2015) study on rough semigroups [1]

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^{*}Corresponding author.

and Biswas and Nanda's exploration of rough groups [2] further enrich the literature.

The concepts of relations and mappings are fundamental in disciplines such as mathematics and computer science. Let S be a set and denote $\wp(S) = \{L \mid L \subseteq S\}$ and $\wp^*(S) = \{L \mid L \subseteq S, \text{ and } L \neq \emptyset\}$. If U and W are nonempty sets and T is a mapping from U to $\wp(W)$, then T is called a *power mapping* from U to $\wp(W)$. In this paper, we demonstrate the existence of a natural one-to-one correspondence between all relations from U to W and all power mappings from U to $\wp(W)$. We denote the set of all power mappings from U to $\wp^*(W)$ by $\langle U, W \rangle$ and the set of all power mappings from U to $\wp(W)$ by $\langle U, W \rangle_0$.

Let $T : U \rightarrow \wp(W)$ be a power mapping and let $S \subseteq W$ and define $\overline{S}_T := \{x \in U \mid T(x) \cap S \neq \emptyset\}$ and $\underline{S}_T := \{x \in U \mid T(x) \subseteq S\}$ (see [4]). The concepts $\overline{S}_T$ and $\underline{S}_T$ are studied in T-rough sets theory [4]. For a subset K of U , we define $T'(K)$ as $T'(K) := \bigcup_{x \in K} T(x)$. Also, we denote K as a proper subset of U by $K \subset U$ if $K \subseteq U$ and $K \neq U$. Additionally, $K \subset_n U$ signifies that K is a nontrivial subset of U .

Lemma 1.1. *Let U and W be sets, then*

- (1) $|\langle U, W \rangle| = (2^{|W|} - 1)^{|U|}$.
- (2) $|\langle U, W \rangle_0| = 2^{|U||W|}$.

Proof. It is straightforward. □

Lemma 1.2. *Let U and W be nonempty sets. Let $S \subseteq W$ and $T \in \langle U, W \rangle$. Then*

- (1) $T'(\overline{S}_T) = \bigcup_{\substack{x \in U \\ T(x) \cap S \neq \emptyset}} T(x)$.
- (2) $T'(\underline{S}_T) = \bigcup_{\substack{x \in U \\ T(x) \subseteq S}} T(x)$.

Proof. It is straightforward. □

Lemma 1.3. *Let U and W be nonempty sets and $2 \leq |U|$. Let S_1, S_2 be nonempty subsets of W and $S \subset_n W$. Then there exists $T \in \langle U, W \rangle$ such that $T'(\overline{S}_T) = S_2$ and $T'(\underline{S}_T) = S_1$ if and only if $S_1 \subseteq S \cap S_2$.*

Proof. ($\Rightarrow$) Suppose that $T : U \rightarrow \wp^*(W)$ such that $T'(\underline{S}_T) = S_1$ and $T'(\overline{S}_T) = S_2$. By Lemma 1.2, we have

$$S_2 = T'(\overline{S}_T) = \bigcup_{\substack{x \in U \\ T(x) \cap S \neq \emptyset}} T(x) \supseteq \bigcup_{\substack{x \in U \\ T(x) \subseteq S}} T(x) = T'(\underline{S}_T) = S_1,$$

and

$$S \supseteq \bigcup_{\substack{x \in U \\ T(x) \subseteq S}} T(x) = S_1.$$

So $S_1 \subseteq S \cap S_2$.

($\Leftarrow$) Conversely, there are nonempty subsets S , S_1 and S_2 of W such that $S_1 \subseteq S \cap S_2$ and $2 \leq |U|$. Assume that $x, y \in U$ and define

$$T(u) = \begin{cases} S_1 & \text{if } u = x, \\ S_2 & \text{if } u = y, \\ T(u) \in \wp^*(W \setminus S) & \text{if } u \in U \setminus \{x, y\}, \end{cases}$$

then $T \in \langle U, W \rangle$. By Lemma 1.2, we have

$$T'(\overline{S}_T) = T(x) \cup T(y) = S_1 \cup S_2 = S_2,$$

and $T'(\underline{S}_T) = T(x) = S_1$. □

Lemma 1.4. *If $T \in \langle U, W \rangle_0$, then*

- (1) $\overline{\emptyset}_T = \emptyset$.
- (2) $\underline{\emptyset}_T = \emptyset$ if and only if $T \in \langle U, W \rangle$.
- (3) $\underline{\emptyset}_T \subseteq \underline{S}_T$ for every $S \subseteq W$.
- (4) $T'(\underline{\emptyset}_T) = T'(\overline{\emptyset}_T) = \emptyset$.

Proof. It is obvious. □

Lemma 1.5. *Let $T \in \langle U, W \rangle_0$, then*

- (1) $T \in \langle U, W \rangle$ if and only if for every $S \subseteq W$, we have $\underline{S}_T \subseteq \overline{S}_T$.
- (2) If $S \subseteq W$, then $T'(\underline{S}_T) \subseteq T'(\overline{S}_T)$.

Proof. (1) It is obvious.

(2) It follows from Lemma 1.2. □

Now, we show that there exists a natural one to one correspondence between $\langle U, W \rangle_0$ and $\mathcal{R}(U, W)$, where $\mathcal{R}(U, W) = \{R \mid R \subseteq U \times W\}$.

Theorem 1.6 (The fundamental theorem of power mappings). *If U and W are sets, then there exists a natural one to one correspondence between $\langle U, W \rangle_0$ and $\mathcal{R}(U, W)$.*

Proof. The mapping

$$\begin{aligned} \rho : \langle U, W \rangle_0 &\longrightarrow \mathcal{R}(U, W) \\ T &\longrightarrow \{(u, w) \in U \times W \mid w \in T(u)\} \end{aligned}$$

is a natural one to one correspondence, and $\mu : \mathcal{R}(U, W) \longrightarrow \langle U, W \rangle_0$ is a natural inverse such that $R \longrightarrow T_R$, where for any $u \in U$,

$$T_R(u) = \{w \in W \mid uRw\}.$$

□

Example 1.7. Let $U = \{x, y, z\}$ and $W = \{a, b, c, d\}$. If

$$R = \{(x, a), (x, b), (z, c)\},$$

then $\mu(R) = T_R$, where

$$T_R(u) = \begin{cases} \{a, b\} & u = x, \\ \emptyset & u = y, \\ \{c\} & u = z, \end{cases}$$

and if

$$\Lambda(u) = \begin{cases} \{b, d\} & u = x, \\ \{a\} & u = y, \\ \{c, d\} & u = z. \end{cases}$$

Then $\rho(\Lambda) = \{(x, b), (x, d), (y, a), (z, c), (z, d)\}$.

2. SOME BINARY OPERATIONS OVER SETS $\langle U, W \rangle$ AND $\langle U, W \rangle_0$

In this section, first we introduce several binary operations on sets $\langle U, W \rangle$ and $\langle U, W \rangle_0$ and investigate their properties.

Definition 2.1. Let $T_1, T_2 \in \langle U, W \rangle_0$ and $x \in U$. We define

- (1) $(T_1 \cup T_2)(x) = T_1(x) \cup T_2(x)$.
- (2) $(T_1 \cap T_2)(x) = T_1(x) \cap T_2(x)$.
- (3) $(T_1 \Delta T_2)(x) = T_1(x) \Delta T_2(x)$.
- (4) $(T_1 - T_2)(x) = T_1(x) - T_2(x)$.

Lemma 2.2. If U and W are sets, then

- (1) $\langle U, W \rangle_0$ is closed under operations $\cup$, $\cap$ and Δ .
- (2) $\langle U, W \rangle$ is closed under the operation $\cup$.
- (3) $\langle U, W \rangle$ and $\langle U, W \rangle_0$, are associative and commutative under the operations $\cup$, $\cap$ and Δ .

- (4) On $\langle U, W \rangle_0$, the operations $\cup$ and $\cap$ are distributive over each other and operation $\cap$ is distributive over operation Δ .

Proof. The proof is straightforward and follows easily from provided Lemmas. $\square$

Lemma 2.3. *The following statements hold.*

- (1) *The power mapping $\Phi : U \rightarrow \wp(W)$ such that $\Phi(u) = \emptyset$ for any $u \in U$, is an identity under operations $\cup$ and Δ .*
- (2) *The power mapping $\Psi : U \rightarrow \wp(W)$ such that $\Psi(u) = W$ for any $u \in U$, is an identity under operation $\cap$.*
- (3) *Any power mapping is its own inverse under operation Δ .*

Proof. It is straightforward. $\square$

Theorem 2.4. *Let U and W be nonempty sets, then*

- (1) *$(\langle U, W \rangle_0, \cup)$ and $(\langle U, W \rangle_0, \cap)$ are monoids.*
- (2) *$(\langle U, W \rangle_0, \Delta)$ is an abelian group.*
- (3) *$(\langle U, W \rangle_0, \Delta, \cap)$ is a commutative boolean ring with identity.*

Proof. This conclusion follows from Lemma 2.2 and Lemma 2.3. $\square$

3. THE POWER SPACE $\langle U, W \mid K = \bar{S} \rangle$ AND ITS UNIQUE BASIS

In this section, we define several significant subsets of $\langle U, W \rangle$ and examine their properties in detail.

Definition 3.1. Let U and W be sets and $\Sigma \subseteq \langle U, W \rangle_0$.

- (1) We define Σ as a power space if, for any $\Sigma_0 \subseteq \Sigma$, it holds that $\bigcup \Sigma_0 - \{\emptyset\} \in \Sigma$.
- (2) We define Σ' as a sub-power space of Σ if $\Sigma' \subseteq \Sigma$ and Σ' is itself a power space.

Definition 3.2. Let U and W be nonempty sets, $K \subseteq U$ and $S \subseteq W$. We define

$$\langle U, W \mid K = \bar{S} \rangle = \{T : U \rightarrow \wp^*(W) \mid T \text{ is a power mapping and } \bar{S}_T = K\}.$$

$$\langle U, W \mid K = \bar{S} \rangle_0 = \{T : U \rightarrow \wp(W) \mid T \text{ is a power mapping and } \bar{S}_T = K\}.$$

Lemma 3.3. *Let $\Sigma \subseteq \langle U, W \rangle$ and $S \subseteq W$, then*

$$(1) \bar{S}_{\bigcup_{T \in \Sigma} T} = \bigcup_{T \in \Sigma} \bar{S}_T,$$

$$(2) \underline{S}_{\bigcup_{T \in \Sigma} T} = \bigcap_{T \in \Sigma} \underline{S}_T.$$

Proof. (1)

$$\begin{aligned} x \in \overline{S}_{\bigcup_{T \in \Sigma}} &\iff \left(\bigcup_{T \in \Sigma} T \right)(x) \cap S \neq \emptyset \\ &\iff \bigcup_{T \in \Sigma} (T(x) \cap S) \neq \emptyset \\ &\iff \exists T_\alpha \in \Sigma : T_\alpha(x) \cap S \neq \emptyset \\ &\iff \exists T_\alpha \in \Sigma : x \in \overline{S}_{T_\alpha} \\ &\iff x \in \bigcup_{T \in \Sigma} \overline{S}_T. \end{aligned}$$

(2)

$$\begin{aligned} x \in \bigcap_{T \in \Sigma} \underline{S}_T &\iff \forall T \in \Sigma : x \in \underline{S}_T \\ &\iff \forall T \in \Sigma : T(x) \subseteq S \\ &\iff \bigcup_{T \in \Sigma} T(x) \subseteq S \\ &\iff x \in \underline{S}_{\bigcup_{T \in \Sigma} T}. \end{aligned}$$

□

Example 3.4. Assume that $U = \{x, y, z, u, v\}$, $W = \{a, b, c, d, e\}$ and $S = \{a, b, c\}$. Let

$$T_1(t) = \begin{cases} \{a\} & t = x, \\ \{b, d\} & t = y, \\ \{e\} & t = z, \\ \{c\} & t = u, \\ \{e\} & t = v, \end{cases} \quad \text{and} \quad T_2(t) = \begin{cases} \{d\} & t = x, \\ \{b\} & t = y, \\ \{c, e\} & t = z, \\ \{b, c\} & t = u, \\ \{d\} & t = v. \end{cases}$$

Then

$$(T_1 \cup T_2)(t) = \begin{cases} \{a, d\} & t = x, \\ \{b, d\} & t = y, \\ \{c, e\} & t = z, \\ \{b, c\} & t = u, \\ \{d, e\} & t = v, \end{cases} \quad \text{and} \quad (T_1 \cap T_2)(t) = \begin{cases} \emptyset & t = x, \\ \{b\} & t = y, \\ \{e\} & t = z, \\ \{c\} & t = u, \\ \emptyset & t = v. \end{cases}$$

So,

- (1) $\underline{S}_{T_1} = \{x, u\}$ and $\underline{S}_{T_2} = \{y, u\}$.
- (2) $\overline{S}_{T_1} = \{x, y, u\}$ and $\overline{S}_{T_2} = \{y, z, u\}$.
- (3) $\underline{S}_{T_1 \cup T_2} = \{u\}$ and $\overline{S}_{T_1 \cup T_2} = \{x, y, z, u\}$.
- (4) $\underline{S}_{T_1 \cap T_2} = \{x, y, u, v\}$ and $\overline{S}_{T_1 \cap T_2} = \{y, u\}$.
- (5) $\underline{S}_{T_1} \cap \underline{S}_{T_2} = \{u\}$ and $\overline{S}_{T_1} \cup \overline{S}_{T_2} = \{x, y, z, u\}$.
- (6) $\overline{S}_{T_1} \cup \overline{S}_{T_2} = \{x, y, z, u\}$ and $\underline{S}_{T_1} \cup \underline{S}_{T_2} = \{x, y, u\}$.
- (7) $\emptyset_{T_1} = \emptyset_{T_2} = \{\emptyset\}$ and $\emptyset_{T_1 \cap T_2} = \{x, v\}$.

If $T_3 = T_1 \cap T_2$, then $\underline{S}_{T_3} \not\subseteq \overline{S}_{T_3}$. Hence $T_3 \notin \langle U, W \rangle$, by Lemma 1.5.

Lemma 3.5. $\langle U, W \mid K = \overline{S} \rangle$ is a power space.

Proof. Let $\emptyset \neq \Sigma \subseteq \langle U, W \mid K = \overline{S} \rangle$. Then

$$\overline{S}_{\bigcup_{T \in \Sigma} T} = \bigcup_{T \in \Sigma} \overline{S}_T = \bigcup_{T \in \Sigma} K = K$$

by Lemma 3.3. So $\bigcup \Sigma \in \langle U, W \mid K = \overline{S} \rangle$. □

Lemma 3.6. Suppose that U and W are nonempty sets. Let $K \subseteq U$ and $S \subseteq W$ and let $\theta \in \langle U, W \rangle_0$ such that

- (1) for any $x \in K$, we have $|\theta(x) \cap S| = 1$ and $|\theta(x) \cap S^c| \leq 1$,
- (2) for any $x' \in K^c$, we have $|\theta(x') \cap S| = 0$ and $|\theta(x') \cap S^c| = 1$.

Then $\theta \in \langle U, W \mid K = \overline{S} \rangle$.

Proof. It is straightforward. □

The power mapping θ introduced in Lemma 3.6 is called a g -mapping. We denote the set of all g -mapping by $g_{K,S}$. For $S \subseteq W$, set

$$P_n(S) = \{S_0 \mid S_0 \subseteq S, |S_0| = n\} \text{ and } P_{\leq n}(S) = \{S_0 \mid S_0 \subseteq S, |S_0| \leq n\}.$$

Lemma 3.7. Let $T \in \langle U, W \mid K = \overline{S} \rangle$, then $x \in K^c$ if and only if $T(x) \subseteq S^c$.

Proof. Let $x \in K^c$. So

$$\begin{aligned} \emptyset \neq T(x) &= T(x) \cap W = T(x) \cap (S \cup S^c) \\ &= (T(x) \cap S) \cup (T(x) \cap S^c) \\ &= \emptyset \cup (T(x) \cap S^c) \\ &= T(x) \cap S^c \\ &\subseteq S^c. \end{aligned}$$

□

Theorem 3.8. $T \in \langle U, W \mid K = \overline{S} \rangle$ if and only if there exists a nonempty subset of $g_{K,S}$ such as g_T , such that $T = \bigcup_{\theta \in g_T} \theta$.

Proof. ($\Leftarrow$) It follows from Lemma 3.5 and Lemma 3.6.

($\Rightarrow$) For any $T \in \langle U, W \mid K = \overline{S} \rangle$. Let A is a set with the following properties:

- (1) $|T'(K) \cap S| + 1 \leq |A|$,
- (2) $|T'(U) \cap S^c| + 1 \leq |A|$.

For example, if $|A| = |W| + 1$, then A satisfies conditions (1) and (2).

Next, we proceed with the following steps.

- (1) For any $x \in K$ we consider the surjective mappings:

$$\Lambda_x : A \xrightarrow{on} P_1(T(x) \cap S) \quad \text{and} \quad \Lambda'_x : A \xrightarrow{on} P_{\leq 1}(T(x) \cap S^c).$$

- (2) For any $x \in K^c$, we consider the surjective mappings:

$$\Lambda_x : A \xrightarrow{on} \{\emptyset\} \quad \text{and} \quad \Lambda'_x : A \xrightarrow{on} P_1(T(x) \cap S^c).$$

Now for any $\alpha \in A$ and $x \in U$, set

$$\theta_\alpha(x) = (\Lambda_x \cup \Lambda'_x)(\alpha) = \Lambda_x(\alpha) \cup \Lambda'_x(\alpha).$$

If $\alpha \in A$, then $\theta_\alpha \in g_{K,S}$ and $T = \bigcup_{\alpha \in A} \theta_\alpha$. Let $g_T = \{\theta_\alpha \mid \alpha \in A\}$, then $g_T \subseteq g_{K,S}$ and $T = \bigcup_{\theta_\alpha \in g_T} \theta_\alpha$. □

Example 3.9. Suppose that $U = \{x, y, z, u\}$, $W = \{a, b, c, d, e, f\}$ and $S = \{a, b, c\}$. Define $T : U \rightarrow \wp^*(W)$ such that

$$T := \begin{cases} x \rightarrow \{a\}, \\ y \rightarrow \{c, d\}, \\ z \rightarrow \{a, b, e\}, \\ u \rightarrow \{d, e\}. \end{cases}$$

So $K = \overline{S}_T = \{x, y, z\}$ and $T \in \langle U, W \mid K = \overline{S} \rangle$. Also, $T'(K) \cap S = \{a, b, c\}$ and $T'(U) \cap S^c = \{d, e\}$. consider $A = \{1, 2, 3, 4\}$, then $3 + 1 = 4 \leq |A|$. Therefore

$$\Lambda_x := \begin{cases} 1 \rightarrow \{a\}, \\ 2 \rightarrow \{a\}, \\ 3 \rightarrow \{a\}, \\ 4 \rightarrow \{a\}. \end{cases} \quad \Lambda_y := \begin{cases} 1 \rightarrow \{c\}, \\ 2 \rightarrow \{c\}, \\ 3 \rightarrow \{c\}, \\ 4 \rightarrow \{c\}. \end{cases} \quad \Lambda_z := \begin{cases} 1 \rightarrow \{a\}, \\ 2 \rightarrow \{a\}, \\ 3 \rightarrow \{b\}, \\ 4 \rightarrow \{a\}. \end{cases} \quad \Lambda_u := \begin{cases} 1 \rightarrow \emptyset, \\ 2 \rightarrow \emptyset, \\ 3 \rightarrow \emptyset, \\ 4 \rightarrow \emptyset, \end{cases}$$

and

$$\Lambda'_x := \begin{cases} 1 \rightarrow \emptyset, \\ 2 \rightarrow \emptyset, \\ 3 \rightarrow \emptyset, \\ 4 \rightarrow \emptyset. \end{cases} \quad \Lambda'_y := \begin{cases} 1 \rightarrow \emptyset, \\ 2 \rightarrow \{d\}, \\ 3 \rightarrow \emptyset, \\ 4 \rightarrow \emptyset. \end{cases} \quad \Lambda'_z := \begin{cases} 1 \rightarrow \emptyset, \\ 2 \rightarrow \{e\}, \\ 3 \rightarrow \emptyset, \\ 4 \rightarrow \emptyset. \end{cases} \quad \Lambda'_u := \begin{cases} 1 \rightarrow \{d\}, \\ 2 \rightarrow \{e\}, \\ 3 \rightarrow \{d\}, \\ 4 \rightarrow \{e\}. \end{cases}$$

By Theorem 3.8, we have

$$\theta_1 := \begin{cases} x \rightarrow \{a\}, \\ y \rightarrow \{c\}, \\ z \rightarrow \{a\}, \\ u \rightarrow \{d\}. \end{cases} \quad \theta_2 := \begin{cases} x \rightarrow \{a\}, \\ y \rightarrow \{c, d\}, \\ z \rightarrow \{a, e\}, \\ u \rightarrow \{e\}. \end{cases} \quad \theta_3 := \begin{cases} x \rightarrow \{a\}, \\ y \rightarrow \{c\}, \\ z \rightarrow \{b\}, \\ u \rightarrow \{d\}. \end{cases} \quad \theta_4 := \begin{cases} x \rightarrow \{a\}, \\ y \rightarrow \{c\}, \\ z \rightarrow \{a\}, \\ u \rightarrow \{e\}. \end{cases}$$

Therefore $g_T = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and $T = \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$.

3.1. Generator and basis. In this section, we show that $\langle U, W \mid K = \overline{S} \rangle$ has a unique basis. Suppose that U and W are sets and $\Sigma \subseteq \langle U, W \rangle$. Set $gt(\Sigma) := \{\bigcup \sigma \mid \sigma \subseteq \Sigma\} - \{\emptyset\}$ and $gt_0(\Sigma) := \{\bigcup \sigma \mid \sigma \subseteq \Sigma\}$.

Definition 3.10. Let U and W be sets, $K \subseteq U$, $S \subseteq W$ and

$$G \subseteq \langle U, W \mid K = \overline{S} \rangle.$$

G is called a *generator* for $\langle U, W \mid K = \overline{S} \rangle$ if $gt(G) = \langle U, W \mid K = \overline{S} \rangle$.

In the following, we define the concept of basis for $\langle U, W \mid K = \overline{S} \rangle$.

Definition 3.11. Let $B \subseteq \langle U, W \mid K = \overline{S} \rangle$. We call B a *basis* for $\langle U, W \mid K = \overline{S} \rangle$ if

- (1) B is a generator for $\langle U, W \mid K = \overline{S} \rangle$,
- (2) if B_1 is a generator and $B_1 \subseteq B$, then $B_1 = B$.

Example 3.12 illustrates that $g_{K,S}$ does not necessarily be a basis.

Example 3.12. Assume that $U = \{x, y, z\}$, $W = \{a, b, c, d, e\}$, $K = \{x, y\}$, $S = \{a, b, c\}$ and

$$\begin{aligned}\theta_1(u) &:= \begin{cases} \{a, d\} & u = x, \\ \{b, e\} & u = y, \\ \{d\} & u = z. \end{cases} & \theta_2(u) &:= \begin{cases} \{a, d\} & u = x, \\ \{b\} & u = y, \\ \{d\} & u = z. \end{cases} \\ \theta_3(u) &:= \begin{cases} \{a\} & u = x, \\ \{b, e\} & u = y, \\ \{d\} & u = z. \end{cases}\end{aligned}$$

Then $\theta_1, \theta_2, \theta_3 \in g_{K,S}$ and $\theta_1 = \theta_2 \cup \theta_3$.

Definition 3.13. Let $T_1, T_2 \in \langle U, W \rangle_0$.

- (1) $T_1 \leq T_2$ if and only if for any $u \in U$ we have $T_1(u) \subseteq T_2(u)$.
- (2) $T_1 < T_2$ if and only if $T_1 \leq T_2$ and there exists $x \in U$ such that $T_1(x) \subset T_2(x)$.

Definition 3.14. Let U and W be sets, $K \subseteq U$ and $S \subseteq W$.

- (1) If $\Sigma \subseteq \langle U, W \rangle$ and $\delta \in \langle U, W \rangle$, then δ is called a *composite mapping* in Σ , if there exists a subset σ of $\Sigma - \{\delta\}$ such that $\delta = \bigcup \sigma$. δ is called a *simple mapping* in Σ , if it is not composite in Σ .
- (2) $\Delta_{K,S} = \{\delta \mid \delta \text{ is a simple mapping in } \langle U, W \mid K = \overline{S} \rangle\}$.

Lemma 3.15. If $K \subset_n U$, $S \subset_n W$ and B is a basis for $\langle U, W \mid K = \overline{S} \rangle$, then $\Delta_{K,S} \subseteq B$.

Proof. Let $\delta \in \Delta_{K,S} - B$. So, there exists a subset b_δ of B such that $\delta = \bigcup_{\theta \in b_\delta} \theta$.

That is a contradiction. $\square$

Lemma 3.16. If $K \subset_n U$, $S \subset_n W$ and B is a basis for $\langle U, W \mid K = \overline{S} \rangle$, then $B \subseteq \Delta_{K,S}$.

Proof. Let $\beta \in B - \Delta_{K,S}$, so there exists $\Sigma \subseteq \langle U, W \mid K = \overline{S} \rangle - \{\beta\}$ such that $\beta = \bigcup_{\sigma \in \Sigma} \sigma$, hence for any $\sigma \in \Sigma$ we have $\sigma < \beta$, and $2 \leq |\Sigma|$. For any $\sigma \in \Sigma$, there exists $b_\sigma \subseteq B$, such that $\sigma = \bigcup_{\theta \in b_\sigma} \theta$, so for any $\theta \in b_\sigma$ we have $\theta \leq \sigma$. So $\beta = \bigcup_{\sigma \in \Sigma} (\bigcup_{\theta \in b_\sigma} \theta)$. Therefore $B - \{\beta\}$ is a generator, it is a contradiction. $\square$

Theorem 3.17. If U and W are sets, $K \subset_n U$ and $S \subset_n W$, then $\Delta_{K,S}$ is a unique basis for $\langle U, W \mid K = \overline{S} \rangle$.

Proof. It follows from Lemma 3.15 and Lemma 3.16. $\square$

Definition 3.18. Let U and W be sets, $K \subset_n U$ and $S \subset_n W$. We denote the unique basis of $\langle U, W | K = \overline{S} \rangle$ with $B_{K,S}$. Also we define

$$\dim(\langle U, W | K = \overline{S} \rangle) := |B_{K,S}|.$$

Now, we determine the number of members of the basis of $\langle U, W | K = \overline{S} \rangle$.

Lemma 3.19. *If G is a generator for $\langle U, W | K = \overline{S} \rangle$, then $B_{K,S} \subseteq G$.*

Proof. Let $\beta \in B_{K,S} - G$, since G is a generator, there exists a subset G_β of G such that $\beta = \bigcup_{\gamma \in G_\beta} \gamma$, and for every $\gamma \in G_\beta$ we have $\gamma < \beta$, so $|G_\beta| \geq 2$. Since $B_{K,S}$ is a basis, hence for every $\gamma \in G_\beta$, there exists the subset b_γ of B such that $\gamma = \bigcup_{\theta \in b_\gamma} \theta$. Thus for every $\theta \in b_\gamma$ we have $\theta \leq \gamma$. Hence $\beta = \bigcup_{\gamma \in G_\beta} (\bigcup_{\theta \in b_\gamma} \theta)$ and $B_{K,S} - \{\beta\}$ is a generator. This is a contradiction. $\square$

Corollary 3.20. $B_{K,S} \subseteq g_{K,S}$.

Proof. It follows from Theorem 3.8 and Lemma 3.19. $\square$

Lemma 3.21. *Assume that U and W are nonempty sets, $K \subset_n U$, $S \subset_n W$ and Γ is the family of all mappings $\gamma : U \rightarrow \wp * (W)$ such that satisfy in the following conditions:*

- (1) *there exists $x_1 \in K$ such that $|\gamma(x_1) \cap S| = 1$ and $|\gamma(x_1) \cap S^c| \leq 1$,*
- (2) *for any $u \in K - \{x_1\}$, $|\gamma(u) \cap S| = 1$ and $|\gamma(u) \cap S^c| = 0$,*
- (3) *for any $u' \in K^c$, $|\gamma(u') \cap S| = 0$ and $|\gamma(u') \cap S^c| = 1$.*

Then $\Gamma = \Delta_{K,S}$.

Proof. It is clear that $\Gamma \subseteq g_{K,S}$. Let $\gamma \in \Gamma - \Delta_{K,S}$, there exist

$$\alpha, \beta \in \langle U, W | K = \overline{S} \rangle$$

such that $\alpha < \gamma$, $\beta < \gamma$ and $\gamma = \alpha \cup \beta$. We proceed with the proof step by step

- (1) If for any $u \in K$, $\gamma(u) = \alpha(u) = \beta(u)$, then there exists $u' \in K^c$ such that $\alpha(u') \subseteq \gamma(u')$, this is a contradiction.
- (2) Let $x \in K$ such that $\alpha(x) \subset \gamma(x)$.
 - (a) If $\gamma(x) = \beta(x)$, then there exists $x' \in U - \{x\}$ such that $\beta(x') \subset \gamma(x')$, this is a contradiction.
 - (b) If $\beta(x) \subset \gamma(x)$, this is a contradiction.

Therefore $\Gamma \subseteq \Delta_{K,S}$.

Now assume that $\theta \in g_{K,S}$. For any $x \in U$, we define:

$$\gamma_x(u) = \begin{cases} \theta(x) & \text{if } u = x, \\ \theta(u) \cap S & \text{if } u \in K - \{x\} \\ \theta(u) \cap S^c & \text{if } u \in K^c - \{x\}. \end{cases}$$

For any $u \in U$, we have $\gamma_u \in \Gamma$ and $\theta = \bigcup_{u \in U} \gamma_u$. Thus Γ is a generator for $g_{K,S}$, hence Γ is a generator for $\langle U, W | K = S \rangle$ by Theorem 3.8. Therefore $\Delta_{K,S} \subseteq \Gamma$ by Lemma 3.19 and Theorem 3.17. $\square$

Definition 3.22. (1) Let $T : U_0 \subseteq U \longrightarrow \wp(W)$ be a power mapping. We define $\hat{T} : U \longrightarrow \wp(W)$ such that

$$\hat{T}(u) := \begin{cases} T(u) & \text{if } u \in U_0, \\ \emptyset & \text{if } u \in U - U_0, \end{cases}$$

and for $(x, t) \in U \times P_{\leq 1}(W)$ we define

$$(x, t)(u) = \begin{cases} t & \text{if } u = x, \\ \emptyset & \text{if } u \neq x. \end{cases}$$

(2) For $\Sigma \subseteq \{T : U_0 \longrightarrow \wp(W) \mid T \text{ is a power mapping}\}$, define

$$\hat{\Sigma} = \{\hat{T} \mid T \in \Sigma\}.$$

Theorem 3.23. If U and W are nonempty finite sets, $K \subset_n U$ and $S \subset_n W$, then

$$|B_{K,S}| = |K| \cdot (|W| - |S| + 1) \cdot |S|^{|K|} \cdot (|W| - |S|)^{(|U| - |K|)}.$$

Proof. Let's prove the theorem step by step.

(1) We define

$$\begin{aligned} \Sigma_{K,S} &= \{T : K \longrightarrow P_1(S) \mid T \text{ is a mapping}\}, \\ \Sigma_{K^c,S^c} &= \{T : K^c \longrightarrow P_1(S^c) \mid T \text{ is a mapping}\} \end{aligned}$$

and $\Sigma_{K,S^c} = \{(x, t) \mid (x, t) \in K \times P_{\leq 1}(S^c)\}$.

(2) $\gamma \in \Gamma$ if and only if there exist $T_1 \in \Sigma_{K,S}$, $T_2 \in \Sigma_{K^c,S^c}$ and $(x, t) \in \Sigma_{K,S^c}$ such that $\gamma = \hat{T}_1 \cup \hat{T}_2 \cup (x, t)$.

(3) $\hat{\Sigma}_{K,S} \cap \hat{\Sigma}_{K^c,S^c} = \hat{\Sigma}_{K,S} \cap \hat{\Sigma}_{K,S^c} = \hat{\Sigma}_{K,S^c} \cap \hat{\Sigma}_{K^c,S^c} = \emptyset$.

(4) We have $|\Gamma| = |\Sigma_{K,S}| \cdot |\Sigma_{K^c,S^c}| \cdot |\Sigma_{K,S^c}|$.

(5) Use Lemma 3.21 and Theorem 3.17. $\square$

4. THE UNIQUENESS OF THE SET $\langle U, W \mid K = \overline{S} \rangle$

In this section, we prove that $\langle U, W \mid K = \overline{S} \rangle$ is a unique subset of $\langle U, W \rangle$. In other words we show that $\langle U, W \mid K_1 = \overline{S_1} \rangle = \langle U, W \mid K_2 = \overline{S_2} \rangle$ if and only if $K_1 = K_2$ and $S_1 = S_2$.

Suppose that U and W are sets. Let

$$\text{sub}(U, W) := \{(K, S) \mid K \subseteq U \text{ and } S \subseteq W\}$$

and

$$\text{nsb}(U, W) := \{(K, S) \mid K \subset_n U \text{ and } S \subset_n W\}.$$

It is clear that $\text{sub}(U, W) = \wp(U) \times \wp(W)$ and $|\text{sub}(U, W)| = 2^{|U|+|W|}$. Let $K \subseteq U$ and $S \subseteq W$ and $((K, S))_{U, W} := \langle U, W \mid K = \overline{S} \rangle$. If there is no ambiguity, we denote $((K, S))_{U, W}$ by $((K, S))$.

Let $K_1, K_2 \subseteq U$ and $S_1, S_2 \subseteq W$, we write $((K_1, S_1)) =_n ((K_2, S_2))$ if and only if $((K_1, S_1)) = ((K_2, S_2))$, $K_1, K_2 \subset_n U$ and $S_1, S_2 \subset_n W$.

Lemma 4.1. *If $K \subseteq U$ and $S \subseteq W$, then*

(1)

$$\langle U, W \mid \emptyset = \overline{S} \rangle = \langle U, S^c \rangle.$$

(2)

$$\langle U, W \mid K = \overline{\emptyset} \rangle = \begin{cases} \langle U, W \rangle & \text{if } K = \emptyset, \\ \emptyset & \text{if } K \neq \emptyset. \end{cases}$$

(3)

$$\langle U, W \mid K = \overline{W} \rangle = \begin{cases} \langle U, W \rangle & \text{if } K = U, \\ \emptyset & \text{if } K \neq U. \end{cases}$$

(4)

$$\langle U, S \rangle \subseteq \langle U, W \mid U = \overline{S} \rangle \subseteq \langle U, S^c \rangle^c.$$

Proof. (1)

$$\begin{aligned} \langle U, W \mid \emptyset = \overline{S} \rangle &= \{T : U \longrightarrow \wp^*(W) \mid \overline{S}_T = \emptyset\} \\ &= \{T : U \longrightarrow \wp^*(W) \mid \{u \in U \mid T(u) \cap S \neq \emptyset\} = \emptyset\} \\ &= \{T : U \longrightarrow \wp^*(W) \mid T'(U) \cap S = \emptyset\} \\ &= \{T : U \longrightarrow \wp^*(W) \mid T'(U) \subseteq S^c\} \\ &= \{T : U \longrightarrow P^*(S^c) \mid T \text{ is a mapping}\} \\ &= \langle U, S^c \rangle. \end{aligned}$$

(2) Follows from Lemma 1.4.

(3) If $T : U \longrightarrow \wp^*(W)$ is an arbitrary power mapping, then $\overline{W}_T = U$. Therefore the proof is complete

(4) If S is a trivial subset of W , then the proof is complete. Now, Let $S \subset_n W$, then

$$T \in \langle U, S \rangle \implies \forall u \in U : \emptyset \neq T(u) \subseteq S \subseteq W \implies T \in \langle U, W \mid U = \overline{S} \rangle$$

and

$$T \in \langle U, W \mid U = \overline{S} \rangle \implies \forall u \in U : T(u) \cap S \neq \emptyset \implies T \notin \langle U, S^c \rangle.$$

□

Lemma 4.2. *If $2 \leq |U|$ and $S \subset_n W$, then*

$$\langle U, S \rangle \subset \langle U, W \mid U = \overline{S} \rangle \subset \langle U, S^c \rangle^c.$$

Proof. We have $\langle U, S \rangle \subseteq \langle U, W \mid U = \overline{S} \rangle$ by Lemma 4.1. Let $a \in S$, $a' \in S^c$ and $x \in U$. Define

$$T(u) := \begin{cases} \{a, a'\} & \text{if } u = x, \\ \{a\} & \text{if } u \in U - \{x\}. \end{cases}$$

Then $\overline{S}_T = U$ and so $T \in \langle U, W \mid U = \overline{S} \rangle$. Also $T \notin \langle U, S \rangle$.

Now, let $x, x' \in U$, $a \in S$ and $a' \in S^c$. Define

$$\Lambda(u) := \begin{cases} \{a\} & \text{if } u = x, \\ \{a'\} & \text{if } u = x', \\ \{a'\} & \text{if } u \in U - \{x, x'\}. \end{cases}$$

Then $\Lambda \notin \langle U, W \mid U = \overline{S} \rangle$ and $\Lambda \notin \langle U, S^c \rangle$.

□

Lemma 4.3. *If U and W are nonempty sets and $S \subset_n W$, then*

$$\langle U, W \mid U = \overline{S} \rangle = \{T_s \cup T_{s'} \mid T_s \in \langle U, S \rangle \text{ and } T_{s'} \in \langle U, S^c \rangle_0\}.$$

Proof. Let $T_s(x) = T(x) \cap S$ and $T_{s'}(x) = T(x) \cap S^c$ for any $x \in U$. Then

$$T(x) = T(x) \cap (S \cup S^c) = (T(x) \cap S) \cup (T(x) \cap S^c) = T_s(x) \cup T_{s'}(x).$$

□

Corollary 4.4. *If U and W are nonempty sets and $S \subset_n W$, then*

$$|\langle U, W \mid U = \overline{S} \rangle| = ((2^{|S|} - 1)(2^{|W|-|S|}))^{|U|}.$$

Example 4.5. Assume that $|U| = 5$, $|W| = 4$ and $|S| = 3$. We have

$$(1) |\langle U, W \rangle| = (2^4 - 1)^5 = 15^5.$$

- (2) $|\langle U, W \rangle_0| = 2^{4.5} = 2^{20} = 16^5$.
 (3) $|\langle U, W \mid U = \overline{S} \rangle| = ((2^3 - 1)(2^{4-3}))^5 = (7 \cdot 2)^5 = 14^5$.
 (4) $|\langle U, W \mid U = \overline{S^c} \rangle| = ((2^1 - 1)(2^{4-1}))^5 = 8^5$.

Lemma 4.6. *If U and W are sets and $S_1, S_2 \subseteq W$, then*

$$\langle U, W \mid \emptyset = \overline{S_1} \rangle = \langle U, W \mid \emptyset = \overline{S_2} \rangle$$

if and only if $S_1 = S_2$.

Proof. It follows from Lemma 4.1. □

Lemma 4.7. *Let U and W be nonempty sets and $S_1, S_2 \subseteq W$. Then $\langle U, W \mid U = \overline{S_1} \rangle = \langle U, W \mid U = \overline{S_2} \rangle$ if and only if $S_1 = S_2$.*

Proof. ($\Rightarrow$) If $S \subset_n W$, then

$$\langle U, W \mid U = \overline{\emptyset} \rangle = \emptyset \subset \langle U, W \mid U = \overline{S} \rangle \subset \langle U, W \rangle = \langle U, W \mid U = \overline{W} \rangle.$$

($\Leftarrow$) Suppose that $S_1, S_2 \subset_n W$ and $S_1 \neq S_2$. Let $a \in S_2 - S_1$. Define $T \equiv \{a\}$ (on U). Then $T \in \langle U, W \mid U = \overline{S_2} \rangle - \langle U, W \mid U = \overline{S_1} \rangle$. □

Lemma 4.8. *Let $((K_1, S_1)) =_n ((K_2, S_2))$ and $K_1 \neq K_2$, then $S_1 \cap S_2 = \emptyset$.*

Proof. Let $a \in S_1 \cap S_2$, $K_2 - K_1 \neq \emptyset$ and $b \in S_2^c$. Define

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K_2, \\ \{b\} & \text{if } u \in K_2^c. \end{cases}$$

Hence $T \in \langle U, W \mid K_2 = \overline{S_2} \rangle = \langle U, W \mid K_1 = \overline{S_1} \rangle$. Therefore $\overline{S_1}_T = K_1$, this is a contradiction. □

Example 4.9. If $K_1, K_2 \subset U$, then $((K_1, W)) = ((K_2, W))$ and

$$((K_1, W)) \neq_n ((K_2, W)).$$

Lemma 4.10. *If $((K_1, S_1)) = ((K_2, S_2))$ and $S_1 \neq S_2$, then $K_1 \cap K_2 = \emptyset$.*

Proof. Let $x \in K_1 \cap K_2$. If $K_2^c \neq \emptyset$, then there exists $b \in S_2^c$ by Lemma 3.7. Let $a \in S_2 - S_1$. Define

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K_2, \\ \{b\} & \text{if } u \in K_2^c. \end{cases}$$

We have $T \in \langle U, W \mid K_2 = \overline{S_2} \rangle = \langle U, W \mid K_1 = \overline{S_1} \rangle$. This is a contradiction. □

Lemma 4.11. *If U and W are nonempty sets and $((K_1, S_1)) =_n ((K_2, S_2))$, then the following statements are equivalent:*

- (1) $K_1 \neq K_2$,
- (2) $S_1 \neq S_2$,
- (3) $K_1 \cap K_2 = \emptyset$,
- (4) $S_1 \cap S_2 = \emptyset$.

Proof. It follows from Lemma 4.8 and Lemma 4.10. $\square$

Lemma 4.12. *If $((K_1, S_1)) =_n ((K_2, S_2))$ and $K_1 \neq K_2$, then $S_1 \cup S_2 = W$.*

Proof. Since $K_1 \neq K_2$, hence $S_1 \cap S_2 = \emptyset$ by Lemma 4.11. Suppose that $S_1 \cup S_2 \neq W$, $b \in W - (S_1 \cup S_2)$, $x \in K_2$ and $a \in S_2$. Define

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K_2, \\ \{b\} & \text{if } u \in K_2^c. \end{cases}$$

Then $\overline{S_{2T}} = K_2$ and hence $T \in \langle U, W \mid K_2 = \overline{S_{2T}} \rangle = \langle U, W \mid K_1 = \overline{S_{1T}} \rangle$. Since $\overline{S_{1T}} = \emptyset$ and $K_1 \neq \emptyset$, therefore $T \notin \langle U, W \mid K_1 = \overline{S_1} \rangle$, this is a contradiction. $\square$

Lemma 4.13. *If $((K_1, S_1)) =_n ((K_2, S_2))$ and $S_1 \neq S_2$, then $K_1 \cup K_2 = U$.*

Proof. Let $u \in U - (K_1 \cup K_2)$. Since $S_1 \neq S_2$, hence $K_1 \cap K_2 = S_1 \cap S_2 = \emptyset$ by Lemma 4.11. Suppose that $a_2 \in S_2$ and $a_1 \in S_1$. Define

$$T(u) = \begin{cases} \{a_2\} & \text{if } u \in K_2, \\ \{a_1\} & \text{if } u \in K_2^c. \end{cases}$$

Then $\overline{S_{2T}} = K_2$, hence $T \in \langle U, W \mid K_2 = \overline{S_2} \rangle$ and $\overline{S_{1T}} = K_2^c$. Therefore

$$u \in U - (K_1 \cup K_2) = (K_1 \cup K_2)^c \subseteq K_2^c = \overline{S_{1T}} = K_1 \subseteq K_1 \cup K_2,$$

this is a contradiction. $\square$

Lemma 4.14. *If $((K_1, S_1)) =_n ((K_2, S_2))$, then the following statements are equivalent:*

- (1) $K_1 \neq K_2$,
- (2) $S_1 \neq S_2$,
- (3) $K_1 \cap K_2 = \emptyset$,
- (4) $S_1 \cap S_2 = \emptyset$,
- (5) $K_1 \cup K_2 = U$,
- (6) $S_1 \cup S_2 = W$,
- (7) $S_2 = S_1^c$,
- (8) $K_2 = K_1^c$.

Proof. It follows from Lemma 4.11, Lemma 4.12 and Lemma 4.13. $\square$

Lemma 4.15. *If U and W are nonempty sets and $S \subset_n W$, then $\langle U, W \mid K = \overline{S} \rangle \neq \langle U, W \mid K^c = \overline{S^c} \rangle$.*

Proof. If $K = \emptyset$ ($K = U$), then $K^c = U$ ($K^c = \emptyset$) and the proof follows from Lemma 4.1.

Now, assume that $K \subset_n U$, $x \in K$, $a \in S$ and $a' \in S^c$. Define

$$T(u) = \begin{cases} \{a, a'\} & \text{if } u = x, \\ \{a\} & \text{if } u \in K - \{x\}, \\ \{a'\} & \text{if } u \in K^c. \end{cases}$$

Then $\overline{S}_T = K$ and $\overline{S^c}_T = K^c \cup \{x\}$. Therefore

$$T \in \langle U, W \mid K = \overline{S} \rangle - \langle U, W \mid K^c = \overline{S^c} \rangle.$$

□

Lemma 4.16. *Suppose that $|U_1| = 1$ and $S_1, S_2 \subset_n W$. Then*

$$((K_1, S_1)) = ((K_2, S_2))$$

if and only if $(K_1, S_1) = (K_2, S_2)$.

Proof. (1) For $S \subseteq W$, we have $((\emptyset, S)) \neq ((U_1, S))$ by Lemma 4.1.

(2) We have $((\emptyset, S_1)) = ((\emptyset, S_2))$ if and only if $S_1 = S_2$ by Lemma 4.1.

(3) We have $((U_1, S_1)) = ((U_1, S_2))$ if and only if $S_1 = S_2$, by Lemma 4.1.

Therefore the proof is complete. □

Lemma 4.17. *If U and W are nonempty sets, $2 \leq |U|$, $K \subset_n U$ and $S, S_1, S_2 \subset_n W$, then*

(1) $((\emptyset, S_1)) = ((\emptyset, S_2))$ if and only if $S_1 = S_2$.

(2) $((\emptyset, S)) \neq ((U, S))$.

(3) $((U, S_1)) = ((U, S_2))$ if and only if $S_1 = S_2$.

(4) $((\emptyset, S_1)) \neq ((U, S_2))$.

(5) $((\emptyset, S)) \neq ((K, S))$.

(6) $((\emptyset, S_1)) \neq ((K, S_2))$.

(7) $((K, S_1)) \neq ((U, S_2))$.

Proof. (1) Follow from Lemma 4.1.

(2) If $a \in S$ and $T \equiv \{a\}$ (on U), then $T \in ((U, S)) - ((\emptyset, S))$.

(3) Follow from Lemma 4.1.

(4) If $a \in S_1 \cap S_2$, define $T \equiv \{a\}$ (on U), then $T \in ((U, S_2)) - ((\emptyset, S_1))$.

Let $S_1 \cap S_2 = \emptyset$, $a_1 \in S_1$ and $a_2 \in S_2$ and define $T \equiv \{a_1, a_2\}$ (on U). Then $T \in ((U, S_2)) - ((\emptyset, S_1))$.

(5) Assume that $a \in S$, and $a' \in S^c$. Define

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K, \\ \{a'\} & \text{if } u \in K^c. \end{cases}$$

Then $T \in ((K, S)) - ((\emptyset, S))$.

(6) If $S_1 \cap S_2 \neq \emptyset$, $a \in S_1 \cap S_2$ and $a' \in S_2^c$, define

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K, \\ \{a'\} & \text{if } u \in K^c. \end{cases}$$

Then $T \in ((K, S_2)) - ((\emptyset, S_1))$. If $S_1 \cap S_2 = \emptyset$, assume that $a_1 \in S_1$ and $a_2 \in S_2$, and define

$$T(u) = \begin{cases} \{a_2\} & \text{if } u \in K, \\ \{a_1\} & \text{if } u \in K^c. \end{cases}$$

Then $T \in ((K, S_2)) - ((\emptyset, S_1))$.

(7) Assume that $a \in S_2$. Define $T \equiv \{a\}$ (on U). Then

$$T \in ((U, S_2)) - ((K, S_1)).$$

□

Lemma 4.18. *If U and W are nonempty sets, $K \subseteq U$ and $S \subset_n W$, then $((K, S)) \subset_n \langle U, W \rangle$.*

Proof. If $S \subset_n W$, then $((\emptyset, S)) \subset_n \langle U, W \rangle$ and $((U, S)) \subset_n \langle U, W \rangle$ by Lemma 4.1. Now, assume that $K \subset_n U$, $a \in S$ and $a' \in S^c$. Let

$$T(u) = \begin{cases} \{a\} & \text{if } u \in K, \\ \{a'\} & \text{if } u \in K^c. \end{cases}$$

Then $T \in ((K, S))$, therefore $((K, S)) \neq \emptyset$. If $\Lambda \equiv \{a\}$ (on U), then $\Lambda \in \langle U, W \rangle - ((K, S))$. □

Theorem 4.19. *Let U and W be nonempty sets, $K_1, K_2 \subseteq U$ and $S_1, S_2 \subset_n W$. Then $((K_1, S_1)) = ((K_2, S_2))$ if and only if $(K_1, S_1) = (K_2, S_2)$.*

Proof. Assume that $S_1, S_2 \subset_n W$ and K_1 and K_2 are trivial subsets of U , then the proof is obtained from Lemma 4.16 and Lemma 4.17. If $K_1, K_2 \subset_n U$, then the proof is obtained from Lemma 4.14 and Lemma 4.15. □

5. THE CARDINAL NUMBER OF $((K, S))$

In this section, we determine the cardinal number of $\langle U, W \mid K = \bar{S} \rangle$.

Theorem 5.1. *If U and W are nonempty finite sets, $K \subseteq U$ and $S \subset_n W$, then*

$$|((K, S))| = (2^{|S|} - 1)^{|K|} (2^{(|W|-|S|)|K|}) (2^{|W|-|S|} - 1)^{|U|-|K|}.$$

Proof. If $K = \emptyset$, then $((\emptyset, S)) = \langle U, S^c \rangle$ by Lemma 4.1. So

$$|((\emptyset, S))| = |\langle U, S^c \rangle| = (2^{|W|-|S|} - 1)^{|U|}.$$

If $K = U$, then $((U, S)) = \{T_1 \cup T_2 \mid T_1 \in \langle U, S \rangle \text{ and } T_2 \in \langle U, S^c \rangle_0\}$. Since $\langle U, S \rangle \cap \langle U, S^c \rangle_0 = \emptyset$, hence $|((U, S))| = |\langle U, S \rangle| \cdot |\langle U, S^c \rangle_0|$. Now, according to Lemma 1.1, the statement is proved. Now, let $K \subset_n U$. We have $T \in ((K, S))$ if and only if there exist $T_1 \in \langle K, S \rangle$, $T_2 \in \langle K, S^c \rangle_0$ and $T_3 \in \langle K^c, S^c \rangle$ such that $T = \hat{T}_1 \cup \hat{T}_2 \cup \hat{T}_3$. Also,

$$\langle K, S \rangle \cap \langle K, S^c \rangle_0 = \langle K, S \rangle \cap \langle K^c, S^c \rangle = \langle K, S^c \rangle_0 \cap \langle K^c, S^c \rangle = \emptyset.$$

Therefore $|((K, S))| = |\langle K, S \rangle| \cdot |\langle K, S^c \rangle_0| \cdot |\langle K^c, S^c \rangle|$. Therefore, the assertion is proved by Lemma 1.1. $\square$

Corollary 5.2. *If U and W are nonempty finite sets, $K_1, K_2 \subseteq U$ and $S_1, S_2 \subseteq W$, $|K_1| = |K_2|$ and $|S_1| = |S_2|$, then $|((K_1, S_1))| = |((K_2, S_2))|$.*

Corollary 5.3. *If U and W are nonempty finite sets, $K \subseteq U$ and $S \subset_n W$, then*

$$|((K^c, S^c))| = 2^{(|S| \cdot |U| - |K| \cdot |W|)} |((K, S))|.$$

Example 5.4. Assume that $|U| = 5$, $|W| = 4$, $|K| = 3$ and $|S| = 2$. We have

- (1) $|\langle U, W \rangle| = (2^4 - 1)^5 = 15^5$.
- (2) $|\langle U, W \rangle_0| = 2^{(5 \cdot 4)} = 2^{20} = 16^5$.
- (3) $|((K, S))| = (2^2 - 1)^3 \cdot 2^{(4-2)3} \cdot (2^{(4-2)} - 1)^{5-3} = 3^3 \cdot 2^6 \cdot 3^2 = 3^5 \cdot 2^6 = 2 \cdot 6^5$.
- (4) $|((K^c, S^c))| = 2^{(2 \cdot 5 - 3 \cdot 4)} \cdot |((K, S))| = 2^{-2} \cdot (3^5 \cdot 2^6) = 3^5 \cdot 2^4 = \frac{1}{2} \cdot 6^5$.
- (5) $\dim((K, S)) = |B_{K,S}| = 3 \cdot 3 \cdot 2^3 \cdot 2^2 = 2^3 \cdot 6^2$.

6. CONCLUSION

This paper has demonstrated that mappings are as comprehensive as all relations. In particular, It has been verified that there exists a natural one-to-one correspondence between all relations from A to B and all mappings from A to $\wp(B)$. These mappings have been termed as “power mappings”.

Given that the concept of relations is fundamental in disciplines such as mathematics and computer science. As a result, it is an important and foundational subject. This paper explored some of the key properties of this concept.

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Seid Morteza Musavi Ahmadabadi

Department of Mathematical Sciences, Yazd University, Yazd, Iran.

Email: `s.morteza.musavi.a@stu.yazd.ac.ir`

Seid Mohammad Anvariye

Department of Mathematical Sciences, Yazd University, Yazd, Iran.

Email: `anvariye@yazd.ac.ir`

Saeed Mirvakili

Department of Mathematical Sciences, Yazd University, Yazd, Iran.

Email: `saeedmirvakili@yazd.ac.ir`

APPLICATIONS OF ROUGH SET THEORY TO POWER MAPPINGS
AND THEIR RELATED SPACES

S. M. MUSAVI AHMADABADI, S. M. ANVARIYEH AND S. MIRVAKILI

کاربردهای نظریه مجموعه‌های ناهموار در نگاشت‌های توانی و فضاهاى مرتبط با آنها

سید مرتضی موسوی احمد آبادی^۱، سید محمد انوریه^۲ و سعید میروکیلی^۳

^{۱،۲،۳} دانشکده علوم ریاضی، دانشگاه یزد، یزد، ایران

در این مقاله، ابتدا مفهوم نگاشت توانی را معرفی کرده و وجود یک تناظر یک‌به‌یک طبیعی بین تمامی روابط از A به B و تمامی نگاشت‌های توانی از A به $\mathcal{P}(B)$ را نشان می‌دهیم. سپس، چندین عمل دوتایی روی یک نگاشت توانی تعریف کرده و ویژگی‌های بنیادی آنها را بررسی می‌کنیم. در ادامه، فضای توانی $\langle U, W \mid K = \overline{S} \rangle$ را مورد مطالعه قرار داده و ثابت می‌کنیم که این فضا یک پایه یکتا دارد. در پایان، چندین زیرمجموعه‌ی مهم از یک نگاشت توانی را تعریف می‌کنیم، نتایج بنیادی را اثبات کرده و مثال‌های متنوعی ارائه می‌دهیم.

کلمات کلیدی: مجموعه ناهموار، نگاشت توانی، رابطه، فضای توانی، پایه، مولد.