

COMMUTATIVE DIAGRAMS OF EXACT SEQUENCES OF S -ACTS BASED ON CONGRUENCES

R. Khosravi*, M. Roueentan and E. Rahimi

ABSTRACT. In the theory of S -acts, congruences are not generally defined by subacts. However, as factor acts are defined by congruences, the congruences are of as great importance in the theory of S -acts. This paper focuses on studying the generalized notion of a (short) exact sequence for S -acts with respect to a congruence. The behavior of chain conditions for these sequences is examined. The main purpose of this paper is to investigate commutative diagrams of these generalized exact sequences. Finally, we generalize the Snake Lemma for S -acts.

1. INTRODUCTION

The study of S -acts has found applications in various branches of mathematics, including algebraic automata theory. In many instances, the study of acts over monoids seeks to mirror situations encountered in the theory of modules over rings. In 2000, Kilp et al. published a comprehensive study on the properties of S -acts ([10]). Studying the properties of a short exact sequence is a key tool in homological algebra. Also, there are many works on commutative diagrams of exact sequences in module theory (see [2]). In the category of S -acts, Rees short exact sequences of S -acts were first introduced in [3]. Additionally, Chen and Shum, in [4], investigated split Rees short exact sequences of S -acts, uncovering results that differ from the classical findings for split short exact sequences of modules. Also, in [7], the authors considered conditions under which a special property, such as various kinds of flatness properties, can be transferred under Rees exact sequences. More recently, the idea of Rees short exact sequences for S -posets over a pomonoid S has been examined in [6, 8]. In [5], the authors introduced quasi-exact sequences as a generalization of exact sequences for modules. Also, in [1], the authors introduced the concept of quasi exact sequences as a generalization of Rees exact sequences in the category of S -acts. In Section 2, we continue our investigation of Rees short exact sequences based on congruences of S -acts, and obtain properties of these sequences. Specifically, we generalize the

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*Corresponding author.

behavior of chain conditions over these sequences. In Section 3, we focus on some commutative diagrams. For instance, we generalize the Snake Lemma for S -acts.

Throughout this paper, we denote a monoid by S and a right S -act by A_S . First, let us recall some concepts. A non-empty set A is called a *right S -act* if there exists a mapping $A \times S \rightarrow A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. A function $f : A \rightarrow B$ of S -acts is said to be an *S -morphism* if $f(as) = f(a)s$ for each $a \in A$ and $s \in S$. A non-empty subset B of a right S -act A_S is called a *subact* of A if $bs \in B$ for all $s \in S$ and $b \in B$. Also, products and coproducts of non-empty families of S -acts are the cartesian product and disjoint unions, respectively.

Unlike the cases of groups, rings and modules, in the category of S -acts their congruences are not defined by special subacts, and so we have to use congruences for the desired characterizations. Recall that a *congruence* on an S -act A_S is an equivalence relation ρ on A_S where in addition $a \rho a'$ implies $as \rho a's$

for any $a, a' \in A_S$ and $s \in S$. Also for an S -act A_S we denote the diagonal congruence by $\Delta_A = \{(a, a) \mid a \in A\}$. Any subact B_S of A_S defines the *Rees congruence* ρ_B on A , by setting $a \rho_B a'$ if $a, a' \in B$ or $a = a'$.

Let $f : A \rightarrow B$ be an S -morphism, as we know $f(A)$ is a subact of B_S . We denote the Rees congruence $\rho_{f(A)} = (f(A) \times f(A)) \cup \Delta_B$ by $\mathcal{I}_f$. Moreover, any subact A' of A gives rise to a kernel congruence $\ker \pi$ which is precisely $\rho_{A'}$ where $\pi : A \rightarrow A/A'$ is the canonical epimorphism. Conversely, unlike the situation with modules, morphisms cannot be characterized by subacts and therefore we have to use the kernel congruence for the desired characterization. In general, for a right congruence ρ on A , the factor act of A by ρ denoted by A/ρ is a right S -act. Additionally, for an S -morphism $f : A \rightarrow B$, the *kernel congruence of f* is the congruence

$$\ker f = \{(a, a') \in A \times A \mid f(a) = f(a')\}$$

on A . For more information on S -acts, we refer the reader to [10].

Now, let us recall the definition of Rees exact sequences from [4]. Let A_S, B_S and C_S be S -acts. Also, suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are homomorphisms. Then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called a *Rees exact sequence* at B if $\mathcal{I}_f = \ker g$, and this Rees exact sequence is called a *Rees short exact sequence* if f is a monomorphism and g is an epimorphism.

2. ρ -EXACT SEQUENCE

In [1], for a sequence $A \xrightarrow{f} B \xrightarrow{g} C$, the trivial congruence Δ_{C_S} in the definition of Rees exact sequence is substituted by a congruence ρ on C_S . Thus the notion of ρ -exact sequence is defined for a congruence ρ on C , as follows. Recall that $\mathcal{K}_g(\rho) = \{(b, b') \in B \times B \mid g(b) \rho g(b')\}$ where $g : B \rightarrow C$ is an S -morphism. Clearly, $\mathcal{K}_g(\Delta_{C_S}) = \ker g$ which is denoted by $\mathcal{K}_g$. In fact the concept of ρ -exact sequences generalizes the idea of exact sequences in the category of S -acts by incorporating congruences. The key condition is that the image of the first morphism must coincide with the kernel of the second morphism with respect to the congruence ρ .

Definition 2.1. Let A_S, B_S and C_S be S -acts, and ρ be a congruence on C_S . Also, suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$ are homomorphisms. Then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called ρ -exact sequence at B if $\mathcal{I}_f = \mathcal{K}_g(\rho)$, and this ρ -exact sequence is called a *short ρ -exact* if f is a monomorphism and g is an epimorphism.

In [1, Example 1] there are two examples of ρ -exact sequences. We give another example.

Example 2.2. Let A_S be an S -act, and define $f : A \rightarrow A \amalg A$ by $f(a) = (a, a)$ and $i : A \amalg A \rightarrow A \amalg A$ the identity map. Clearly, $B = \Delta_A$ is a subact of $A \amalg A$. Take ρ_B the Rees congruence on $A \amalg A$. Then the sequence

$$A \xrightarrow{f} A \amalg A \xrightarrow{i} A \amalg A$$

is ρ_B -exact. Indeed, $\mathcal{I}_f = \mathcal{K}_i(\rho_B) = \rho_B$.

The following lemma presents the relation between ρ -exact sequence and Rees exact sequence.

Lemma 2.3. Let A_S, B_S and C_S be S -acts, and ρ be a congruence on C_S . Then $A \xrightarrow{f} B \xrightarrow{g} C$ is a ρ -exact sequence at B_S if and only if

$$A \xrightarrow{f} B \xrightarrow{\pi g} C/\rho$$

is a Rees exact sequence at B_S , where $\pi : C \longrightarrow C/\rho$ is the natural epimorphism.

Proof. We have

$$(b, b') \in \mathcal{K}_{\pi g} \Leftrightarrow \pi g(b) = \pi g(b') \Leftrightarrow (g(b), g(b')) \in \rho \Leftrightarrow (b, b') \in \mathcal{K}_g(\rho).$$

Then $\mathcal{K}_{\pi g} = \mathcal{K}_g(\rho)$, and the result follows. $\square$

The following two remarks illustrate how ρ -exact sequences occur naturally.

Remark 2.4. Let A_i, B_i, C_i be S -acts, for $i \in I$, and ρ_i a congruence on C_i . It is easily checked that

$$\rho = \prod_{i \in I} \rho_i = \{(\{c_i\}_{i \in I}, \{d_i\}_{i \in I}) \mid (c_i, d_i) \in \rho_i, \text{ for all } i \in I\}$$

is a congruence on $\prod_{i \in I} C_i$. Suppose that the sequences

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

are ρ_i -exact at B_i for all $i \in I$. Define $\prod_{i \in I} f_i : \prod_{i \in I} A_i \longrightarrow \prod_{i \in I} B_i$ by $\prod_{i \in I} f_i(\{a_i\}_{i \in I}) = \{f_i(a_i)\}_{i \in I}$. The sequence

$$\prod_{i \in I} A_i \xrightarrow{\prod_{i \in I} f_i} \prod_{i \in I} B_i \xrightarrow{\prod_{i \in I} g_i} \prod_{i \in I} C_i$$

is ρ -exact at $\prod_{i \in I} B_i$, because

$$\mathcal{I}_{\prod_{i \in I} f_i} = \prod_{i \in I} \mathcal{I}_{f_i} = \prod_{i \in I} \mathcal{K}_{g_i} = \mathcal{K}_{\prod_{i \in I} g_i}.$$

If ρ is a congruence on C_S , and $h : C \longrightarrow D$ is an S -morphism, then it is easily checked that $h(\rho) = \{(h(a), h(b)) \mid (a, b) \in \rho\} \cup \Delta_D$ is a congruence on D_S .

Remark 2.5. Let ρ and σ be congruences on S -acts C_S and E_S , respectively. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ and $C \xrightarrow{h} D \xrightarrow{k} E$ are ρ -exact and σ -exact at B and D , respectively. If g is an epimorphism and h is a monomorphism, then

$$A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{k} E$$

is $h(\rho)$ -exact and σ -exact at B and D , respectively.

Proposition 2.6. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a ρ -exact sequence at B_S , and g is an epimorphism. If A_S is generated by a set A' and C_S is generated by a set C' , then B_S is generated by $f(A') \cup g^{-1}(C')$.

Proof. Let $A_S = \bigcup_{i \in I} a_i S$ and $C_S = \bigcup_{j \in J} c_j S$. Since g is an epimorphism, there exists $b_j \in B$, $j \in J$, such that $g(b_j) = c_j$. To show that $B_S = (\bigcup_{i \in I} f(a_i)S) \cup (\bigcup_{j \in J} b_j S)$, suppose that $b \in B$. So, for some $k \in J$

and $s \in S$, $g(b) = c_k s = g(b_k s)$. Then $(b, b_k s) \in \mathcal{K}_g(\rho) = \mathcal{I}_f$. Thus, $b = b_k s \in \bigcup_{j \in J} b_j S$ or $b \in f(A) = \bigcup_{i \in I} f(a_i)S$, and the result follows. $\square$

The following corollary can be similarly proved.

Corollary 2.7. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a ρ -exact sequence at B_S , and g is an epimorphism. If A_S and C_S are finitely generated, then so is B_S .*

Recall from [9] that an S -act is called *artinian* (*noetherian*) if it satisfies the descending (ascending) chain condition on its congruences. Moreover, a *Rees artinian* (*Rees noetherian*) S -act is defined as one that satisfies the descending (ascending) chain condition on its subacts. The following theorem presents a generalization of [9, Theorem 2.3] to short ρ -exact sequences.

Theorem 2.8. *Let the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ be a short ρ -exact sequence at B_S . Then B_S is artinian (noetherian) if and only if A_S and C_S are artinian (noetherian).*

Proof. If B is artinian (noetherian), then by [9, Lemma 2.1], A_S and C_S are artinian (noetherian).

Conversely, if A_S and C_S are artinian (noetherian), then so is C/ρ . By Lemma 2.3, $A \xrightarrow{f} B \xrightarrow{g} C/\rho$ is a Rees short exact sequence. Since both A_S and C/ρ are artinian (noetherian), by [9, Theorem 2.3], so is B_S . $\square$

The previous theorem is also valid for Rees artinian and Rees noetherian.

Proposition 2.9. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a ρ -exact sequence at B_S and B' be a subact of B_S such that $B'' = B' \cap f(A) \neq \emptyset$. Take $C' = g(B')$, $A' = f^{-1}(B'')$, $f' = f|_{A'}$, $g' = g|_{B'}$ and $\sigma = \rho \cap \mathcal{I}_{g'}$. Then $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ is a σ -exact sequence at B' .*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a ρ -exact sequence at B_S and B' be a subact of B such that $B' \cap f(A) \neq \emptyset$. By assumption $\mathcal{I}_f = \mathcal{K}_g(\rho)$. We show that $\mathcal{I}_{f'} = \mathcal{K}_{g'}(\sigma)$. Suppose that $b \neq c$ and $(b, c) \in \mathcal{I}_{f'}$. The definition of $\mathcal{I}_{f'}$ implies the existence of $a, d \in A'$ with $b = f(a)$ and $c = f(d)$. So, $(b, c) \in \mathcal{K}_g(\rho)$, and then $(g(b), g(c)) \in \rho$. Moreover, $b, c \in B'$ implies that $(g(b), g(c)) \in \sigma$. Hence, $\mathcal{I}_{f'} \subseteq \mathcal{K}_{g'}(\sigma)$.

To show the converse, let $(b, c) \in \mathcal{K}_{g'}(\sigma)$ for $b \neq c$ in B' . Since $\sigma \subseteq \rho$, $(b, c) \in \mathcal{K}_g(\rho)$. Now, $\mathcal{I}_f = \mathcal{K}_g(\rho)$ implies that $(b, c) \in \mathcal{I}_f$. Then $b = f(a)$ and $c = f(d)$ for some $a, d \in A$. So $f(a), f(d) \in f(A) \cap B' = B''$, and then $a, d \in A'$ which implies that $(b, c) \in \mathcal{I}_{f'}$, as required. $\square$

3. COMMUTATIVE DIAGRAMS OF ρ -EXACT SEQUENCE

In this section, we discuss some properties of commutative diagrams in the sense that their rows are ρ -exact sequences.

Proposition 3.1. *Let the following diagram be commutative such that the upper row is ρ -exact and the lower row is σ -exact, and g , r and ι are epimorphisms:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \iota & & \downarrow \kappa & & \downarrow \lambda \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

Then $\sigma = \lambda(\rho)$.

Proof. We consider the diagram stated in the assumption. First, we show that $\lambda(\rho) \subseteq \sigma$. Suppose that $(n, n') \in \lambda(\rho)$. Then there exists $(c, c') \in \rho$ such that $(n, n') = (\lambda(c), \lambda(c'))$. Since g is an epimorphism, $(c, c') = (g(b), g(b'))$ for some $b, b' \in B$. So $(b, b') \in \mathcal{K}_g(\rho) = \mathcal{I}_f$. Then $b = b'$ or $(b, b') = (f(a), f(a'))$ for some $a, a' \in A$. If $b = b'$, then $c = c'$ and $(n, n') = (\lambda(c), \lambda(c)) \in \sigma$. Otherwise, $(b, b') = (f(a), f(a'))$ for some $a, a' \in A$. From $h\iota = \kappa f$, we get

$$(\kappa(f(a)), \kappa(f(a'))) = (h(\iota(a)), h(\iota(a'))).$$

Hence,

$$(\kappa(b), \kappa(b')) = (\kappa(f(a)), \kappa(f(a'))) \in \mathcal{I}_h = \mathcal{K}_r(\sigma).$$

Thus, $(r(\kappa(b)), r(\kappa(b'))) \in \sigma$. Moreover, $r\kappa = \lambda g$ implies that

$$(n, n') = (\lambda(c), \lambda(c')) = (\lambda(g(b)), \lambda(g(b'))) = (r(\kappa(b)), r(\kappa(b'))) \in \sigma.$$

Therefore, $\lambda(\rho) \subseteq \sigma$.

To prove the inclusion $\sigma \subseteq \lambda(\rho)$, suppose that $(n, n') \in \sigma$. Since r is an epimorphism, there exist $m, m' \in M$ such that $n = r(m)$ and $n' = r(m')$. Then $(r(m), r(m')) \in \sigma$ which implies that (m, m') is an element of $\mathcal{K}_r(\sigma) = \mathcal{I}_h$. Then $m = m'$ or $(m, m') = (h(l), h(l'))$ for some $l, l' \in L$. If $m = m'$, then $n = n'$. By the definition of $\lambda(\rho)$, $\Delta_N \subseteq \lambda(\rho)$ and so $(n, n') \in \lambda(\rho)$. Otherwise, $(m, m') = (h(l), h(l'))$ for some $l, l' \in L$. Since ι is an epimorphism, $l = \iota(a)$ and $l' = \iota(a')$ for some $a, a' \in A$. Now, take $b = f(a)$, $b' = f(a')$, we get $(b, b') \in \mathcal{I}_f = \mathcal{K}_g(\rho)$. Thus $(g(b), g(b')) \in \rho$.

Hence, from $\lambda g = r\kappa$ and $\kappa f = h\iota$, we get

$$\begin{aligned} (n, n') &= (r(m), r(m')) = (r(h(l)), r(h(l'))) = (r(h(\iota(a))), r(h(\iota(a')))) \\ &= (r(\kappa(f(a))), r(\kappa(f(a')))) \\ &= (r(\kappa(b)), r(\kappa(b'))) \\ &= (\lambda(g(b)), \lambda(g(b'))) \\ &\in \lambda(\rho). \end{aligned}$$

Therefore, $\sigma = \lambda(\rho)$. □

Definition 3.2. Let the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ be ρ -exact at B_S and the sequence $L \xrightarrow{h} M \xrightarrow{r} N$ be σ -exact at M_S , where ρ and σ are congruences on C_S and N_S , respectively. These two sequences are said to be *isomorphic* if there exist isomorphisms ι , λ and κ such that the following diagram of S -morphisms is commutative:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \iota & & \downarrow \kappa & & \downarrow \lambda \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

It is routine to check that the isomorphism of ρ -exact sequences forms an equivalence relation. Next, we generalize [3, Lemma 3.2] to ρ -exact sequences.

Theorem 3.3. Suppose that the two sequences $A \xrightarrow{f} B \xrightarrow{g} C$ and $L \xrightarrow{h} M \xrightarrow{r} N$ are isomorphic by the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \iota & & \downarrow \kappa & & \downarrow \lambda \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

Then the following hold.

- (i) The upper row is ρ -exact at B if and only if the lower row is $\lambda(\rho)$ -exact at M .
- (ii) The upper row is a short ρ -exact sequence if and only if the lower row is a short $\lambda(\rho)$ -exact sequence.

Proof. (i) Necessity. Let the upper row be ρ -exact at B_S . Then $\mathcal{I}_f = \mathcal{K}_g(\rho)$. To show that $\mathcal{I}_h \subseteq \mathcal{K}_r(\lambda(\rho))$, let $m \neq m'$ and $(m, m') \in \mathcal{I}_h$. Then $m = h(l)$ and $m' = h(l')$ for some $l, l' \in L$. Since ι is an isomorphism, $l = \iota(a)$ and $l' = \iota(a')$ for some $a, a' \in A$. Now, from $h\iota = \kappa f$, we have $m = \kappa(f(a))$. Also,

$m = \kappa(b)$ for some $b \in B$. Since κ is a monomorphism, $b = f(a)$. Similarly, $m' = \kappa(b')$ for some $b' = f(a') \in B$. Hence, $(b, b') \in \mathcal{I}_f \subseteq \mathcal{K}_g(\rho)$, and so $(g(b), g(b')) \in \rho$. Thus,

$$r(m) = r\kappa(b) = \lambda(g(b)) \lambda(\rho) \lambda(g(b')) = r\kappa(b') = r(m'),$$

and $(m, m') \in \mathcal{K}_r(\lambda(\rho))$. Therefore, $\mathcal{I}_h \subseteq \mathcal{K}_r(\lambda(\rho))$. To show the converse, let $(m, m') \in \mathcal{K}_r(\lambda(\rho))$ for some $m \neq m' \in M$. So $(r(m), r(m'))$ is in $\lambda(\rho)$. Let $m = \kappa(b)$ and $m' = \kappa(b')$ for some b, b' in B . Then

$$\lambda(g(b)) = r\kappa(b) = r(m) \lambda(\rho) r(m') = r\kappa(b') = \lambda(g(b')),$$

and so $(g(b), g(b')) \in \rho$. Now, $\mathcal{K}_g(\rho) \subseteq \mathcal{I}_f$ implies that (b, b') is a member of $\mathcal{I}_f$. Hence, $b = f(a)$ and $b' = f(a')$ for some $a, a' \in A$. We get $\kappa(b) = \kappa(f(a))$, $\kappa(f(a')) = \kappa(b')$, and so $m = \kappa(f(a)) = h(\iota(a))$ and $h(\iota(a')) = \kappa(f(a')) = m'$. Thus, $(m, m') \in \mathcal{I}_h$, and the result follows.

Sufficiency can be proved similarly.

(ii) Let the upper row be a short ρ -exact sequence. By utilizing the fact that the diagram is commutative, it can be easily verified that r is an epimorphism. Now, by part (i) it is enough to show that h is a monomorphism. Let $h(l) = h(l')$ for some $l, l' \in L$. The surjectivity of ι implies the existence of $a, a' \in A$ with $l = \iota(a)$ and $l' = \iota(a')$. So

$$\kappa f(a) = h\iota(a) = h(l) = h(l') = h\iota(a') = \kappa f(a').$$

From the fact that κ and f are monomorphisms, it follows that $a = a'$ and so h is a monomorphism. Using a dual argument, the converse can be proved. $\square$

Let C, N be S -acts, and $\lambda : C \rightarrow N$ be an S -morphism. Define $\text{coker}(\lambda) = C/\mathcal{I}_\lambda$. If σ is a congruence on N , then it is straightforward to verify that

$$\mu = \{([n_1]_{\mathcal{I}_\lambda}, [n_2]_{\mathcal{I}_\lambda}) \mid (n_1, n_2) \in \sigma \vee \mathcal{I}_\lambda\}.$$

is a congruence on $\text{coker}(\lambda)$.

Recall that the join $\rho \vee \sigma$ is the congruence defined by $x (\rho \vee \sigma) y$ if there exist $z_1, z_2, \dots, z_n \in N$ such that $x = z_1$, $y = z_n$ and $(z_i, z_{i+1}) \in \{\rho, \sigma\}$, $i \in \{1, \dots, n-1\}$. In fact, using the definition of $\mathcal{I}_\lambda$, it can be seen that an arbitrary length n can be reduced to $n = 1$. Hence,

$$([n_1]_{\mathcal{I}_\lambda}, [n_2]_{\mathcal{I}_\lambda}) \in \mu \Leftrightarrow (n_1, t), (t, n_2) \in \{\mathcal{I}_\lambda, \sigma\}, \text{ for some } t \in N.$$

Finally, we present a generalization of the Snake Lemma, as the following theorem.

Theorem 3.4. *Let the sequence $L \xrightarrow{h} M \xrightarrow{r} N$ be σ -exact at M_S , where σ is a congruence on N_S . Suppose the following diagram is commutative:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \iota & & \downarrow \kappa & & \downarrow \lambda \\ L & \xrightarrow{h} & M & \xrightarrow{r} & N \end{array}$$

where g is an epimorphism, and $\iota, \kappa, \lambda, f$ are S -morphisms. Then the sequence $\text{coker}(\iota) \longrightarrow \text{coker}(\kappa) \longrightarrow \text{coker}(\lambda)$ is μ -exact at $\text{coker}(\kappa)$ where

$$\mu = \{([n_1]_{\mathcal{I}_\lambda}, [n_2]_{\mathcal{I}_\lambda}) \mid (n_1, n_2) \in \sigma \vee \mathcal{I}_\lambda\}.$$

Proof. Take $\phi : \text{coker}(\iota) \longrightarrow \text{coker}(\kappa)$ by $\phi([l]_{\mathcal{I}_\iota}) = [h(l)]_{\mathcal{I}_\kappa}$ and

$$\chi : \text{coker}(\kappa) \longrightarrow \text{coker}(\lambda) \text{ by } \chi([m]_{\mathcal{I}_\kappa}) = [r(m)]_{\mathcal{I}_\lambda}.$$

First we show that ϕ is well-defined. If $[l]_{\mathcal{I}_\iota} = [l']_{\mathcal{I}_\iota}$, then $(l, l') \in \mathcal{I}_\iota$. So $l = l'$ or $l, l' \in \text{Im} \iota$. If $l = l'$, then $\phi([l]_{\mathcal{I}_\iota}) = \phi([l']_{\mathcal{I}_\iota})$. Now, let $\iota(a) = l$, $\iota(a') = l'$ for some $a, a' \in A$. Then $h(l) = h(\iota(a)) = \kappa(f(a)) \in \kappa(B)$ and

$$h(l') = h(\iota(a')) = \kappa(f(a')) \in \kappa(B).$$

These imply that $[h(l)]_{\mathcal{I}_\kappa} = [h(l')]_{\mathcal{I}_\kappa}$, and ϕ is well-defined. Similarly, one can prove that χ is well-defined. Clearly, ϕ and χ are S -morphisms.

To prove that $\mathcal{K}_\chi(\mu) \subseteq \mathcal{I}_\phi$, let $([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{K}_\chi(\mu)$. So

$$(\chi([m_1]_{\mathcal{I}_\kappa}), \chi([m_2]_{\mathcal{I}_\kappa})) = ([r(m_1)]_{\mathcal{I}_\lambda}, [r(m_2)]_{\mathcal{I}_\lambda}) \in \mu.$$

We deduce, without loss of generality, $(r(m_1), n) \in \mathcal{I}_\lambda, (n, r(m_2)) \in \sigma$ for some $n \in N$, and so $r(m_1) = n$ or $r(m_1), n \in \text{Im} \lambda$. Thus the following two cases occur.

(i). If $r(m_1) = n$, then $(m_1, m_2) \in \mathcal{K}_r(\sigma) = \mathcal{I}_h$. Then $m_1 = m_2$ or $m_1 = h(l_1)$, $h(l_2) = m_2$ for some $l_1, l_2 \in L$. If $m_1 = m_2$, then

$$([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{I}_\phi.$$

If $m_1 = h(l_1)$, $h(l_2) = m_2$, then

$$([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) = ([h(l_1)]_{\mathcal{I}_\kappa}, [h(l_2)]_{\mathcal{I}_\kappa}) \in \mathcal{I}_\phi.$$

Therefore, $\mathcal{K}_\chi(\mu) \subseteq \mathcal{I}_\phi$.

(ii). If $r(m_1), n \in \text{Im} \lambda$, then $r(m_1) = \lambda(c)$, $\lambda(c') = n$, for some $c, c' \in C$. As g is an epimorphism, we have $c = g(b)$, $c' = g(b')$, for some $b, b' \in B$. Thus

$$r(m_1) = \lambda(c) = \lambda g(b) = r\kappa(b) \text{ and } n = \lambda(c') = \lambda g(b') = r\kappa(b').$$

Then $(r\kappa(b'), r(m_2)) \in \sigma$, and so $(\kappa(b'), m_2), (\kappa(b), m_1) \in \mathcal{I}_h$. We get

$$([m_1]_{\mathcal{I}_\kappa}, [\kappa(b)]_{\mathcal{I}_\kappa}), ([\kappa(b')]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{I}_\phi,$$

and therefore $([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa})$ is a member of $\mathcal{I}_\phi$, and we have done.

Now, we show that $\mathcal{I}_\phi \subseteq \mathcal{K}_\chi(\mu)$. Let $([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{I}_\phi$. So $[m_1]_{\mathcal{I}_\kappa} = [m_2]_{\mathcal{I}_\kappa}$ or $[m_1]_{\mathcal{I}_\kappa} = [h(l_1)]_{\mathcal{I}_\kappa}$, $[h(l_2)]_{\mathcal{I}_\kappa} = [m_2]_{\mathcal{I}_\kappa}$ for some l_1, l_2 in L .

If $[m_1]_{\mathcal{I}_\kappa} = [m_2]_{\mathcal{I}_\kappa}$, then $m_1 = m_2$ or $m_1 = \kappa(b_1)$, $\kappa(b_2) = m_2$ for some $b_1, b_2 \in B$. If $m_1 = m_2$, then $([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{K}_\chi(\mu)$.

If $[m_1]_{\mathcal{I}_\kappa} = [h(l_1)]_{\mathcal{I}_\kappa}$, $[h(l_2)]_{\mathcal{I}_\kappa} = [m_2]_{\mathcal{I}_\kappa}$ for some $l_1, l_2 \in L$, then

$$(h(l_1), h(l_2)) \in \mathcal{I}_h = \mathcal{K}_r(\sigma).$$

Now,

$$(r(m_1), r(m_2)) = (r(h(l_1)), r(h(l_2))) \in \sigma.$$

Hence,

$$(\chi([m_1]_{\mathcal{I}_\kappa}), \chi([m_2]_{\mathcal{I}_\kappa})) = ([r(m_1)]_{\mathcal{I}_\lambda}, [r(m_2)]_{\mathcal{I}_\lambda}) \in \mu$$

which implies that $([m_1]_{\mathcal{I}_\kappa}, [m_2]_{\mathcal{I}_\kappa}) \in \mathcal{K}_\chi(\mu)$. Therefore, $\mathcal{I}_\phi = \mathcal{K}_\chi(\mu)$, and the result follows. $\square$

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REFERENCES

1. R. Aminizadeh, H. Rasouli and A. Tehranian, Quasi-exact Sequences of S -Acts, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 2225–2235.
2. F. Anderson and K. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.
3. Y. Chen, Projective S -acts and exact functors, *Algebra Colloq.*, **7**(1) (2000), 113–120.
4. Y. Chen and K. P. Shum, Rees short exact sequences of S -systems, *Semigroup Forum*, **65** (2002), 141–148.
5. B. Davvaz and Y. A. Parnian-Garamaleky, A note on exact sequences, *Bull. Malays. Math. Sci. Soc.*, **22** (1999), 53–56.
6. S. Irannezhad and A. Madanshekar, Projective S -posets and hom functor, *Algebra Colloq.*, **22**(1) (2015), 975–984.
7. M. Jafari, A. Golchin and H. Mohammadzadeh Saany, Rees short exact sequence and flatness properties, *Semigroup Forum*, **99** (2019), 32–46.
8. R. Khosravi, Rees short exact sequences of S -posets, *J. Algebr. Syst.*, **4**(2) (2017), 123–134.
9. R. Khosravi and M. Roueentan, Chain conditions on (Rees) congruences of S -acts, *J. Algebra Appl.*, **23**(5) (2024), Article ID: 2450085.

10. M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, W. de gruyter. Berlin, 2000.

Roghaieh Khosravi

Department of Mathematics, Faculty of Sciences, Fasa University, Fasa, Iran.

Email: khosravi@fasau.ac.ir

Mohammad Roueentan

College of Engineering, Lamerd Higher Education Center, Shiraz University of Technology, Lamerd, Iran.

Email: m.rooeintan@yahoo.com

Elham Rahimi

Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran.

Email: rahimie@shirazu.ac.ir

COMMUTATIVE DIAGRAMS OF EXACT SEQUENCES OF S -ACTS
BASED ON CONGRUENCES

R. KHOSRAVI, M. ROUEENTAN AND E. RAHIMI

نمودارهای جابجایی دنباله‌های دقیق S -سیستم‌ها بر اساس هم‌نهشتی‌ها

رقیه خسروی^۱، محمد روئین تن^۲ و الهام رحیمی^۳

^۱بخش ریاضی، دانشکده علوم، دانشگاه فسا، فسا، ایران

^۲دانشکده مهندسی، مرکز آموزش عالی لامرد، دانشگاه صنعتی شیراز، لامرد، ایران

^۳بخش ریاضی، دانشگاه آزاد اسلامی شیراز، شیراز، ایران

در نظریه S -سیستم‌ها، هم‌نهشتی‌ها به‌طور کلی توسط زیر-سیستم‌ها تعریف نمی‌شوند. با این وجود، از آنجایی که سیستم‌های خارج‌قسمتی توسط هم‌نهشتی‌ها تعریف می‌شوند، هم‌نهشتی‌ها در نظریه S -سیستم‌ها از اهمیت زیادی برخوردارند. این مقاله، مفهوم تعمیم‌یافته از یک دنباله دقیق (کوتاه) برای S -سیستم‌ها نسبت به یک هم‌نهشتی را بررسی می‌کند. رفتار شرایط زنجیری برای این دنباله‌ها مورد بررسی قرار می‌گیرد. هدف اصلی این مقاله، بررسی نمودارهای جابجایی این دنباله‌های دقیق تعمیم داده شده است. در نهایت، لم مار را برای S -سیستم‌ها تعمیم می‌دهیم.

کلمات کلیدی: S -سیستم‌ها، تکواره‌ها، دنباله‌های ρ -کامل.