

WEIGHTED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN TOPOLOGY

P. Bal* and P. Das

ABSTRACT. This research extends the concept of double weighted statistical density of two-dimensional natural number sequences, or double sequences, utilizing two weight functions g and h all within the context of topological spaces. We explore the product space of double weighted statistical convergence, providing a comprehensive framework for analyzing convergence behavior in this more complex setting. Additionally, we introduce and examine the concepts of double weighted limit points and double weighted cluster points. This work broadens the understanding of convergence in topological spaces, offering new tools and perspectives for dealing with sequences influenced by multiple weighting schemes.

1. INTRODUCTION

Both Fast [10] and Schoenberg [15] explored the idea as a summability technique and offered some basic properties of statistical convergence. Assume that $S \subseteq \mathbb{N}$ and that the set containing all natural numbers is $\mathbb{N}$. The natural density (also known as the asymptotic density) of the set S is represented by the symbol $\delta(S)$, where the expression for $\delta(S)$ is

$$\delta(S) = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : k \in S\}|}{n}.$$

The concept of convergence was broadened in 1951 by Fast [10] to encompass statistical convergence for real number sequences. The notion of statistical convergence in a topological space was first developed by Di Maio and Kočinac [12]. In a topological space (X, τ) , a sequence $\{a_n : n \in \mathbb{N}\}$ is said to be statistically convergent to $a \in X$ if the natural density of the set $\{n \in \mathbb{N} : a_n \notin U\}$, for any neighborhood U of a , is zero.

A modified form of asymptotic density was proposed by the authors of [3], whereby n^α , $0 < \alpha < 1$ was substituted for n , which is defined as the statistical convergence of order α . The natural density of the set S of order

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*Corresponding author.

α is represented by the symbol $\delta^\alpha(S)$, which is as follows:

$$\delta^\alpha(S) = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : k \in S\}|}{n^\alpha}.$$

By replacing n^α with a weight function g , the authors of [1] defined a more general type of natural density in the manner $g : \mathbb{N} \rightarrow [0, \infty)$ such that $\lim_{n \rightarrow \infty} g(n) = \infty$ and $\lim_{n \rightarrow \infty} \frac{n}{g(n)} \neq 0$. The weighted density is defined by the notation $\delta_g(S)$ of the same set $S \subset \mathbb{N}$ using the following formula:

$$\delta_g(S) = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : k \in S\}|}{g(n)}.$$

Mursaleen and Edely[13] contributed to the development of double statistical convergence in the early 2000s. The researchers extend the idea of double convergence when there exists a $n_0 \in \mathbb{N}$ such that $|x_{m,n} - x_0| < \epsilon$ for all $m, n \geq n_0$ then a double sequence $\{x_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ is said to be double statistical convergence and converges to x_0 . Some studies in this direction can be found in [2, 8, 11].

Later Rakshit et. al.[14] bring out an idea of double convergence for double sequence under the effect of two parameters α and β where $0 < \alpha, \beta < 1$.

Following the works of [1, 6, 14, 13], in our investigation we adopt a new notion called double weighted statistical convergence of double sequence with two weight functions g and h that defines with a mapping $g : \mathbb{N} \rightarrow [0, \infty)$ such that $\lim_{m \rightarrow \infty} \frac{m}{g(m)} \neq 0$ and $\lim_{m \rightarrow \infty} g(m) = \infty$; and $h : \mathbb{N} \rightarrow [0, \infty)$ such that $\lim_{n \rightarrow \infty} \frac{n}{h(n)} \neq 0$ and $\lim_{n \rightarrow \infty} h(n) = \infty$.

2. PRELIMINARIES

The underlying ideas and definitions required to comprehend the more complex subjects covered in the paper on double weighted statistical convergence in topological space (X, τ) are laid forth in this preliminary part. There are no separation axioms offered, unless noted otherwise. We also adhere to accepted ideas, terminology, and symbols from [9]. Some required certain necessary definitions and ideas are included here for the convenience of the reader.

Definition 2.1. [13] Let B be a subset of two-dimensional set of natural number $\mathbb{N} \times \mathbb{N}$. Then the double density of the set B is denoted by $\delta(B)$ and

given by

$$\delta(B) = \lim_{m,n \rightarrow \infty} \frac{|\{(k,l) \in B : k \leq m, l \leq n\}|}{mn} = \lim_{m,n \rightarrow \infty} \frac{B_{mn}}{mn}.$$

Definition 2.2. [13] A real double sequence $x = \{x_{mn}\}$ is said to be statistically convergent to x_0 if for each $\epsilon > 0$ the set

$$\{(m,n), k \leq m, l \leq n : |x_{mn} - x_0| \geq \epsilon\}$$

has double density zero.

Definition 2.3. [3, 4] Let $x = \{x_k : k \in \mathbb{N}\}$ be a real sequence and $0 < \alpha < 1$ be given. The sequence $\{x_k : k \in \mathbb{N}\}$ is said to be statistically convergent of order α if there is a real number L such that

$$\lim_{n \rightarrow \infty} \frac{|\{k \leq n : |x_k - L| \geq \epsilon\}|}{n^\alpha} = 0.$$

And we write $s^\alpha\text{-}\lim_{n \rightarrow \infty} x_k = L$.

Definition 2.4. [7] A sequence $\{a_n : n \in \mathbb{N}\}$ in a topological space X is said to be statistically convergent of order α ($0 < \alpha < 1$) to $a \in X$, if for every neighborhood U of a , $\delta^\alpha(\{n \in \mathbb{N} : a_n \notin U\}) = 0$.

Definition 2.5. [14] Let $B \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of natural numbers. Then the double asymptotic density of order α, β of the set B is denoted by $\delta^{\alpha,\beta}(B)$ is given by

$$\delta^{\alpha,\beta}(B) = \lim_{m,n \rightarrow \infty} \frac{B_{mn} = \{(k,l) \in B : k \leq m, l \leq n\}}{m^\alpha n^\beta}.$$

Definition 2.6. [14] A double sequence $\{x_{mn}\}$ in a topological space (X, τ) is said to be statistically convergent of order α, β ($0 < \alpha < 1, 0 < \beta < 1$) to $x_0 \in X$, if for every neighborhood U of x_0 , the double asymptotic density of order α, β of the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \notin U\}$ is zero.

Definition 2.7. [1] Let $P \subset \mathbb{N}$, g be a weight function and

$$P_n = \{k \leq n : k \in P\}.$$

Then the notation $\delta_g(P)$ defines the weight density of the set P is given by

$$\delta_g(P) = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : k \in P\}|}{g(n)}.$$

Definition 2.8. [6] Let g be a weight function. A sequence $\{x_n : n \in \mathbb{N}\}$ is said to be g -statistically convergent to x_0 in a topological space (X, τ) , if for every neighborhood U of x_0 , $\delta_g(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.

Definition 2.9. [6] A subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of a sequence $\{x_n : n \in \mathbb{N}\}$ is said to be s_g -dense if $\{n \in \mathbb{N} : x_n \in \{x_{n_k} : k \in \mathbb{N}\}\}$ is s_g -dense.

Definition 2.10. [14] A subsequence $\{x_{m_k, n_l} : (k, l) \in \mathbb{N} \times \mathbb{N}\}$ of a sequence $\{x_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ is called $s^{\alpha, \beta}$ -dense if

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_{m, n} \in \{x_{m_k, n_l} : (k, l) \in \mathbb{N} \times \mathbb{N}\}\}$$

is $s^{\alpha, \beta}$ -dense.

Definition 2.11. [14] $x \in X$ is said to be a α, β -statistical limit point of a double sequence (x_{mn}) in a topological space (X, τ) if there is a subsequence $(x_{m_k n_l})$ of (x_{mn}) such that $\delta^{\alpha, \beta}(\{(m_k n_l) : (k, l) \in \mathbb{N} \times \mathbb{N}\}) \neq 0$, where $\lim_{m, n \rightarrow \infty} x_{m_k n_l} = x$.

Definition 2.12. [14] $x \in X$ is said to be a α, β -statistical cluster point of a double sequence (x_{mn}) in a topological space (X, τ) if for each neighborhood U of x $\delta^{\alpha, \beta}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \in U\}) \neq 0$.

The weight functions $g, h : \mathbb{N} \rightarrow [0, \infty)$, defined as $g(m) = \ln(1 + m)$ and $g(m) = m^\alpha, 0 < \alpha < 1$; and $h(n) = \ln(1 + n)$ and $h(n) = n^\alpha, 0 < \alpha < 1$ are easily verifiable. These weight functions will be used in the development of various examples in this study.

3. $s_{g,h}$ -CONVERGENCE IN TOPOLOGY

For two ordered pairs (p, q) and (m, n) of natural numbers $(p, q) \leq (m, n)$ if and only if $p \leq m$ and $q \leq n$.

Definition 3.1. Let $S \subset \mathbb{N} \times \mathbb{N}$ be a two dimensional set of natural numbers. Then the double weighted density of the set S is utilized by the notation $\delta_{g,h}(S)$ and it is defined by

$$\delta_{g,h}(S) = \lim_{m, n \rightarrow \infty} \frac{|\{(p, q) \leq (m, n) : (p, q) \in S\}|}{g(m)h(n)}.$$

Definition 3.2. Let g, h be two weight functions. A double sequence $\{a_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ is said to be double weighted statistical convergence (or shortly $s_{g,h}$ -convergence) to a_0 in a topological space (X, τ) , if for every neighborhood U of a_0 $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m, n} \notin U\}) = 0$.

For this study, we establish

$$s_{g,h} - \lim_{m, n \rightarrow \infty} a_{m, n} = a_0.$$

Theorem 3.3. Every $s_{g,h}$ -convergent double sequence is s -convergent.

Proof. Let $\{a_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ be an $s_{g,h}$ -convergent sequence in a topological space (X, τ) such that

$$s_{g,h} - \lim_{m,n \rightarrow \infty} a_{m,n} = a_0.$$

for every neighborhood U of a_0 , $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U\}) = 0$.

Let us consider the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U\} = B$ then $\delta_{g,h}(B) = 0$.
i.e.

$$\lim_{m,n \rightarrow \infty} \frac{|\{(p,q) \in B : (p,q) \leq (m,n)\}|}{g(m)h(n)} = 0.$$

i.e.

$$\lim_{m,n \rightarrow \infty} \frac{|\{(p,q) \in B : (p,q) \leq (m,n)\}|}{mn} \times \lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)} = 0.$$

As, $\lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)} \neq 0$. So,

$$\lim_{m,n \rightarrow \infty} \frac{|\{(p,q) \in B : (p,q) \leq (m,n)\}|}{mn} = 0.$$

Therefore, $\delta(\{m,n \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U\}) = \delta(B) = 0$. Thus,

$$s - \lim_{m,n \rightarrow \infty} a_{m,n} = a_0.$$

So, every $s_{g,h}$ -convergent double sequence is s -convergent. $\square$

Every weighted statistical convergence and double statistical convergence be equate to each other when the weight functions $g(m) = m$ and $h(n) = n$. Hence $s_{g,h}$ -convergence and s -double convergence coincides. But the converse of the above theorem may not be true so we will build up a counterexample.

Example 3.4. Let (X, τ) be a topological space where $X = \{0, 1\}$ and $\tau = \{\emptyset, X, \{0\}, \{1\}\}$. Let us consider the double sequence

$$\{a_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$$

and two weight functions $g(m) = \ln(1+m)$ and $h(n) = \ln(1+n)$ such that

$$a_{m,n} = \begin{cases} 1, & \text{if } m = p^2 \text{ and } n = q^3, \text{ for some } (p,q) \in \mathbb{N} \times \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Open neighborhoods of 1 are $U_1 = X$ and $U_2 = \{1\}$. So for the neighborhood U_2 of 1, the set

$$\begin{aligned} & \{(m,n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_2\} \\ &= \{(1,1), (1,8), (1,27), (1,64), \dots, (4,1), (4,8), (4,27), \dots\}. \end{aligned}$$

That implies the double weighted density of the set

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_2\}$$

is

$$\begin{aligned} & \delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_2\}) \\ &= \delta_{g,h}(\{(1, 1), (1, 8), (1, 27), (1, 64), \dots, (4, 1), (4, 8), (4, 27), \dots\}) \\ &= \lim_{m,n \rightarrow \infty} \frac{mn}{\ln(1+m^2) \ln(1+n^3)} \\ &= \infty \\ &\neq 0. \end{aligned}$$

Therefore, $a_{m,n} \xrightarrow{s_{g,h}\text{-lim}} 1$.

Similarly open neighborhoods of 0 are $V_1 = \{X\}$ and $V_2 = \{0\}$. So for the neighborhood V_2 of 0, the set

$$\begin{aligned} & \{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin V_2\} \\ &= \{(1, 2), (1, 3), (1, 4), \dots, (2, 1), (2, 2), (2, 3), \dots\} \end{aligned}$$

which implies the double weighted density of the set is

$$\begin{aligned} & \delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin V_2\}) \\ &= \delta_{g,h}(\{(1, 2), (1, 3), (1, 4), \dots, (2, 1), (2, 2), (2, 3), \dots\}) \\ &= \infty \\ &\neq 0. \end{aligned}$$

So $a_{m,n} \xrightarrow{s_{g,h}\text{-lim}} 0$.

Thus the double sequence $\{a_{m,n}\}$ is not $s_{g,h}$ -convergent. But for the neighborhood U_1 of 1, the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_1\} = \emptyset$ i.e. the asymptotic density $\delta(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_1\}) = 0$. Also for the neighborhood U_2 of 1 the set

$$\begin{aligned} & \{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_2\} \\ &= \{(1, 1), (1, 4), (1, 27), \dots, (4, 1), (4, 8), (4, 27), \dots\} \end{aligned}$$

i.e. the asymptotic density $\delta(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin U_2\}) = 0$. Therefore,

$$a_{m,n} \xrightarrow{s\text{-lim}} 1.$$

Hence the double sequence $\{a_{m,n}\}$ is s -double convergent but not $s_{g,h}$ -convergent.

Theorem 3.5. *In a first countable space, if a sequence*

$$\{a_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$$

is $s_{g,h}$ -convergent to a_0 , then there exist

$$\Omega = \{(K_1, L_1) < (K_2, L_2) < (K_3, L_3) < \dots\} \subseteq \mathbb{N} \times \mathbb{N}$$

such that $\delta_{g,h}(\Omega) = \lim_{n \rightarrow \infty} \frac{n^2}{g(n)h(n)}$ and $\lim_{m,n \rightarrow \infty, (m,n) \in \Omega} a_{m,n} = a_0$.

Proof. Let $\{a_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ be a double sequence in a first countable space X such that $s_{g,h} - \lim_{m,n \rightarrow \infty} a_{m,n} = a_0$. Let $B_{1,a_0} \supseteq B_{2,a_0} \supseteq B_{3,a_0} \supseteq \dots$ be countable decreasing local base of a_0 . For every $r \in \mathbb{N}$, we set $\Omega_r = \{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \in B_{r,a_0}\}$. Clearly $\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots$. But

$$s_{g,h} - \lim_{m,n \rightarrow \infty} a_{m,n} = a_0.$$

So $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : a_{m,n} \notin B_{r,a_0}\}) = 0 \forall r \in \mathbb{N}$. i.e. $\delta_{g,h}(\Omega_r^c) = 0 \forall r \in \mathbb{N}$. i.e. $\delta_{g,h}(\Omega_r) = \lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)} \forall r \in \mathbb{N}$. Let $\{\frac{i^2}{g(i)h(i)} : i \in \mathbb{N}\}$ be a monotone increasing unbounded sequence. Take $(p_1, q_1) \in \Omega_1$ arbitrarily and as $\delta_{g,h}(\Omega_2) = \lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)}$. So there exists $(p_2, q_2) \in \Omega_2$ such that $(p_2, q_2) > (p_1, q_1)$ and for all $m > p_2$ and $n > q_2$ that implies

$$\frac{|\{(i, j) \in \Omega_2 : i \leq m, j \leq n\}|}{g(m)h(n)} > \frac{2 \times 2}{g(2)h(2)}.$$

As $\delta_{g,h}(\Omega_3) = \lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)}$. So there exists $(p_3, q_3) \in \Omega_3$ such that $(p_3, q_3) > (p_2, q_2)$ and for all $m > p_3$ and $n > q_3$ that implies

$$\frac{|\{(i, j) \in \Omega_3 : i \leq m, j \leq n\}|}{g(m)h(n)} > \frac{3 \times 3}{g(3)h(3)}.$$

Proceeding in this way we obtain $(p_r, q_r) \in \Omega_r$ such that

$$(p_1, q_1) < (p_2, q_2) < (p_3, q_3) < \dots < (p_r, q_r)$$

and for all $m > p_r$ and $n > q_r$ that implies

$$\frac{|\{(i, j) \in \Omega_r : i \leq m, j \leq n\}|}{g(m)h(n)} > \frac{r^2}{g(r)h(r)}.$$

Let Ω be the set which contains all the two-dimensional natural number $\{(i, j) \in \mathbb{N} \times \mathbb{N} : (i, j) \in [1, p_1] \times [1, q_1]\}$ and also contains all the two-dimensional natural number $\{(i, j) \in \mathbb{N} \times \mathbb{N} : (i, j) \in [p_r, p_{r+1}] \times [q_r, q_{r+1}]\}$ where $(r = 1, 2, 3, \dots)$ which belong to Ω_r . So

$$\frac{|\{(i, j) \in \Omega : i \leq m, j \leq n\}|}{g(m)h(n)} \geq \frac{|\{(i, j) \in \Omega_r : i \leq m, j \leq n\}|}{g(m)h(n)} > \frac{r^2}{g(r)h(r)}.$$

for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ and $p_r \leq m \leq p_{r+1}$, $q_r \leq n \leq q_{r+1}$. Therefore,

$$\delta_{g,h}(\Omega) = \lim_{n \rightarrow \infty} \frac{n^2}{g(n)h(n)}.$$

Let Y be the neighborhood of a_0 and $B_{r,a_0} \subset Y$. If $(i, j) \in \Omega$ such that $i \geq p_r$ and $j \geq q_r$ then there exist a $t > r$ such that $p_t \leq i \leq p_{t+1}$ and $q_t \leq j \leq q_{t+1}$. So $(i, j) \in \Omega_t$. Thus for every $(i, j) \in \Omega$, $i \geq p_r$ and $j \geq q_r$. Therefore, $a_{i,j} \in B_{t,a_0} \subset B_{r,a_0} \subset Y$. Thus $\lim_{m,n \rightarrow \infty, (m,n) \in \Omega} a_{m,n} = a_0$. $\square$

Example 3.6. Limit of an $s_{g,h}$ -convergent sequence may not be unique. Let (X, τ) be a topological space where $X = \{0, 1, 2\}$ and $\tau = \{\emptyset, X, \{0, 1\}, \{2\}\}$. Consider the weight functions $g, h : \mathbb{N} \rightarrow \mathbb{N}$ such that $g(i) = \ln(1 + i)$, $h(j) = \ln(1 + j)$ and the double sequence $\{\alpha_{i,j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ where

$$\alpha_{i,j} = \begin{cases} 0, & \text{if } i = p^m \text{ and } j = q^m, \text{ for some } (p, q) \in \mathbb{N} \times \mathbb{N} \text{ and } m \in \mathbb{N} \\ 1, & \text{otherwise.} \end{cases}$$

Open neighborhoods of 0 are $V_1 = X$ and $V_2 = \{0, 1\}$. So for the neighborhood V_1 of 0, the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V_1\} = \emptyset$ which implies the double weighted density of the set is $\delta_{g,h}(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V_1\}) = 0$. Also the set

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V_2\} = \{(1, 1), (4, 4), (27, 27), (256, 256), \dots\}$$

has the double weighted density zero. i.e.

$$\begin{aligned} & \delta_{g,h}(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V_2\}) \\ &= \delta_{g,h}(\{(1, 1), (4, 4), (27, 27), (256, 256), \dots\}) \\ &= \lim_{i,j \rightarrow \infty} \frac{ij}{\ln(1 + i^i) \ln(1 + j^j)} \\ &= 0. \end{aligned}$$

Thus for every neighborhood V of 0, $\delta_{g,h}(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V\}) = 0$. Therefore,

$$\alpha_{i,j} \xrightarrow{s_{g,h}-lim} 0.$$

But the neighborhoods of 0 are the only neighborhoods of 1. So for every neighborhood of 1 the double weighted density

$$\delta_{g,h}(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V\}) = 0.$$

Therefore,

$$\alpha_{i,j} \xrightarrow{s_{g,h}-lim} 1.$$

Thus the limit of the $s_{g,h}$ -convergence sequence $\{\alpha_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ is not unique.

Theorem 3.7. *In a Hausdörff space, the limit of an double weighted convergent sequence is unique.*

Proof. Let $\{\alpha_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ be a double sequence in a Hausdörff space X such that

$$\alpha_{i,j} \xrightarrow{s_{g,h}-lim} P \text{ and } \alpha_{i,j} \xrightarrow{s_{g,h}-lim} Q \text{ with } P \neq Q.$$

Since X is a Hausdörff space, there exist two open neighborhoods $U, V \in \tau$ such that $P \in U$ and $Q \in V$. Also $U \cap V = \emptyset$. Now,

$$\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin U\}) = 0.$$

i.e. $\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \in X \setminus U\}) = 0$. But $V \subseteq X \setminus U$ and since, $U \cap V = \emptyset$. So

$$\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \in V\}) \leq \delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \in X \setminus U\}) = 0.$$

i.e. $\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \in V\}) = 0$. i.e.

$$\delta_{g,h}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \alpha_{i,j} \notin V\}) = \lim_{i,j \rightarrow \infty} \frac{ij}{g(i)h(j)} \neq 0,$$

which contradicts the fact that

$$\alpha_{i,j} \xrightarrow{s_{g,h}-lim} Q.$$

Therefore, $P = Q$. Hence the limit of double weighted convergence sequence is unique in Hausdörff space X . $\square$

Example 3.8. Subsequence of $s_{g,h}$ -convergent sequence may not be $s_{g,h}$ -convergent. Let $X = \{p, q\}$ and $\tau = \{\emptyset, X, \{p\}\}$. Then (X, τ) is a topological space. Let us define the weight functions $g(m) = \ln(1 + m)$ and $h(n) = \ln(1 + n)$ to solve this example. Considering the double sequence $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ where

$$x_{m,n} = \begin{cases} q, & \text{if } m = i^i \text{ and } n = j^j, (i,j) \in \mathbb{N} \times \mathbb{N} \\ p, & \text{otherwise.} \end{cases}$$

for every open neighborhood W of p , $\delta_{g,h}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : x_{m,n} \notin W\}) = 0$. So $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent sequence. On the other

hand considering the subsequence $\{x_{m_k, n_l} : (k, l) \in \mathbb{N} \times \mathbb{N}\}$ of the sequence $\{x_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ such that

$$x_{m_k, n_l} = \begin{cases} x_{k^k, l^l}, & \text{if both } k \text{ and } l \text{ are odd.} \\ x_{k^{k+1}, l^{l+1}}, & \text{if both } k \text{ and } l \text{ are even.} \end{cases}$$

Thus, for every open neighborhood U of p , the double weighted density

$$\begin{aligned} \delta_{g,h}(\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{m_k, n_l} \notin U\}) &= \lim_{k, l \rightarrow \infty} \frac{|\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{m_k, n_l} \notin U\}|}{g(k)h(l)} \\ &= \lim_{k, l \rightarrow \infty} \frac{kl}{\ln(1+2k)\ln(1+2l)} \\ &= \infty \\ &\neq 0. \end{aligned}$$

Therefore, the subsequence $\{x_{m_k, n_l} : (k, l) \in \mathbb{N} \times \mathbb{N}\}$ is not $s_{g,h}$ -convergent sequence.

We adopt the following notations:-

M_0 : the collection of all statistically convergent sequences in a space (X, τ) .

${}_gM_0$: the collection of all s_g -convergent sequences in a space (X, τ) .

${}_{g,h}M_0$: the collection of all $s_{g,h}$ -convergent double sequence in a space (X, τ) .

Theorem 3.9. *Let $g_1(x)$, $g_2(x)$, $h_1(x)$ and $h_2(x)$ be weight functions. If $g_1(n) < g_2(n)$ and $h_1(n) < h_2(n) \forall n \in \mathbb{N}$. Then ${}_{g_1, h_1}M_0 \subseteq {}_{g_2, h_2}M_0$.*

Proof. Let $\{B_{p,q} : (p, q) \in \mathbb{N} \times \mathbb{N}\}$ be an s_{g_1, h_1} -convergent sequence in a topological space (X, τ) and for all $n \in \mathbb{N}$ $g_1(n) < g_2(n)$ and $h_1(n) < h_2(n)$. Then there exist a b_0 such that for every neighborhood W of b_0 ,

$$\lim_{m, n \rightarrow \infty} \frac{|\{(p, q) \in \mathbb{N} \times \mathbb{N} : B_{p,q} \notin W\}|}{g_1(n)h_1(n)} = 0.$$

Now,

$$\begin{aligned} 0 &= \lim_{m, n \rightarrow \infty} \frac{|\{(p, q) \in \mathbb{N} \times \mathbb{N} : B_{p,q} \notin W\}|}{g_1(n)h_1(n)} \\ &\geq \lim_{m, n \rightarrow \infty} \frac{|\{(p, q) \in \mathbb{N} \times \mathbb{N} : B_{p,q} \notin W\}|}{g_2(n)h_2(n)} \\ &\implies \lim_{m, n \rightarrow \infty} \frac{|\{(p, q) \in \mathbb{N} \times \mathbb{N} : B_{p,q} \notin W\}|}{g_2(n)h_2(n)} \\ &= 0. \end{aligned}$$

i.e. $\{B_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is an s_{g_2,h_2} -convergent sequence. Therefore, $_{g_1,h_1}M_0 \subseteq _{g_2,h_2}M_0$. $\square$

Definition 3.10. A subset $D \subseteq \mathbb{N} \times \mathbb{N}$ is said to be double weighted statistically g, h -dense (or shortly $s_{g,h}$ -dense) if

$$\delta_{g,h}(D) = \lim_{m,n \rightarrow \infty} \frac{mn}{g(m)h(n)}.$$

It is obvious that union of two $s_{g,h}$ -dense set is $s_{g,h}$ -dense. But the intersection of two $s_{g,h}$ -dense set is not $s_{g,h}$ -dense since, $(\mathbb{N} \setminus 2\mathbb{N}) \cap 2\mathbb{N} = \emptyset$ while both the set of odd natural numbers $(\mathbb{N} \setminus 2\mathbb{N})$ and the set of even natural numbers $(2\mathbb{N})$ are both $s_{g,h}$ -dense.

Definition 3.11. Let $\{a_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ be a double sequence in a topological space (X, τ) . Then a subsequence $\{a_{m_p,n_q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is said to be double weighted statistically g, h -dense (or shortly $s_{g,h}$ -dense) if $\{(m,n) \in \mathbb{N} \times \mathbb{N} : x_{m,n} \in \{x_{m_p,n_q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}\}$ is $s_{g,h}$ -dense.

Theorem 3.12. In a topological space (X, τ) , a sequence is $s_{g,h}$ -convergent if and only if every $s_{g,h}$ -dense subsequence is $s_{g,h}$ -convergent.

Proof. Let $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ be a double sequence in a topological space (X, τ) and every $s_{g,h}$ -dense subsequence of the sequence $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent. Now, $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is a subsequence to itself. Also,

$$\begin{aligned} & \lim_{p,q \rightarrow \infty} \frac{|\{(m,n) \in \mathbb{N} \times \mathbb{N} : a_{p_m,q_n} \in \{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}\}|}{g(p)h(q)} \\ &= \lim_{p,q \rightarrow \infty} \frac{pq}{g(p)h(q)} \\ &\neq 0. \end{aligned}$$

Therefore, the double sequence $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -dense in itself. So $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent.

Conversely, let $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent sequence and the subsequence $\{a_{p_m,q_n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ be an $s_{g,h}$ -dense subsequence of $\{a_{p,q} : (p,q) \in \mathbb{N} \times \mathbb{N}\}$. If possible let us assume that the subsequence is not $s_{g,h}$ -convergent i.e. for each $x \in X$ there exist a neighborhood W of x such that $\delta_{g,h}(\{(p_m,q_n) \in \mathbb{N} \times \mathbb{N} : a_{p_m,q_n} \notin W\}) \neq 0$. Also

$$\{(p,q) \in \mathbb{N} \times \mathbb{N} : a_{p,q} \notin W\} \supseteq \{(p_m,q_n) \in \mathbb{N} \times \mathbb{N} : a_{p_m,q_n} \notin W\}.$$

i.e

$$\begin{aligned} & \delta_{g,h}(\{(p, q) \in \mathbb{N} \times \mathbb{N} : a_{p,q} \notin W\}) \\ & \geq \delta_{g,h}(\{(p_m, q_n) \in \mathbb{N} \times \mathbb{N} : a_{p_m,q_n} \notin W\}) \\ & \neq 0. \end{aligned}$$

i.e. $\delta_{g,h}(\{(p, q) \in \mathbb{N} \times \mathbb{N} : a_{p,q} \notin W\}) \neq 0$ which contradicts the fact that $\{a_{p,q} : (p, q) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent. Hence every $s_{g,h}$ -dense subsequence of a $s_{g,h}$ -convergent sequence must be $s_{g,h}$ -convergent. $\square$

4. $s_{g,h}$ -CONVERGENCE OF DOUBLE SEQUENCE IN PRODUCT SPACE

Theorem 4.1. *If $\{a_m : m \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ be two sequences in two topological spaces (X, τ) and (Y, σ) respectively such that $a_m \xrightarrow{s_g-lim} a$ and $b_n \xrightarrow{s_h-lim} b$ with $\lim_{m \rightarrow \infty} \frac{m}{g(m)}$ and $\lim_{n \rightarrow \infty} \frac{n}{h(n)}$ are finite. Then the double sequence $\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent to (a, b) in the product space $X \times Y$.*

Proof. $\mathcal{B} = \{P \times Q : P \in \tau \text{ and } Q \in \sigma\}$ is the base of the product space $X \times Y$. Let U be an arbitrary open neighborhood of (a, b) . Then there exist $P \times Q \in \mathcal{B}$ such that $(a, b) \in P \times Q \subset U$ where $a \in P \in \tau$ and $b \in Q \in \sigma$. But $a_m \xrightarrow{s_g-lim} a$ and $b_n \xrightarrow{s_h-lim} b$. i.e. $\lim_{n \rightarrow \infty} \frac{|\{m \in \mathbb{N} : a_m \notin P\}|}{g(m)} = 0$ and $\lim_{n \rightarrow \infty} \frac{|\{n \in \mathbb{N} : b_n \notin Q\}|}{h(n)} = 0$. Now,

$$\begin{aligned} & = \lim_{m,n \rightarrow \infty} \frac{|\{(m, n) \in \mathbb{N} \times \mathbb{N} : (a_m, b_n) \notin P \times Q\}|}{g(m)h(n)} \\ & \leq \lim_{m,n \rightarrow \infty} \frac{|\{m \in \mathbb{N} : a_m \notin P\}|n + |\{n \in \mathbb{N} : b_n \notin Q\}|m}{g(m)h(n)} \\ & \leq \lim_{m \rightarrow \infty} \frac{|\{m \in \mathbb{N} : a_m \notin P\}|}{g(m)} \times \lim_{n \rightarrow \infty} \frac{n}{h(n)} \\ & \quad + \lim_{n \rightarrow \infty} \frac{|\{n \in \mathbb{N} : b_n \notin Q\}|}{h(n)} \times \lim_{m \rightarrow \infty} \frac{m}{g(m)}. \end{aligned}$$

Therefore, $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : (a_m, b_n) \notin p \times Q\}) = 0$. So the double sequence $\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ is $s_{g,h}$ -convergent to (a, b) in the product space $X \times Y$. $\square$

But the converse of the above theorem 4.1 does not guaranteed to be true.

Example 4.2. The $s_{g,h}$ -convergence of the double sequence

$$\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$$

in the product space $X \times Y$ does not guaranteed that the sequences $\{a_m : m \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ are weighted statistical convergence in the respective topological spaces. Let $X = \{r, s\}$ and $\tau = \{\emptyset, \{r\}, \{s\}, X\}$. Then (X, τ) is a topological space. Consider a double sequence

$$\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$$

in the product space $X \times X$ where,

$$a_m = \begin{cases} r, & \text{if } n = k^k, \text{ for some } k \in \mathbb{N} \\ s, & \text{otherwise.} \end{cases}$$

and

$$b_n = \begin{cases} r, & \text{if } n = k^2, \text{ for some } k \in \mathbb{N} \\ s, & \text{otherwise.} \end{cases}$$

If we take the weight functions $g(m) = \ln(1 + m)$ and $h(n) = n^{\frac{1}{2}}$ for the sequences $\{a_m : m \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ respectively. Then $\{a_m : m \in \mathbb{N}\}$ is s_g -convergent but $\{b_n : n \in \mathbb{N}\}$ is not s_h -convergent. Now,

$$\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$$

is a double sequence in the product space $X \times X$, the smallest open set containing $\{s\} \times \{s\} = \{s, s\} = U$. For every open set V containing $\{s, s\}$,

$$\begin{aligned} & \lim_{m,n \rightarrow \infty} \frac{|\{(l, k) : l \leq m, k \leq n \text{ and } (a_l, b_k) \notin U\}|}{g(m)h(n)} \\ &= \lim_{m,n \rightarrow \infty} \frac{mn}{\ln(1 + m^m)n^{\frac{1}{2}}} \\ &= 0. \end{aligned}$$

Thus,

$$a_{m,n} \xrightarrow{s_{g,h}-lim} (s, s).$$

Hence the double sequence $\{(a_m, b_n) : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ in the product space $X \times X$ is $s_{g,h}$ -convergent but its projection $\{b_n : n \in \mathbb{N}\}$ is not s_h -convergent.

5. $s_{g,h}$ -LIMITS, $s_{g,h}$ -LIMIT POINTS AND $s_{g,h}$ -CLUSTER POINTS

Definition 5.1. In a topological space (X, τ) , a point $t \in X$ is said to be a g, h -weighted statistical limit point of a double sequence

$$\{t_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$$

if there exist a subsequence $\{t_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ of $\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ such that the double weighted density of the set $\{(m_i, n_j) : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ is non zero.

Definition 5.2. In a topological space (X, τ) , a point $t \in X$ is said to be a g, h -weighted statistical cluster point of a double sequence

$$\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\},$$

if for every neighborhood W of t the double weighted density of the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{m, n} \in W\}$ is non-zero.

For our research, we use the following notations:

$L_{g, h}(t_{mn})$: the set of all g, h -weighted statistical limits of

$$\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}.$$

$\Lambda_{g, h}(t_{mn})$: the set of all g, h -weighted statistical limit points of

$$\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}.$$

$\Gamma_{g, h}(t_{mn})$: the set of all g, h -weighted statistical cluster points of

$$\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}.$$

Theorem 5.3. In a topological space (X, τ) , for any double sequence $\{t_{mn}\}$, the set $\Gamma_{g, h}(t_{mn})$ is a closed set.

Proof. Let $\{t_{m, n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ be a double sequence in a topological space (X, τ) with the set of g, h -weighted statistical cluster point $\Gamma_{g, h}(t_{mn})$. Let W be an arbitrary neighborhood of $t \in cl(\Gamma_{g, h}(t_{mn}))$. So,

$$(W \cap \Gamma_{g, h}(t_{mn})) \setminus \{t\} \neq \emptyset.$$

Also let us consider another point $t' \in W \cap \Gamma_{g, h}(t_{mn})$. Then there exist a neighborhood V of t' such that $V \subset W$ and

$$\delta_{g, h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in V\}) = p \neq 0.$$

Now, as $W \subset V$ and t' is a cluster point. So

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in V\} \subseteq \{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in W\}.$$

i.e.,

$$\begin{aligned} 0 \neq p &= \delta_{g, h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in V\}) \\ &\leq \delta_{g, h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in W\}). \end{aligned}$$

Therefore, $\delta_{g, h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in W\}) = p \neq 0$ and $t \in \Gamma_{g, h}(t_{mn})$. Thus, $\Gamma_{g, h}(t_{mn}) = cl(\Gamma_{g, h}(t_{mn}))$. Hence $\Gamma_{g, h}(t_{mn})$ is a closed set. $\square$

Theorem 5.4. *In a topological space (X, τ) , for any double sequence $\{t_{mn}\}$, the set of all g, h -weighted statistical limit points of $\{t_{mn}\}$ is a subset of the set of all g, h -weighted statistical cluster points of $\{t_{mn}\}$.*

Proof. Let t be g, h -weighted statistical limit point of a double sequence $\{t_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$. Then there exists a subsequence

$$\{t_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$$

of $\{t_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ such that $\delta_{g,h}(\{(m_i, n_j) : (i, j) \in \mathbb{N} \times \mathbb{N}\}) \neq 0$. Also $\lim_{m,n \rightarrow \infty} t_{m_i, n_j} = t$. Let W be a neighborhood of the point t and G be any finite set with $\delta_{g,h}(G) = 0$. Now,

$$\{(m_i, n_j) : (i, j) \in \mathbb{N} \times \mathbb{N}\} \setminus G \subseteq \{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{m,n} \in W\}.$$

i.e.

$$\begin{aligned} 0 &\neq \delta_{g,h}(\{(m_i, n_j) : (i, j) \in \mathbb{N} \times \mathbb{N}\} \setminus G) \\ &\leq \delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{m,n} \in W\}). \end{aligned}$$

Thus, $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{m,n} \in W\}) \neq 0$. So $t \in \Gamma_{g,h}(t_{mn})$. Hence, $\Lambda_{g,h}(t_{mn}) \subseteq \Gamma_{g,h}(t_{mn})$. $\square$

Theorem 5.5. *In a topological space (X, τ) , if there exists two double sequences $\{p_{mn}\}$ and $\{q_{mn}\}$ such that $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : p_{mn} \neq q_{mn}\}) = 0$. Then $\Lambda_{g,h}(p_{mn}) = \Lambda_{g,h}(q_{mn})$ and $\Gamma_{g,h}(p_{mn}) = \Gamma_{g,h}(q_{mn})$.*

Proof. Let $\{p_{mn} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ and $\{q_{mn} : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ be two double sequences of the topological space (X, τ) . Let $p \in \Lambda_{g,h}(p_{mn})$. Therefore, there exists a subsequence $\{p_{m_i, n_j}\}$ of $\{p_{mn}\}$ such that

$$\delta_{g,h}(\{p_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}) \neq 0.$$

Now,

$$\begin{aligned} &\{q_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\} \\ &\supseteq \{p_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\} \setminus \{(m, n) \in \mathbb{N} \times \mathbb{N} : p_{mn} \neq q_{mn}\} \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &\neq \delta_{g,h}(\{q_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}) \\ &\geq \delta_{g,h}(\{p_{m_i, n_j} : (i, j) \in \mathbb{N} \times \mathbb{N}\} \setminus \{(m, n) \in \mathbb{N} \times \mathbb{N} : p_{mn} \neq q_{mn}\}). \end{aligned}$$

But, $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : p_{mn} \neq q_{mn}\}) = 0$. So $p \in \Lambda_{g,h}(q_{mn})$. Thus, $\Lambda_{g,h}(p_{mn}) \subseteq \Lambda_{g,h}(q_{mn})$, also it is clear from symmetry $\Lambda_{g,h}(q_{mn}) \subseteq \Lambda_{g,h}(p_{mn})$ which implies $\Lambda_{g,h}(p_{mn}) = \Lambda_{g,h}(q_{mn})$. Hence, $\Gamma_{g,h}(p_{mn}) = \Gamma_{g,h}(q_{mn})$ can also be easily shown in a similar way. $\square$

Theorem 5.6. *In a topological space (X, τ) , for any double sequence $\{t_{mn}\}$, $L_{g,h}(t_{mn}) \subseteq \Gamma_{g,h}(t_{mn})$.*

Proof. Let $\{t_{mn}\}$ is g, h -weighted statistically convergent to t i.e. $t \in L_{g,h}(t_{mn})$. Then for every neighborhood V of t , $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \notin V\}) = 0$. i.e. $\delta_{g,h}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : t_{mn} \in V\}) \neq 0$. Thus, $t \in \Gamma_{g,h}(t_{mn})$. Hence $L_{g,h}(t_{mn}) \subseteq \Gamma_{g,h}(t_{mn})$. $\square$

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REFERENCES

1. A. A. Adem and M. Altinok, Weighted statistical convergence of real valued sequences, *Facta Univ. Ser. Math. Inform.*, **35**(3) (2020), 887–898.
2. P. Bal and L. D. R. Kočinac, On selectively star-ccc spaces, *Topology Appl.*, **281** (2020), Article No. 107184.
3. S. Bhunia, P. Das and S. K. Pal, Restricting statistical convergence, *Acta Math. Hungar.*, **134**(1-2) (2012), 153–161.
4. R. Çolak and Ç. A. Bektaş, λ -statistical convergence of order α , *Acta Math. Sci. Ser. B.*, **31**(3) (2010), 953–959.
5. J. Connor, The statistical and strong p -Cesaro convergence of sequences, *Analysis*, **8** (1988), 47–63.
6. P. Das and P. Bal, Statistical convergence restricted by weight functions and its application in the variation of γ -covers, *Rend. Istit. Mat. Univ. Trieste*, **56** (2024), Article No. 11.
7. T. Datta, P. Bal and P. Das, Statistical convergence of order α and it's application to selection principles, *Comment. Math. Univ. Carolin.*, (In press).
8. D. Djurčić, Lj. D. R. Kočinac and M. R. Žižović, Double sequences and selections, *Abstr. Appl. Anal.*, **2012** (2012), Article ID: 497594.
9. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
10. H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.
11. L. D. R. Kočinac and M. H. M. Rashid, On ideal convergence of double sequences in the topology induced by a fuzzy 2-norm, *TWMS J. Pure Appl. Math.*, **8**(1) (2017), 97–111.
12. G. D. Maio and L. D. R. Kočinac, Statistical Convergence in Topology, *Topology Appl.*, **156** (2008), 28–45.
13. M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288** (2003), 223–231.
14. D. Rakshit, P. Bal and T. Datta, Statistical convergence of order (α, β) of double sequence in topology, (submitted).
15. I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361–375.

Prasenjit Bal

Department of Mathematics, ICFAI University Tripura, Kamalghat, West Tripura, INDIA-799210.
Email: balprasenjit177@gmail.com

Parthiba Das

Department of Mathematics, ICFAI University Tripura, Kamalghat, West Tripura, INDIA-799210.
Email: parthivdas1999@gmail.com

WEIGHTED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES
IN TOPOLOGY

P. BAL AND P. DAS

همگرایی آماری وزن دار دنباله های دوگانه در توپولوژی

پراسن جیت بال^۱ و پارتیبا داس^۲

^{۱,۲}گروه ریاضی، دانشگاه ICFAI تریپورا، کمالگات، تریپورای غربی، هند

این پژوهش مفهوم چگالی آماری وزن دار دوگانه دنباله های اعداد طبیعی دوبعدی، یا همان دنباله های دوگانه، را با استفاده از دو تابع وزن g و h و در چارچوب فضاهای توپولوژیکی گسترش می دهد. در این مقاله، فضای حاصل ضرب همگرایی آماری وزن دار دوگانه را بررسی کرده و چارچوبی جامع برای تحلیل رفتار همگرایی در این محیط پیچیده تر ارائه می کنیم. همچنین، مفاهیم نقاط حدی وزن دار دوگانه و نقاط تجمع وزن دار دوگانه را معرفی و بررسی می کنیم. این پژوهش درک ما از همگرایی در فضاهای توپولوژیکی را گسترش داده و ابزارها و دیدگاه های جدیدی را برای بررسی دنباله هایی که تحت تأثیر چندین طرح وزن دهی قرار دارند، فراهم می کند.

کلمات کلیدی: چگالی طبیعی، همگرایی آماری وزن دار دوگانه، فضای حاصل ضرب، نقطه حدی آماری وزن دار، نقطه تجمع آماری وزن دار.