

MONOCHROMATIC ZERO DIVISOR SUMS IN CERTAIN FINITE COMMUTATIVE RINGS

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ABSTRACT. For a finite commutative ring R , let $a, b, c \in R$ be fixed elements. Consider the equation $a\alpha + b\beta = c\gamma$ where α, β and γ are zero divisors in R . We say that R satisfies the r -monochromatic zero divisor sum if for any r -colouring χ of the elements of the ring R , there exist $\alpha, \beta, \gamma \in R$ with $\chi(\alpha) = \chi(\beta) = \chi(\gamma)$ such that the equation holds. We define $\zeta_{(a,b,c)}(R)$ to be the least positive integer r such that R does not satisfy the r -monochromatic zero divisor sum. In this paper, we prove some results on $\zeta_{(a,b,c)}(R)$ for various values of a, b, c . Special focus is given to the case where $a = b = c = 1$.

1. INTRODUCTION

Ramsey theory is the study of preservation of certain properties under set partitions. It has undergone a great deal of research in the past century. In this paper, we attempt to extend the investigation of Ramsey-theoretic numbers to finite commutative rings. We begin with some definitions. An r -colouring χ of a set S is a surjective function $\chi : S \rightarrow \{1, \dots, r\}$. Given an r -colouring χ of a set S and an equation f in the variables of elements of S , we say that f has a *monochromatic solution* under χ if there exist elements $x_1, \dots, x_n \in S$ that satisfy f and also are monochromatic under χ , i.e., $\chi(x_1) = \dots = \chi(x_n)$. By considering different sets S and different equations f , the question of whether there exists a monochromatic solution for a given equation has led to many different lines of investigation. Below, we present one such investigation, which is the Schur's theorem, a well-known Ramsey-type theorem in integers.

Theorem 1.1. [7] [Schur's Theorem] *For any $r \geq 1$, there exists a least positive integer $s = s(r)$ such that for every r -colouring of $[1..s] \subseteq \mathbb{Z}$ there exists a monochromatic solution to $x + y = z$.*

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The numbers $s(r)$ are called the *Schur numbers* where r represents the number of colours. Schur numbers are significant for being one of the earliest results in Ramsey theory in integers. For example, it is known that $s(2) = 5$. This means that for the set $\{1, 2, 3, 4\}$, there exists a 2-colouring of its elements such that there is no monochromatic solution to $x + y = z$. As it turns out, the only Schur numbers that are currently known are $s(1) = 2, s(2) = 5, s(3) = 14, s(4) = 45$ (see [7]), and most recently, $s(5) = 161$ (see [6]). Some very recent results in this direction can be seen in [2, 5].

The concept of a modular Schur number, denoted by $h(r)$, was introduced by Abbott and Wang in [1]. Below, we present a slightly adjusted definition of the modular Schur number:

Definition 1.2. [1] The *modular Schur number* $h(r)$ is the smallest integer m_0 such that for all $m \geq m_0$, any partitioning of $\{1, 2, \dots, m\}$ into r sets implies that one of the r sets admits a solution to $x + y \equiv z \pmod{m+1}$, i.e., x, y, z are contained in one of the r sets.

By Definition 1.2, there exists a partition of $\{1, 2, \dots, m_0 - 1\}$ into r sets such that each of the r sets do not admit a solution to $x + y \equiv z \pmod{m_0}$. This line of research was further investigated by Chappelton, Marchena and Dominguez [4]. Similar to the Schur numbers, determining the modular Schur number is a very difficult endeavour and is known only for small values of modulo m . Abbott and Wang [1] determined that $h(r) = s(r)$ for $r \in \{1, 2, 3, 4\}$, and they conjectured that the equality is true for all positive integers r .

Ramsey theoretic investigation isn't just limited to integers. There has been some development of Ramsey Theory in abstract algebra recently. Sim et al. [8] investigated the existence of monochromatic solutions to the equation $ax + by = cz$ under r -colourings of a finite commutative ring R , where $a, b, c \in R$ are fixed elements and x, y and z are the idempotent, unit and any element in the ring R , respectively.

In this article, we investigate a variant of Schur's theorem in commutative rings. Henceforth, by a *ring* we always assume it is a commutative ring. Throughout this paper, the letter R will always refer to a ring. We remind the reader of the following definition.

Definition 1.3. In a ring R , an element $x \in R$ is said to be a *zero divisor* if there exists a non-zero element y such that $xy = 0$.

In particular, if $R \neq \{0\}$, then 0 is a zero divisor of R . Let $a, b, c \in R$ be fixed and consider the equation

$$aX + bY = cZ, \quad (1.1)$$

where X, Y and Z are unknowns. We say equation (1.1) has a solution if there exist zero divisors α, β and γ in R such that the equation holds, i.e., $a\alpha + b\beta = c\gamma$. Since 0 is a zero divisor, we always have the solution $(0, 0, 0)$ to equation (1.1), i.e.,

$$a \cdot 0 + b \cdot 0 = 0 = c \cdot 0.$$

So, $(0, 0, 0)$ will be called the trivial solution to equation (1.1). We say R satisfies the *r-monochromatic zero divisor sum* if for any r -colouring χ of the elements of the ring R , there exists zero divisors α, β, γ where

$$\chi(\alpha) = \chi(\beta) = \chi(\gamma)$$

such that (α, β, γ) is a non-trivial solution to equation (1.1). In other words, R satisfies the *r-monochromatic zero divisor sum*, if equation (1.1) has a monochromatic non-trivial solution (α, β, γ) for which α, β, γ are zero divisors. Note that there is no restriction on the colour of the fixed elements a, b, c . If R does not satisfy the *r-monochromatic zero divisor sum*, then there exists at least one r -colouring χ of the elements of the ring R such that there is no monochromatic non-trivial solution to equation (1.1). Recall that the sum of two zero divisors is not necessarily a zero divisor, e.g. $2 + 3 = 5$ in $\mathbb{Z}_{12}$.

Definition 1.4. A triple $\{x, y, z\}$ that satisfies

$$x + y \equiv z \pmod{n}$$

for some $n \in \mathbb{Z}^+$ is called a *Schur triple in modulo n*.

Definition 1.5. Let $\zeta_{(a,b,c)}(R)$ be the least positive integer r such that R does not satisfy the *r-monochromatic zero divisor sum*.

Note that $\zeta(R)_{(a,b,c)} = 1$ if equation (1.1) has no non-trivial solution in R . In particular, if the only zero divisor in R is the zero element, then $\zeta(R)_{(a,b,c)} = 1$. For example, in the ring $\mathbb{Z}_p$ where p is a prime, the equation (1.1) has no non-trivial solution. Next, in the ring $\mathbb{Z}_4$ of integers modulo 4; there are two zero divisors, namely 0 and 2, and we see that $1 \cdot 0 + 1 \cdot 2 = 1 \cdot 2$, i.e., $(0, 2, 2)$ is a non-trivial solution. This implies that $\zeta(\mathbb{Z}_4)_{(1,1,1)} \geq 2$. If the ring R does not satisfy the *r-monochromatic zero divisor sum* for all integers $r \geq 1$, then we set $\zeta_{(a,b,c)}(R) = \infty$. An example would be the case

where $a = 0, b = c = 1$ in the ring $\mathbb{Z}_4$ of integers modulo 4; the solution $\alpha = \beta = \gamma = 2$ is always monochromatic.

2. MAIN RESULTS

The main results will be divided into two sections, one for the case where $a = b = c = 1$ and the other for the remaining cases. Recall that if the only zero divisor in R is the zero element, then $\zeta(R)_{(a,b,c)} = 1$. So, hereafter, we may assume that R has at least two zero divisors, unless stated otherwise.

2.1. Case $(a, b, c) \neq (1, 1, 1)$.

Theorem 2.1. $\zeta_{(a,b,c)}(R) = \infty$ if and only if $a + b - c$ is a zero divisor of the ring R .

Proof. Suppose $\zeta_{(a,b,c)}(R) = \infty$. This implies that for every r -colouring χ of the elements of the ring R , equation (1.1) has a monochromatic non-trivial solution. In particular, we may choose χ such that each element in R is coloured with different colours. Since equation (1.1) has a monochromatic non-trivial solution, the only possibility would be $a\alpha + b\alpha = c\alpha$ for some non-zero zero divisor α . Thus, $(a + b - c)\alpha = 0$ and $a + b - c$ is a zero divisor of the ring R .

For the converse, suppose $a + b - c$ is a zero divisor of the ring R . Then $(a + b - c)\beta = 0$ for some non-zero element β . Clearly, β is also a zero divisor. Now, we have $a\beta + b\beta = c\beta$, which means equation (1.1) has a non-trivial solution (β, β, β) . Therefore, given any r -colouring χ of the elements of the ring R , equation (1.1) always has the monochromatic solution (β, β, β) . Hence, $\zeta_{(a,b,c)}(R) = \infty$. $\square$

Theorem 2.2. $\zeta_{(a,b,c)}(R) = 1$ if and only if equation (1.1) does not have a non-trivial solution.

Proof. Suppose $\zeta_{(a,b,c)}(R) = 1$. If equation (1.1) has a non-trivial solution (α, β, γ) , then R satisfies the 1-monochromatic zero divisor sum, a contradiction. Thus, equation (1.1) does not have a non-trivial solution.

If equation (1.1) does not have a non-trivial solution, then R does not satisfy the 1-monochromatic zero divisor sum. $\square$

Lemma 2.3. If d is a unit in R , then $\zeta_{(da,db,dc)}(R) = \zeta_{(a,b,c)}(R)$.

Proof. This follows by noting that equation (1.1) has a non-trivial solution (α, β, γ) if and only if $da\alpha + db\beta = dc\gamma$. $\square$

Theorem 2.4. *Let R be a ring such that $2d \neq 0$ for all $d \in R \setminus \{0\}$. If a is a unit in R , then $\zeta_{(a,a,0)}(R) = 3$.*

Proof. By Lemma 2.3, it is sufficient to show that $\zeta_{(1,1,0)}(R) = 3$. Let α be a non-zero zero divisor in R . Clearly, $-\alpha$ is also a non-zero zero divisor in R . Then $(\alpha, -\alpha, 0)$ is a non-trivial solution to equation (1.1) with $(a, b, c) = (1, 1, 0)$. Thus, $\zeta_{(1,1,0)}(R) \geq 2$.

Suppose $\zeta_{(1,1,0)}(R) = 2$. Then there is a 2-colouring χ_1 on R such that equation (1.1) has no monochromatic non-trivial solution. Let α be a non-zero zero divisor in R . Then $(0, 0, \alpha)$ and $(0, 0, -\alpha)$ are non-trivial solutions to equation (1.1). So, $\chi_1(0) \neq \chi_1(\alpha)$ and $\chi_1(0) \neq \chi_1(-\alpha)$. This implies that $\chi_1(\alpha) = \chi_1(-\alpha)$, but then $(\alpha, -\alpha, 0)$ is a monochromatic non-trivial solution to equation (1.1), a contradiction. Hence, $\zeta_{(1,1,0)}(R) \geq 3$.

Since $2d \neq 0$ for all $d \in R \setminus \{0\}$, we see that $d \neq -d$ for all $d \in R \setminus \{0\}$. So, we can partition $R \setminus \{0\}$ into union of the disjoint sets:

$$R \setminus \{0\} = \bigcup_{d \in R \setminus \{0\}} \{d, -d\},$$

where either $\{d, -d\} = \{d', -d'\}$ or $\{d, -d\} \cap \{d', -d'\} = \emptyset$. So, we may define a 3-colouring χ on R , such that $\chi(d) = 1$ and $\chi(-d) = 2$ for each partition $\{d, -d\}$ and set $\chi(0) = 3$. Suppose equation (1.1) with $(a, b, c) = (1, 1, 0)$ has a monochromatic non-trivial solution (α, β, γ) . Then $\alpha + \beta = 0$ and this implies that $\beta = -\alpha$. If $\alpha = 0$, then $\gamma \neq 0$. This means $\chi(\gamma) \in \{1, 2\}$ and $\chi(\alpha) = 3 \neq \chi(\gamma)$. If $\alpha \neq 0$, then $\chi(\alpha) \neq \chi(-\alpha)$ because $\{\chi(\alpha), \chi(-\alpha)\} = \{1, 2\}$. Hence, equation (1.1) has no monochromatic non-trivial solution, and $\zeta_{(1,1,0)}(R) = 3$. $\square$

Corollary 2.5. *If a is a unit in $\mathbb{Z}_n$, then*

$$\zeta_{(a,a,0)}(\mathbb{Z}_n) = \begin{cases} \infty, & \text{for } n \text{ even} \\ 3, & \text{for } n \text{ odd} \end{cases}$$

Proof. By Theorem 2.4, $\zeta_{(a,a,0)}(\mathbb{Z}_n) = 3$ if n is odd. Suppose n is even. By Lemma 2.3, it is sufficient to show that $\zeta_{(1,1,0)}(\mathbb{Z}_n) = \infty$. Let $n = 2m$ for some positive integer m . Clearly, $(m, m, 0)$ is a non-trivial solution to equation (1.1) with $(a, b, c) = (1, 1, 0)$. Hence, $\zeta_{(1,1,0)}(\mathbb{Z}_n) = \infty$. $\square$

2.2. Case $(a, b, c) = (1, 1, 1)$.

By Lemma 2.3, $\zeta_{(a,a,a)}(R) = \zeta_{(1,1,1)}(R)$ provided that a is a unit. So, we shall only consider $\zeta_{(1,1,1)}(R)$. From now onwards, we shall refer to equation (1.1) with $(a, b, c) = (1, 1, 1)$.

We begin by stating the following obvious lemma.

Lemma 2.6. *Let χ be an arbitrary r -colouring of R . If $\chi(0) = \chi(\alpha)$ for some non-zero zero divisor $\alpha \in R$, then $(\alpha, 0, \alpha)$ is a monochromatic non-trivial solution to equation (1.1).*

The following corollary is a consequence of Lemma 2.6.

Corollary 2.7. *Let χ be an r -colouring of R . If equation (1.1) does not have a monochromatic non-trivial solution, then $\chi(0) \neq \chi(\alpha)$ for all non-zero zero divisors α of R .*

Corollary 2.8. $\zeta_{(1,1,1)}(R) \geq 2$ for any ring R with at least two zero divisors.

Proof. Let r be least positive integer such that R does not satisfy the r -monochromatic zero divisor sum. This means there is an r -colouring χ such that equation (1.1) does not have a monochromatic non-trivial solution. By Corollary 2.7, $\chi(0) \neq \chi(\alpha)$ for all non-zero zero divisors α of R . Therefore, $\zeta_{(1,1,1)}(R) \geq 2$. $\square$

Theorem 2.9. $\zeta_{(1,1,1)}(R) = 2$ if and only if equation (1.1) does not have any solution (α, β, γ) where all α, β, γ are non-zero zero divisors.

Proof. Suppose $\zeta_{(1,1,1)}(R) = 2$. There exists a 2-colouring χ of R such that equation (1.1) does not have a monochromatic non-trivial solution. By Corollary 2.7, $\chi(0) \neq \chi(d)$ for all non-zero zero divisors d of R . This implies that $\chi(d_1) = \chi(d_2)$ all non-zero zero divisors d_1, d_2 of R . If equation (1.1) has solution (α, β, γ) where all α, β, γ are non-zero zero divisors, then (α, β, γ) is a monochromatic non-trivial solution, a contradiction. Hence, equation (1.1) does not have any solutions (α, β, γ) where all α, β, γ are non-zero zero divisors.

Suppose equation (1.1) does not have any solutions (α, β, γ) where all α, β, γ are non-zero zero divisors. Define a 2-colouring χ such that $\chi(0) = 1$ and $\chi(d) = 2$ for all non-zero zero divisors d of R . If equation (1.1) has a monochromatic non-trivial solution (α, β, γ) , then all α, β, γ are non-zero zero divisors, a contradiction. Hence, equation (1.1) does not have monochromatic non-trivial solution and $\zeta_{(1,1,1)}(R) = 2$. $\square$

We end this subsection with the following technical lemma which will be used in later subsections.

Lemma 2.10. *Let $J \subseteq \mathbb{R}$ and $0 \notin J$. Let χ' be an r' -colouring of J . If $|J| \geq r \geq r'$, then there is a r -colouring χ of J and a $J_1 \subseteq J$ such that*

- (a) $|J_1| = r - r'$,
- (b) *the restriction of χ to J_1 is a bijection onto $\{r' + 1, \dots, r\}$,*
- (c) $\chi(a) = \chi'(a)$ for all $a \in J \setminus J_1$,
- (d) *if (α, β, γ) is a solution to equation (1.1) under χ , then it is also a solution under χ' .*

Proof. If $r = r'$, then $J_1 = \emptyset$ and so, we may set $\chi = \chi'$. Thus, let $r > r'$.

Note that χ' is a surjective function onto $\{1, \dots, r'\}$. For each $1 \leq i \leq r'$, let

$$J'(i) = \{a \in J : \chi'(a) = i\}.$$

Now, $J'(i) \neq \emptyset$ for all i . For those i , with $|J'(i)| \geq 2$, we shall move some of these elements to the set J_1 . Eventually, we shall obtain a set J_1 such that (a) holds and $|J(i)| \geq 1$ for all $1 \leq i \leq r'$, where $J(i) = J'(i) \setminus J_1$. Let χ be a bijection of J_1 onto $\{r' + 1, \dots, r\}$. Next, set $\chi(a) = \chi'(a)$ for all $a \in \bigcup_{1 \leq i \leq r'} J(i)$. Then, χ a surjection of J onto $\{1, \dots, r\}$. By this construction, (a), (b) and (c) are satisfied.

Suppose (α, β, γ) is a solution to equation (1.1) under χ . If

$$\chi(\alpha) \in \{r' + 1, \dots, r\},$$

then from $\chi(\alpha) = \chi(\beta) = \chi(\gamma)$ and (b), we have $\alpha = \beta = \gamma$, from which it follows from equation (1.1) that $\alpha = 0$, a contradiction as $0 \notin J$. Thus, $\chi(\alpha) \in \{1, \dots, r'\}$. This means that $\alpha \in J \setminus J_1$ and $\chi(\alpha) = \chi'(\alpha)$. Similarly, $\beta, \gamma \in J \setminus J_1$, $\chi(\beta) = \chi'(\beta)$ and $\chi(\gamma) = \chi'(\gamma)$. Hence, (α, β, γ) is a solution to equation (1.1) under χ' and (d) holds. $\square$

2.2.1. Ring $\mathbb{Z}_n$ of integers modulo n .

We remind readers of the well-known property that x is a zero divisor in the ring $\mathbb{Z}_n$ of integers modulo n if and only if x and n are not coprime.

Lemma 2.11. *If equation (1.1) has a solution (α, β, γ) in $\mathbb{Z}_n$, then either α , β and γ have a non-unit common divisor, or there is no non-unit common pairwise divisor between any of the two integers in $\{\alpha, \beta, \gamma\}$.*

Proof. If α , β and γ have a non-unit common divisor, then we are done. Otherwise, there are two possibilities: either there is a non-unit common divisor between any of the two integers in $\{\alpha, \beta, \gamma\}$, or all such common divisors are a unit. In the former case, we consider the following cases.

Case 1. Let p be a non-unit common divisor such that $p|\alpha$, $p|\beta$ and $p \nmid \gamma$. This clearly implies that $p|(\alpha + \beta)$ as well. Since $\alpha + \beta = \gamma$, we have that $p|\gamma$, which is a contradiction.

Case 2. Let q be a non-unit common divisor such that $q|\alpha$, $q \nmid \beta$ and $q|\gamma$. Now, $\alpha + \beta = \gamma$ implies $\alpha - \gamma = \beta$. So, $q|(\alpha - \gamma)$, which subsequently means that $q|\beta$, which is a contradiction.

Hence, there are no non-unit common pairwise divisors between any two integers in $\{\alpha, \beta, \gamma\}$. $\square$

Lemma 2.12. *Let $n = m_1 m_2$ where m_1 and m_2 are positive integers and $m_2 \geq 2$. Then $I = \{1, \dots, m_2 - 1\}$ contains a monochromatic Schur triple in modulo m_2 under any r -colouring if and only if $J = \{m_1, 2m_1, \dots, (m_2 - 1)m_1\}$ contains a monochromatic Schur triple in modulo n under any r -colouring.*

Proof. ($\Rightarrow$) Let χ^* be an r -colouring on J . Now, define a r -colouring χ on I by $\chi(a) = \chi^*(m_1 a)$ for all $a \in I$. By the assumption, there exists a monochromatic Schur triple in I modulo m_2 , i.e., $\{x, y, z\} \subseteq I$ with $x + y \equiv z \pmod{m_2}$ and $\chi(x) = \chi(y) = \chi(z)$. Multiplying both sides of the equation by m_1 gives us $xm_1 + ym_1 \equiv zm_1 \pmod{n}$. So, $\{m_1 x, m_1 y, m_1 z\} \subseteq J$ is a monochromatic Schur triple modulo n under χ^* .

($\Leftarrow$) Let χ be a r -colouring on I . Now, define an r -colouring χ^* on J by $\chi^*(m_1 a) = \chi(a)$ for all $a \in I$. By the assumption, there exists a monochromatic Schur triple in J modulo n , i.e., $\{m_1 x, m_1 y, m_1 z\} \subseteq J$ with $m_1 x + m_1 y \equiv m_1 z \pmod{n}$ and $\chi^*(m_1 x) = \chi^*(m_1 y) = \chi^*(m_1 z)$. There exists an integer u such that $m_1 x + m_1 y = m_1 z + un$. Dividing both sides of the equation by m_1 gives us $x + y = z + um_2$. So, $\{x, y, z\} \subseteq I$ is a monochromatic Schur triple modulo m_2 under χ . $\square$

Lemma 2.13. *If $h(r)$ is the modular Schur number for r colours, then $h(r) \geq r + 1$.*

Proof. Assign each of the elements in $S = \{1, 2, \dots, r\}$ with different colours. If S admits a monochromatic Schur triple in modulo $r + 1$, then $a + a \equiv a \pmod{r + 1}$ for some $a \in S$, which implies that $a \equiv 0 \pmod{r + 1}$, a contradiction. Hence, $h(r) \geq r + 1$. $\square$

Lemma 2.14. *Let m_1 and m_2 be positive integers and $n = m_1 m_2$. If $m_2 \geq h(r) + 1$, then $\zeta_{(1,1,1)}(\mathbb{Z}_n) \geq r + 2$.*

Proof. Note that the non-zero zero divisors in $\mathbb{Z}_n$ include the non-zero elements that are multiples of m_1 . Let χ^* be a $(r + 1)$ -colouring on $\mathbb{Z}_n$. If $\chi^*(0) = \chi^*(d)$ for some non-zero zero divisors $d \in \mathbb{Z}_n$, then by Lemma 2.6,

$(d, 0, d)$ is a monochromatic non-trivial solution to equation (1.1). So, we may assume that $\chi^*(0) \neq \chi^*(d)$ for all non-zero zero divisors d of $\mathbb{Z}_n$. Now, $\mathbb{Z}_n$ contains the set $J = \{m_1, 2m_1, \dots, (m_2 - 1)m_1\}$ and $0 \notin J$. The restriction of χ^* on J is an r' -colouring on J where $r' \leq r$. Next, $m_2 - 1 \geq h(r) \geq r + 1$, where the last inequality follows from Lemma 2.13. Therefore, by Lemma 2.10, there is an r -colouring χ^{**} of J such that the conclusions in Lemma 2.10 hold.

Now, $I = \{1, 2, \dots, m_2 - 1\}$ contains a monochromatic Schur triple in modulo m_2 under any r -colouring, because $m_2 - 1 \geq h(r)$. It follows from Lemma 2.12 that J contains a monochromatic Schur triple in modulo n under χ^{**} . By Lemma 2.10, J contains a monochromatic Schur triple in modulo n under χ^* , i.e., there exist $\alpha, \beta, \gamma \in J$ such that $\alpha + \beta \equiv \gamma \pmod{n}$ and $\chi^*(\alpha) = \chi^*(\beta) = \chi^*(\gamma)$. Hence, $\zeta_{(1,1,1)}(\mathbb{Z}_n) \geq r + 1$. $\square$

Theorem 2.15. *Let $\mathbb{Z}_n$ be a ring with $n = h(r)p$ where $r \geq 2$, p is a prime and $h(r)$ is the modular Schur number for r colours. If $2h(r) \leq p$ and $h(r - 1) < h(r)$, then $\zeta_{(1,1,1)}(\mathbb{Z}_n) = r + 1$.*

Proof. Since $h(r - 1) < h(r)$, it follows from Lemma 2.14 that

$$\zeta_{(1,1,1)}(\mathbb{Z}_n) \geq r + 1.$$

It is left to show that $\zeta_{(1,1,1)}(\mathbb{Z}_n) \leq r + 1$. Let

$$P = \{jp : 1 \leq j \leq h(r) - 1\};$$

$$D = \{d : d > 1 \text{ and } d \text{ is a divisor of } h(r)\}.$$

Since $p \geq 2h(r)$, we have $\gcd(h(r), p) = 1$. Therefore, $P \cap D = \emptyset$. Note that all zero divisors of $\mathbb{Z}_n$ are contained in $P \cup D \cup \{0\}$. By Definition 1.2, there exists a r -colouring χ^* of $\{1, 2, \dots, h(r) - 1\}$ that does not admit a monochromatic Schur triple in modulo $h(r)$. Define an $(r + 1)$ -colouring χ of $\mathbb{Z}_n$, where $\chi(0) = r + 1 = \chi(u)$ for all $1 \leq u \leq n - 1$ with $\gcd(u, n) = 1$, $\chi(jp^i) = \chi^*(j)$ for all $jp^i \in P$, $\chi(d) = \chi^*(d)$ for all $d \in D$ and $d \neq h(r)$, finally set $\chi(h(r)) = 1$ if $h(r)$ is odd and set $\chi(h(r)) = 1$ if $\chi^*\left(\frac{h(r)}{2}\right) \neq 1$ and $h(r)$ is even or $\chi(h(r)) = 2$ if $\chi^*\left(\frac{h(r)}{2}\right) = 1$ and $h(r)$ is even.

Suppose equation (1.1) has a monochromatic non-trivial solution (α, β, γ) , i.e., $\alpha + \beta \equiv \gamma \pmod{n}$. Since $\chi(0) = r + 1 \neq \chi(u)$ for all non-zero zero divisors u in $\mathbb{Z}_n$, we have α, β, γ are non-zero zero divisors. By Lemma 2.11, it is sufficient to consider the following four cases:

- (a) $\alpha, \beta, \gamma \in P$, or

- (b) $\alpha, \beta, \gamma \in D$, or
- (c) $\alpha \in P$ and $\beta, \gamma \in D$, or
- (d) $\gamma \in P$ and $\alpha, \beta \in D$.

Case 1. Suppose (a) holds. Then $\alpha = j_1p$, $\beta = j_2p$ and $\gamma = j_3p$ where $j_1, j_2, j_3 \in \{1, 2, \dots, h(r) - 1\}$ and

$$\frac{\alpha}{p} + \frac{\beta}{p} \equiv \frac{\gamma}{p} \pmod{\left(\frac{n}{p}\right)},$$

which implies that

$$j_1 + j_2 \equiv j_3 \pmod{h(r)}.$$

Since $\chi^*(j_1) = \chi^*(j_2) = \chi^*(j_3)$, $\{1, 2, \dots, h(r) - 1\}$ admits a monochromatic Schur triple in modulo $h(r)$, a contradiction.

Case 2. Suppose (b) holds. If none of the elements in $\{\alpha, \beta, \gamma\}$ is equal to $h(r)$, then $\alpha, \beta, \gamma \in \{1, 2, \dots, h(r) - 1\}$. Since $\chi(d) = \chi^*(d)$ for all $d \in \{1, 2, \dots, h(r) - 1\}$, we see that $\{1, 2, \dots, h(r) - 1\}$ admits a monochromatic Schur triple $\{\alpha, \beta, \gamma\}$ in modulo $h(r)$ under χ^* , a contradiction. If $(\alpha, \beta, \gamma) = (h(r), h(r), h(r))$, then $h(r) \equiv 0 \pmod{n}$, a contradiction. So, we may assume that $(\alpha, \beta, \gamma) \neq (h(r), h(r), h(r))$. This means at most one element in $\{\alpha, \beta, \gamma\}$ can be equal to $h(r)$. It is sufficient to consider these two possibilities $\alpha = h(r)$ or $\gamma = h(r)$. If the former holds, then $h(r) + \beta \equiv \gamma \pmod{n}$. Since $\gamma \in D$, we have γ divides n and $h(r)$. So, γ divides β . Similarly, β divides γ . Thus, $\beta = \gamma$ and $h(r) \equiv 0 \pmod{n}$, a contradiction. If the latter holds, then $\alpha + \beta \equiv h(r) \pmod{n}$. Now,

$$2 - h(r) = 1 + 1 - h(r) \leq \alpha + \beta - h(r) < h(r) + h(r) - h(r) = h(r),$$

and n divides $\alpha + \beta - h(r)$ imply that $\alpha + \beta = h(r)$. As before, from $\alpha, \beta \in D$, we deduce that $\alpha = \beta$. This means $\alpha = \beta = \frac{h(r)}{2}$, a contradiction, as $\chi(h(r)) \neq \chi\left(\frac{h(r)}{2}\right) = \chi^*\left(\frac{h(r)}{2}\right)$.

Suppose (c) holds. Then $\beta \equiv \gamma \pmod{p}$. Now,

$$1 - h(r) < \beta - \gamma < h(r) - 1,$$

and $p \geq 2h(r) > h(r)$ imply that $\beta = \gamma$. This means $\alpha \equiv 0 \pmod{n}$, a contradiction.

Suppose (d) holds. Then $\alpha + \beta \equiv 0 \pmod{p}$. Now,

$$2 \leq \alpha + \beta < 2h(r),$$

and $p \geq 2h(r)$ imply that $\alpha + \beta \not\equiv 0 \pmod{p}$, a contradiction.

Hence, equation (1.1) does not have a monochromatic non-trivial solution and $\zeta_{(1,1,1)}(\mathbb{Z}_n) \leq r + 1$. $\square$

2.2.2. Rings $C_n(0)$ with cyclic additive structure and trivial multiplication.

The ring $C_n(0)$ may be written with the ring presentation

$$\langle a; na = 0, a^2 = 0 \rangle.$$

Its additive structure is isomorphic to $\mathbb{Z}_n$, but every multiplication in the ring is trivial, that is, $\forall x, y \in C_n(0) [x \cdot y = 0]$. Hence, every non-zero element is a zero divisor. We start with a concrete example: since the modular Schur number for two colours is $h(2) = 5$, we conclude that $C_n(0)$ satisfies the 2-monochromatic zero divisor sum for $n > 5$, and $C_n(0)$ does not satisfy the 2-monochromatic zero divisor sum for $n \leq 5$. Indeed, we can generalise this result below.

Theorem 2.16. *The ring $C_n(0)$ satisfies the $(r + 1)$ -monochromatic zero divisor sum for $n > h(r)$ and does not satisfy the $(r + 1)$ -monochromatic zero divisor sum for $n = h(r)$.*

Proof. If $n > h(r)$, then the ring $C_n(0)$ contains the set

$$J = \{a, 2a, \dots, (n - 1)a\}$$

where a is the generator of $C_n(0)$ and $0 \notin J$. In fact,

$$C_n(0) = J \cup \{0\}.$$

Let χ be an $(r + 1)$ -colouring of $C_n(0)$. By Lemma 2.6, if $\chi(0) = \chi(d)$ for some $d \in J$, then equation (1.1) has a monochromatic non-trivial solution. This means $C_n(0)$ satisfies the $(r + 1)$ -monochromatic zero divisor sum. So, we may assume that $\chi(0) \neq \chi(d)$ for all $d \in J$. Now, the restriction of χ to J is a r -colouring. Consider the set $I = \{1, 2, \dots, n - 1\}$. Let χ^* be an r -colouring of I defined by $\chi^*(u) = \chi(ua)$ for all $u \in I$. By Definition 1.2, I contains a monochromatic Schur triple in modulo n under any r -colouring. So, there are $\alpha, \beta, \gamma \in I$ such that

$$\alpha + \beta \equiv \gamma \pmod{n},$$

and $\chi^*(\alpha) = \chi^*(\beta) = \chi^*(\gamma)$. Now, $\alpha + \beta = \gamma + kn$ for some integer k implies that

$$\alpha a + \beta a = \gamma a + kna = \gamma a,$$

in $C_n(0)$. Since $\chi(da) = \chi^*(d)$ for all $d \in I$, we have $\chi(\alpha a) = \chi(\beta a) = \chi(\gamma a)$. Hence, $C_n(0)$ satisfies the $(r + 1)$ -monochromatic zero divisor sum.

Suppose $n = h(r)$. By Definition 1.2, there is an r -colouring χ^* of $I = \{1, 2, \dots, n-1\}$ such that I does not contain any monochromatic Schur triples in modulo n . Define a $(r+1)$ -colouring χ of $C_n(0)$ where $\chi(0) = r+1$ and $\chi(da) = \chi^*(d)$ for all $d \in I$. If equation (1.1) has a monochromatic non-trivial solution (α, β, γ) under χ in $C_n(0)$, then α, β, γ are non-zero, because $\chi(0) = r+1 \neq \chi(da)$ for all $d \in I$. Let $\alpha = d_1a$, $\beta = d_2a$ and $\gamma = d_3a$ for some $d_1, d_2, d_3 \in I$. Then $d_1a + d_2a = d_3a$, in $C_n(0)$. This implies that $(d_1 + d_2 - d_3)a = 0$ and it follows that $d_1 + d_2 - d_3 \equiv 0 \pmod{n}$, i.e., $d_1 + d_2 \equiv d_3 \pmod{n}$. Since $\chi^*(d) = \chi(da)$ for all $d \in I$, we have

$$\chi^*(d_1) = \chi^*(d_2) = \chi^*(d_3).$$

Thus, $\{d_1, d_2, d_3\}$ is a monochromatic Schur triple in modulo n under χ^* , a contradiction. Hence, equation (1.1) does not have any monochromatic non-trivial solutions, that is $C_n(0)$ does not satisfy the $(r+1)$ -monochromatic zero divisor sum. $\square$

Corollary 2.17. *If $n = h(r) > h(r-1)$, then $\zeta_{(1,1,1)}(C_n(0)) = r+1$.*

Proof. Since $n > h(r-1)$, it follows from Theorem 2.16 that $C_n(0)$ satisfies the r -monochromatic zero divisor sum. Thus, $\zeta_{(1,1,1)}(C_n(0)) \geq r+1$. Again, by Theorem 2.16, $C_n(0)$ does not satisfy the $(r+1)$ -monochromatic zero divisor sum for $n = h(r)$. Hence, $\zeta_{(1,1,1)}(C_n(0)) \leq r+1$ and the corollary follows. $\square$

3. CONCLUDING REMARKS

We have been able to obtain some results on the equation $a\alpha + b\beta = c\gamma$, but it might be possible to extend this investigation to also include more general equations involving zero divisors. An example would be an equation of the form $a_1\alpha_1 + \dots + a_{n-1}\alpha_{n-1} = a_n\alpha_n$ where $\alpha_1, \dots, \alpha_n$ are all zero divisors in a ring and $a_1, \dots, a_n$ are fixed elements. It might also be of interest to investigate equations involving units, idempotents and nilpotents in rings.

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MONOCHROMATIC ZERO DIVISOR SUMS IN CERTAIN FINITE
COMMUTATIVE RINGS

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مجموع‌های تک‌رنگ مقسوم‌علیه‌های صفر در برخی حلقه‌های جابه‌جایی متناهی

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برای یک حلقه‌ی جابه‌جایی متناهی R ، فرض کنید $a, b, c \in R$ عناصر ثابتی باشند. معادله

$$a\alpha + b\beta = c\gamma$$

را در نظر بگیرید که در آن α, β و γ مقسوم‌علیه‌های صفر در R هستند. می‌گوییم R دارای مجموع r -تک‌رنگی مقسوم‌علیه صفر است، هرگاه برای هر r -رنگ‌آمیزی χ از عناصر حلقه‌ی R ، عناصری $\alpha, \beta, \gamma \in R$ وجود داشته باشند به‌طوری‌که $\chi(\alpha) = \chi(\beta) = \chi(\gamma)$ و معادله فوق برقرار باشد. همچنین، $\zeta_{(a,b,c)}(R)$ را به‌صورت کوچک‌ترین عدد صحیح مثبت r تعریف می‌کنیم که R دارای مجموع r -تک‌رنگی مقسوم‌علیه صفر نباشد. در این مقاله، نتایجی درباره‌ی $\zeta_{(a,b,c)}(R)$ برای مقادیر مختلف a, b, c به دست می‌آوریم. به‌علاوه، تمرکز ویژه‌ای نیز بر حالت $a = b = c = 1$ داریم.

کلمات کلیدی: مجموع تک‌رنگی مقسوم‌علیه صفر، حلقه‌های جابه‌جایی متناهی، نظریه رمزی
تعمیم‌یافته.