

SINGLE-VALUED NEUTROSOPHIC S-APPROXIMATION SPACES

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ABSTRACT. This paper investigates the concept of single-valued neutrosophic S-approximation spaces (SVNSs) and explores their fundamental properties. It is demonstrated that SVNSs provide a unified framework for representing two well-established models of single-valued neutrosophic rough sets. Furthermore, the properties of the lower and upper approximation operators defined in an SVNS are examined with respect to two essential conditions: (1) monotonicity and (2) compatibility with complementation.

INTRODUCTION

S-approximation spaces were introduced in [8] as a generalization of rough set theory and the Dempster-Shafer theory of evidence. Subsequently, Shakiba and Hooshmandasl studied S-approximation spaces from the perspective of three-way decision theory in [17], following the outline presented in [30]. Furthermore, they extended S-approximation spaces to the context of neighborhood systems in [18], based on the relational framework discussed in [31]. Additional extensions to fuzzy sets [20] and intuitionistic fuzzy sets [19] have also been proposed. The relationship between belief structures and S-approximation spaces has been further explored in [21]. Moreover, S-approximation spaces have been shown to express T -rough sets [5] as well as neighborhood systems [9]. Their application in distributed decision-making was investigated by Shakiba in [16]. A recent comprehensive survey on S-approximation spaces is presented in [15].

Smarandache introduced the concept of neutrosophic set theory as a novel branch of philosophy and employed it to develop a tri-component framework for logic, set, and probability theories [22, 23]. The fundamental principle of neutrosophy can be viewed as a generalization of many-valued logic, where each logical statement is characterized independently by a degree of truth, a degree of falsity, and a degree of indeterminacy. Notably, these three degrees are mutually independent. Each membership degree in a neutrosophic set (NS) is a real standard or non-standard subset of the non-standard unit

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interval $]0^-, 1^+[$ [13, 23]. To facilitate practical applications, Wang et al. introduced the concept of single-valued neutrosophic sets (SVNSs) as a further generalization of fuzzy sets and intuitionistic fuzzy sets [26], where three membership functions are employed independently, each ranging within the unit interval $[0, 1]$.

The organization of the paper is as follows. Section 1 reviews the necessary preliminaries and establishes the notation used throughout the paper. Section 2 presents the proposed extension of intuitionistic S-approximation spaces to single-valued neutrosophic S-approximation spaces (SVNSs) through the introduction of an uncertainty mapping. In Section 3, we propose a monotonicity condition for SVNSs to ensure the preservation of several critical properties. However, one of the most significant properties cannot be guaranteed unless the SVNS satisfies the complement compatibility condition, which we introduce in Section 4. Finally, Section 5 concludes the paper and outlines potential directions for future research.

1. PRELIMINARIES

We remind some basic concepts of S-approximation spaces and intuitionistic S-approximation spaces in Sections 1.1 and 1.2, respectively. Then, we set our notation for single-valued neutrosophic sets in Section 1.3. In Section 1.4, we will review two basic approaches for neutrosophic fusion of rough set theory.

1.1. S-approximation spaces. The concept of S-approximation spaces was introduced in [8] as a generalization of Shafer-Dempster belief functions [14] as well as rough set theory [11]. Then, it was extended to three-way decisions in [30, 17].

Formally, an *S-approximation space* is constructed over two finite non-empty sets U and W which are connected by a knowledge mapping $T : U \rightarrow \mathcal{P}(W)$ and a decision mapping over the set W , e.g.

$$S : \mathcal{P}(W) \times \mathcal{P}(W) \rightarrow \{0, 1\}.$$

These four components define an S-approximation space $G = (U, W, T, S)$.

Given an S-approximation space G , the lower and the upper approximations of a set $X \subseteq W$ are defined as

$$\underline{G}(X) = \{x \in U \mid S(T(x), X) = 1\}, \quad (1.1)$$

and

$$\overline{G}(X) = \{x \in U \mid S(T(x), X^c) = 0\}, \quad (1.2)$$

respectively, where $X^c = W \setminus X$.

While designing S-approximation spaces, there are various different choices for the decider mapping. Many different decider mappings are individually treated in the literature, e.g. [12, 7]. However, some intrinsic properties of these decider mappings can be used to characterize the properties for the lower and the upper approximation operators. S-min deciders are identified and studied in [8]. Partial monotone deciders are studied in [17] and shown to be a generalization of S-min deciders.

Definition 1.1 (Partial monotone decider [17]). Let

$$S : \mathcal{P}(W) \times \mathcal{P}(W) \rightarrow \{0, 1\}$$

and $A, B \subseteq W$. We use $A \preceq_S B$ to denote that $S(A, B) = 1$. A decider S is called *partial monotone* if $X \subseteq Y$ and $A \preceq_S X$ imply that $A \preceq_S Y$ for any $A, X, Y \subseteq W$. An S-approximation space whose decider mapping is partial monotone is called a partial monotone S-approximation space.

Another class of deciders are complement compatible deciders, identified and studied in [17].

Definition 1.2 (Complement compatible decider [17]). A decider mapping $S : \mathcal{P}(W) \times \mathcal{P}(W) \rightarrow \{0, 1\}$ is called complement compatible if either $S(T(x), X) = 0$ or $S(T(x), X^c) = 0$ for all $x \in U$ and $X \subseteq W$. An S-approximation space is called a complement compatible S-approximation space if its decider is complement compatible.

Proposition 1.3 ([17]). Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. For all sets $X, Y \subseteq W$, the followings hold:

- (1) $X \subseteq Y$ implies $\overline{G}(X) \subseteq \overline{G}(Y)$,
- (2) $X \subseteq Y$ implies $\underline{G}(X) \subseteq \underline{G}(Y)$,
- (3) $\overline{G}(X \cup Y) \supseteq \overline{G}(X) \cup \overline{G}(Y)$,
- (4) $\overline{G}(X \cap Y) \subseteq \overline{G}(X) \cap \overline{G}(Y)$,
- (5) $\underline{G}(X \cup Y) \supseteq \underline{G}(X) \cup \underline{G}(Y)$,
- (6) $\underline{G}(X \cap Y) \subseteq \underline{G}(X) \cap \underline{G}(Y)$,
- (7) $\overline{G}(X) = (\underline{G}(X^c))^c$,
- (8) $\underline{G}(X) = (\overline{G}(X^c))^c$.

Proposition 1.4 ([17]). Let $G = (U, W, T, S)$ be a complement compatible S-approximation space. Then for any $X \subseteq W$, we have $\underline{G}(X) \subseteq \overline{G}(X)$.

1.2. Intuitionistic S-approximation Spaces. The concept of S-approxim-

ation spaces are generalized to intuitionistic fuzzy set theory [1] in [19]. In intuitionistic fuzzy, or IF theory, the notion of a subset A of a set W is written as $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in W\}$ where $\mu_A, \nu_A : W \rightarrow [0, 1]$ are degrees of membership and non-membership of an element $x \in W$ in A , respectively. Note that it is required that for every $x \in W$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. With these two degree functions, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is defined as the degree of non-determinacy of or hesitation of the membership of an element $x \in W$ in A . We use $\mathcal{IF}(W)$ to denote the collection of all the intuitionistic subsets of W . For $A \in \mathcal{IF}(W)$, its necessity and possibility are defined as

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in U\}, \quad (1.3)$$

and

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in U\}, \quad (1.4)$$

respectively. The complement of A in the IF theory is

$$A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in U\}. \quad (1.5)$$

An *intuitionistic fuzzy S-approximation space*, or IF S -approximation space for short, $IFG = (U, W, T, S)$, is the same as the standard S -approximation spaces with the following exceptions:

- the knowledge mapping is changed to $T : U \rightarrow \mathcal{IF}(W)$
- the decider mapping is changed to $S : \mathcal{IF}(W) \times \mathcal{IF}(W) \rightarrow [0, 1] \times [0, 1]$ with $0 \leq \text{Proj}_1^2(S(T(x), X)) + \text{Proj}_2^2(S(T(x), X)) \leq 1$, for all $x \in U$ and $X \in \mathcal{IF}(W)$. Note that $\text{Proj}_i^n((x_1, \dots, x_n)) = x_i$.

For an $IFG = (U, W, T, S)$, the *intuitionistic fuzzy lower and upper approximations* of X , or IF lower and upper approximations, with respect to IFG are defined as

$$\underline{IFG}(X) = \{\langle x, S_\mu(T(x), X), S_\nu(T(x), X) \rangle \mid x \in U\}, \quad (1.6)$$

and

$$\overline{IFG}(X) = \{\langle x, S_\nu(T(x), X^c), S_\mu(T(x), X^c) \rangle \mid x \in U\}, \quad (1.7)$$

respectively. The IF lower and upper approximation operators are connected in terms of the possibility and necessity operators.

Proposition 1.5 ([19]). *Let $IFG = (U, W, T, S)$ be an IF S -approximation space and $X \in \mathcal{IF}(W)$. Then, the followings hold:*

- (1) $\Box \overline{IFG}(X) = (\Diamond \underline{IFG}(X^c))^c$,
- (2) $\Box \underline{IFG}(X) = (\Diamond \overline{IFG}(X^c))^c$,
- (3) $\Diamond \overline{IFG}(X) = (\Box \underline{IFG}(X^c))^c$,
- (4) $\Diamond \underline{IFG}(X) = (\Box \overline{IFG}(X^c))^c$.

The IF lower and upper approximation operators are turned into the lower and the upper approximations by employing a pair of thresholds $(a, r) \in [0, 1]^2$ as follows: Let $X \in \mathcal{IF}(W)$. Then, its *lower and the upper approximations* with respect to IFG and (a, r) are defined as

$$\begin{aligned} \underline{IFG}_{(a,r)}(X) &= \{x \in U \mid S_\mu(T(x), X) \geq a \\ &\quad \text{and } S_\nu(T(x), X) < r\}, \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} \overline{IFG}_{(a,r)}(X) &= \{x \in U \mid S_\mu(T(x), X^c) < a \\ &\quad \text{and } S_\nu(T(x), X^c) \geq r\}, \end{aligned} \quad (1.9)$$

respectively.

Similar to standard S-approximation spaces, an IFS is called *monotone* if $A \subseteq B$ implies $S_\mu(T(x), A) \leq S_\mu(T(x), B)$ and $S_\nu(T(x), A) \geq S_\nu(T(x), B)$ for every $x \in U$ and $A, B \in \mathcal{IF}(W)$. When an IFS is monotone, then its IFS lower and IFS upper approximations satisfy certain properties.

Proposition 1.6 ([19]). *Let $IFG = (U, W, T, S)$ be a monotone IF S-approximation space, (a, r) a pair of thresholds, and $X, Y \in \mathcal{IF}(W)$. Then,*

- (M1): $X \subseteq Y$ implies $\overline{IFG}(X) \subseteq \overline{IFG}(Y)$,
- (M2): $X \subseteq Y$ implies $\underline{IFG}(X) \subseteq \underline{IFG}(Y)$,
- (M3): $\overline{IFG}(X \cup Y) \supseteq \overline{IFG}(X) \cup \overline{IFG}(Y)$,
- (M4): $\overline{IFG}(X \cap Y) \subseteq \overline{IFG}(X) \cap \overline{IFG}(Y)$,
- (M5): $\underline{IFG}(X \cup Y) \supseteq \underline{IFG}(X) \cup \underline{IFG}(Y)$,
- (M6): $\underline{IFG}(X \cap Y) \subseteq \underline{IFG}(X) \cap \underline{IFG}(Y)$.
- (M7): $X \subseteq Y$ implies $\overline{IFG}_{(a,r)}(X) \subseteq \overline{IFG}_{(a,r)}(Y)$,
- (M8): $X \subseteq Y$ implies $\underline{IFG}_{(a,r)}(X) \subseteq \underline{IFG}_{(a,r)}(Y)$,
- (M9): $\overline{IFG}_{(a,r)}(X \cup Y) \supseteq \overline{IFG}_{(a,r)}(X) \cup \overline{IFG}_{(a,r)}(Y)$,
- (M10): $\overline{IFG}_{(a,r)}(X \cap Y) \subseteq \overline{IFG}_{(a,r)}(X) \cap \overline{IFG}_{(a,r)}(Y)$,
- (M11): $\underline{IFG}_{(a,r)}(X \cup Y) \supseteq \underline{IFG}_{(a,r)}(X) \cup \underline{IFG}_{(a,r)}(Y)$,
- (M12): $\underline{IFG}_{(a,r)}(X \cap Y) \subseteq \underline{IFG}_{(a,r)}(X) \cap \underline{IFG}_{(a,r)}(Y)$.

Note that in general we don't have the inclusion $\underline{IFG}(X) \subseteq \overline{IFG}(X)$ for every $X \in \mathcal{IF}(W)$. The concept of complement compatible which is introduced [19], guarantees this inclusion. An IFG is called *complement compatible* if for every $x \in U$ and $A \in \mathcal{IF}(W)$, we have

$$S_\mu(T(x), A) \leq S_\nu(T(x), A^c).$$

For complement IFS, we have $\underline{IFG}(X) \subseteq \overline{IFG}(X)$ for every $X \in \mathcal{IF}(W)$.

1.3. Single-valued Neutrosophic Sets. The single-valued neutrosophic sets (SVNS) as a subclass of neutrosophic sets, is presented by Wang et al. [26] and can be used to represent uncertainty and inconsistency in the data present at the science and engineering applications [27, 3, 32, 6, 29]. In this section, we review basic definitions of single-valued neutrosophic sets and basic set operations among them, which are used throughout this paper.

Definition 1.7 ([26]). A single-valued neutrosophic set (abbreviated as SVNS) A on a universal set U is defined as

$$A = \{ \langle x, \mu_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in U \}, \quad (1.10)$$

where $\mu_A : U \rightarrow [0, 1]$ is a truth-membership function, $\nu_A : U \rightarrow [0, 1]$ is an indeterminacy-in-membership (a.k.a. uncertainty-in-membership) function, and $\omega_A : U \rightarrow [0, 1]$ is a falsity-membership function. Note that it is also required that for all $x \in U$, we have $0 \leq \mu_A(x) + \nu_A(x) + \omega_A(x) \leq 3$. The class of single-valued neutrosophic sets on U is denoted by $SVN(U)$.

The basic set operations are also defined for SVNS. We review some related operations here. An interested reader can consult [26] for more operations.

Definition 1.8 ([26]). Let $A = \{ \langle x, \mu_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in U \}$ and $B = \{ \langle x, \mu_B(x), \nu_B(x), \omega_B(x) \rangle \mid x \in U \}$ be two SVNS on U . Then,

- Subset: $A \subseteq B \Leftrightarrow \forall x \in U, \mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \omega_A(x) \geq \omega_B(x)$,
- Equality:

$$A = B \Leftrightarrow \forall x \in U \mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \omega_A(x) = \omega_B(x),$$

- Complement : $\sim A = \{ \langle x, \omega_A(x), 1 - \nu_A(x), \mu_A(x) \rangle \mid x \in U \}$,
- Union: $A \cup B = \{ \langle x, \mu_{A \cup B}(x), \nu_{A \cup B}(x), \omega_{A \cup B}(x) \rangle \mid x \in U \}$ where

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},$$

$$\nu_{A \cup B}(x) = \min\{\nu_A(x), \nu_B(x)\}$$

$$\text{and } \omega_{A \cup B}(x) = \min\{\omega_A(x), \omega_B(x)\},$$

- Intersection: $A \cap B = \{ \langle x, \mu_{A \cap B}(x), \nu_{A \cap B}(x), \omega_{A \cap B}(x) \rangle \mid x \in U \}$ where

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\},$$

$$\nu_{A \cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$$

$$\text{and } \omega_{A \cap B}(x) = \max\{\omega_A(x), \omega_B(x)\},$$

- Necessity: $\Box A = \{ \langle x, \mu_A(x), \nu_A(x), 1 - \mu_A(x) \rangle \mid x \in U \}$,
- Possibility: $\Diamond A = \{ \langle x, 1 - \omega_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in U \}$.

We can also define relations on SVNSSs, as is stated in the following definition.

Definition 1.9 ([28]). Let U and W be two sets. A single-valued neutrosophic relation, SVN R , from U to W is a single-valued neutrosophic subset of $U \times W$. More formally,

$$R = \{ \langle (x, y), \mu_R(x, y), \nu_R(x, y), \omega_R(x, y) \rangle \mid x, y \in U \times W \}. \quad (1.11)$$

The set of single-valued neutrosophic relations from U to W is denoted by $SVNR(U \times W)$. For simplicity, if $U = W$, we use $SVNR(U)$.

Similarly, the inverse relation of $R \in SVNR(U \times W)$ is defined as

$$R^{-1} = \{ \langle (y, x), \mu_{R^{-1}}(y, x), \nu_{R^{-1}}(y, x), \omega_{R^{-1}}(y, x) \rangle \mid (y, x) \in W \times U \}, \quad (1.12)$$

where $\mu_{R^{-1}}(y, x) = \mu_R(x, y)$, $\nu_{R^{-1}}(y, x) = \nu_R(x, y)$ and $\omega_{R^{-1}}(y, x) = \omega_R(x, y)$.

The properties of the relations on ordinary sets are also extended to SVNS as is given next.

Definition 1.10 ([28]). Let $R \in SVNR(U)$ be a SVN R over U . Then,

- R is called a reflexive SVN R , if for every $u \in U$, $\mu_R(u, u) = 1$ and $\nu_R(u, u) = \omega_R(u, u) = 0$.
- R is called a anti-reflexive SVN R , if for every $u \in U$, $\mu_R(u, u) = 0$ and $\nu_R(u, u) = \omega_R(u, u) = 1$.
- R is called a symmetric SVN R , if for every $u, v \in U$,

$$\mu_R(u, v) = \mu_R(v, u), \nu_R(u, v) = \nu_R(v, u)$$

and $\omega_R(u, v) = \omega_R(v, u)$.

- R is called a transitive SVN R , if for every $u, w \in U$, we have

$$\bigvee_{v \in U} (\mu_R(u, v) \wedge \mu_R(v, w)) \leq \mu_R(u, w), \quad (1.13)$$

$$\bigwedge_{v \in U} ((\nu_R(u, v) \vee \nu_R(v, w))) \leq \nu_R(u, w), \quad (1.14)$$

$$\bigwedge_{v \in U} ((\omega_R(u, v) \vee \omega_R(v, w))) \leq \omega_R(u, w), \quad (1.15)$$

where $\vee$ and $\wedge$ denote the maximum and the minimum operators, respectively.

1.4. Rough Neutrosophic Sets. In [2, 25], two foundational approaches to the neutrosophic fusion of rough set theory were proposed: (1) rough neutrosophic sets (RNSs) [2], and (2) neutrosophic rough sets (NRSs) [25]. Furthermore, Zhang et al. [33] provided a comprehensive analysis of neutrosophic models within the framework of rough set theory.

Definition 1.11 (RNSs [2]). Let U be a universal set and E be an equivalence relation over U . For a SVNS $A \in SVN(U)$, the lower approximation $\underline{E}(A)$

and the upper approximation $\overline{E}(A)$ are defined as follows:

$$\underline{E}(A) = \{ \langle x, \mu_{\underline{E}(A)}(x), \nu_{\underline{E}(A)}(x), \omega_{\underline{E}(A)}(x) \rangle \mid x \in U \}, \quad (1.16)$$

and

$$\overline{E}(A) = \{ \langle x, \mu_{\overline{E}(A)}(x), \nu_{\overline{E}(A)}(x), \omega_{\overline{E}(A)}(x) \rangle \mid x \in U \}, \quad (1.17)$$

where

$$\mu_{\underline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \mu_A(y), \quad \nu_{\underline{E}(A)}(x) = \bigvee_{y \in [x]_E} \nu_A(y), \quad \omega_{\underline{E}(A)}(x) = \bigvee_{y \in [x]_E} \omega_A(y), \quad (1.18)$$

and

$$\mu_{\overline{E}(A)}(x) = \bigvee_{y \in [x]_E} \mu_A(y), \quad \nu_{\overline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \nu_A(y), \quad \omega_{\overline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \omega_A(y). \quad (1.19)$$

The pair $(\underline{E}(A), \overline{E}(A))$ is called a rough neutrosophic set of A in (U, E) .

The rough complement of $E(A)$ is defined as $\sim E(A) = (\underline{E}^c(A), \overline{E}^c(A))$, where

$$\underline{E}^c(A) = \{ \langle x, \omega_{\underline{E}(A)}(x), 1 - \nu_{\underline{E}(A)}(x), \mu_{\underline{E}(A)}(x) \rangle \mid x \in U \}, \quad (1.20)$$

and

$$\overline{E}^c(A) = \{ \langle x, \omega_{\overline{E}(A)}(x), 1 - \nu_{\overline{E}(A)}(x), \mu_{\overline{E}(A)}(x) \rangle \mid x \in U \}. \quad (1.21)$$

Definition 1.12 (NRSs [25]). Let U be a universal set and $R \in SVN R(U)$ be a neutrosophic relation on U . For a SVNS $A \in SVN(U)$, the lower approximation $\underline{R}(A)$ and the upper approximation $\overline{R}(A)$ are defined as follows

$$\underline{R}(A) = \{ \langle x, \mu_{\underline{R}(A)}(x), \nu_{\underline{R}(A)}(x), \omega_{\underline{R}(A)}(x) \rangle \mid x \in U \}, \quad (1.22)$$

and

$$\overline{R}(A) = \{ \langle x, \mu_{\overline{R}(A)}(x), \nu_{\overline{R}(A)}(x), \omega_{\overline{R}(A)}(x) \rangle \mid x \in U \}, \quad (1.23)$$

where

$$\mu_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\omega_R(x, y) \vee \mu_A(y)], \quad \nu_{\underline{R}(A)}(x) = \bigvee_{y \in U} [(1 - \nu_R(x, y)) \vee \nu_A(y)], \quad (1.24)$$

$$\omega_{\underline{R}(A)}(x) = \bigvee_{y \in U} [\mu_R(x, y) \wedge \omega_A(y)], \quad (1.25)$$

and

$$\mu_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\mu_R(x, y) \wedge \mu_A(y)], \quad \nu_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\nu_R(x, y) \wedge \nu_A(y)], \quad (1.26)$$

$$\omega_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\omega_R(x, y) \vee \omega_A(y)]. \quad (1.27)$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called a neutrosophic rough set, an NRS for short, of A in the pair (U, R) .

2. SINGLE-VALUED NEUTROSOPHIC S-APPROXIMATION SPACES

To extend S-approximation spaces, we generalize the knowledge and decider mappings to support single-valued neutrosophic subsets, as formalized in Definition 2.1. While it is also possible to approximate single-valued neutrosophic (SVN) sets using intuitionistic fuzzy sets or crisp sets, these alternatives are naturally encompassed by the more general framework. Consequently, we focus on extending S-approximation spaces directly to single-valued neutrosophic sets, which we term Single-Valued Neutrosophic S-Approximation Spaces (SVN- S spaces), and conduct a detailed investigation of approximation operators and their fundamental properties within this generalized setting.

Definition 2.1 (Single-valued Neutrosophic S-approximation Space). A Single-valued Neutrosophic S-approximation space, or *SVN- S -space*, is a quadruple $\text{SVNG} = (U, W, T, S)$ where U and W are non-empty sets, T is a mapping of the form $T : U \rightarrow \mathcal{SVNS}(W)$, and S is a mapping of the form $S : \mathcal{SVNS}(W) \times \mathcal{SVNS}(W) \rightarrow [0, 1] \times [0, 1] \times [0, 1]$.

Given the definition of SVN- S -space (Definition 2.1), we can re-define the SVN lower and upper approximation operators as Definition 2.2.

Definition 2.2 (The SVN lower and upper approximation operators). Let $\text{SVNG} = (U, W, T, S)$ be an SVN- S -space and $X \in \mathcal{SVNS}(W)$. Then, the lower and the upper approximations of X with respect to SVNG are defined as

$$\underline{\text{SVNG}}(X) = \{ \langle x, S^1(T(x), X), S^2(T(x), X), S^3(T(x), X) \rangle \mid x \in U \}, \quad (2.1)$$

and

$$\overline{\text{SVNG}}(X) = \{ \langle x, S^3(T(x), X^c), \sim S^2(T(x), X^c), S^1(T(x), X^c) \rangle \mid x \in U \}, \quad (2.2)$$

respectively. Note that

$S(X, Y) = (S^1(X, Y), S^2(X, Y), S^3(X, Y)) \in [0, 1] \times [0, 1] \times [0, 1]$
for $X, Y \in \mathcal{SVNS}(W)$.

Remark 2.3. Note that the Definition 2.2 captures the lower and the upper approximation operators defined in [4, 10, 24, 19, 20], as we discuss in Corollary 2.4.

Corollary 2.4. Let $SVNG = (U, W, T, S)$ be an SVN-S-space. Then,

- For the approximation pair in [10], we define
 $T(u) = R_s(u)$, $S^{(1)}(A, B) = \bigwedge_{y \in A} \mu_B(y)$, $S^{(2)}(A, B) = \bigvee_{y \in A} \sigma_B(y)$ and
 $S^{(3)}(A, B) = \bigvee_{y \in A} \gamma_B(y)$ for $u \in U$. This way, the lower and the upper approximation operators coincide.
- To obtain the approximation pair in [4], we can set $T(u) = R(u)$, and

$$S^{(1)}(A, B) = \inf_{x \in B} \mathcal{I}(\mu_A(x), \mu_B(x)), \quad (2.3)$$

$$S^{(2)}(A, B) = \sup_{x \in B} \mathcal{T}(\sigma_A(x), \sigma_B(x)), \quad (2.4)$$

$$S^{(3)}(A, B) = \sup_{x \in B} \mathcal{T}(\gamma_A(x), \gamma_B(x)). \quad (2.5)$$

- For the pair in [24], it is enough to consider $T(u) = R(u)$, and

$$S^{(1)}(A, B) = \bigwedge_{w \in W} (\mu_B(w) \vee (1 - \mu_A(w))), \quad (2.6)$$

$$S^{(2)}(A, B) = \bigvee_{w \in W} (\sigma_B(w) \wedge \sigma_A(w)), \quad (2.7)$$

$$S^{(3)}(A, B) = \bigvee_{w \in W} (\gamma_B(w) \wedge \gamma_A(w)). \quad (2.8)$$

In the following theorems (Theorems 2.5 and 2.6), corresponding to NRS and RNS models, we define the SVN S-approximation spaces that achieve these models.

Theorem 2.5 (Expressing the NRS with S-neutrosophic). Let U be a universal set and $R \in \text{SVNR}(U)$ be a neutrosophic relation on U . Then, there is a neutrosophic S-approximation space G such that $\underline{R}(A) = \underline{G}(A)$ and $\overline{R}(A) = \overline{G}(A)$, where $(\underline{R}(A), \overline{R}(A))$ is the neutrosophic rough set corresponding to $A \in \text{SVN}(U)$ in the NRSs model.

Proof. Let $R \in \text{SVNR}(U)$ be as

$$R = \{ \langle (u, v), \mu_R(u, v), \nu_R(u, v), \omega_R(u, v) \rangle \mid (u, v) \in U \times U \}. \quad (2.9)$$

According to R , we define S-neutrosophic $G = (U, U, T, S)$ as follows:

Let $T : U \rightarrow \text{SVNS}(U)$ be defined as

$$T(u) = \{ \langle v, \mu_R((u, v), \nu_R((u, v), \omega_R((u, v))) \mid v \in U \} \quad (2.10)$$

and $S(A, B) = (S^1(A, B), S^2(A, B), S^3(A, B))$ for $A, B \in \mathcal{SVNS}(U)$ be defined as

$$\begin{aligned} S^1(T(u), A) &= \bigwedge_{v \in U} [\omega_R(u, v) \vee \mu_A(v)], \\ S^2(T(u), A) &= \bigwedge_{v \in U} [(1 - \nu_R(u, v)) \vee \nu_A(v)], \\ S^3(T(u), A) &= \bigvee_{v \in U} [\mu_R(u, v) \wedge \omega_A(v)]. \end{aligned} \quad (2.11)$$

For an $A = \{\langle v, \mu_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in U\}$, the lower and the upper approximations of A with respect to model NRS (U, R) are defined as

$$\begin{aligned} \underline{R}(A) &= \{\langle u, \mu_{\underline{R}(A)}(u), \nu_{\underline{R}(A)}(u), \omega_{\underline{R}(A)}(u) \rangle \mid u \in U\}, \\ &= \{ \langle u, \bigwedge_{v \in U} [\omega_R(u, v) \vee \mu_A(v)], \bigwedge_{v \in U} [(1 - \nu_R(u, v)) \vee \nu_A(v)], \\ &\quad \bigvee_{v \in U} [\mu_R(u, v) \wedge \omega_A(v)] \rangle \mid u \in U \}, \\ &= \{\langle u, S^1(T(u), A), S^2(T(u), A), S^3(T(u), A) \rangle \mid u \in U\} \\ &= \underline{\text{SVNG}}(A), \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \overline{R}(A) &= \{\langle u, \mu_{\overline{R}(A)}(u), \nu_{\overline{R}(A)}(u), \omega_{\overline{R}(A)}(u) \rangle \mid u \in U\}, \\ &= \{ \langle u, \bigvee_{v \in U} [\mu_R(u, v) \wedge \mu_A(v)], \bigvee_{v \in U} [\nu_R(u, v) \wedge \nu_A(v)], \\ &\quad \bigwedge_{v \in U} [\omega_R(u, v) \vee \omega_A(v)] \rangle \mid u \in U \}, \\ &= \{\langle u, S^3(T(u), A^c), \neg S^2(T(u), A^c), S^1(T(u), A^c) \rangle \mid u \in U\} \\ &= \overline{\text{SVNG}}(A), \end{aligned} \quad (2.13)$$

respectively. So, the NRSs model is expressible in terms of the neutrosophic S-approximation spaces. $\square$

Theorem 2.6 (Expressing the RNS with S-neutrosophic). *Let U be a universal set and E be an equivalence relation over U and $A \in \mathcal{SVN}(U)$. Then there is a neutrosophic S-approximation spaces G such that $\underline{E}(A) = \underline{G}(A)$ and $\overline{E}(A) = \overline{G}(A)$, where $(\underline{E}(A), \overline{E}(A))$ rough neutrosophic set of A in the model RNSs.*

Proof. In the model RNSs, lower approximation $\underline{E}(A)$ and upper approximation $\overline{E}(A)$ are defined as follows

$$\underline{E}(A) = \{ \langle x, \mu_{\underline{E}(A)}(x), \nu_{\underline{E}(A)}(x), \omega_{\underline{E}(A)}(x) \rangle \mid x \in U \}$$

and

$$\overline{E}(A) = \{ \langle x, \mu_{\overline{E}(A)}(x), \nu_{\overline{E}(A)}(x), \omega_{\overline{E}(A)}(x) \rangle \mid x \in U \}$$

where

$$\mu_{\underline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \mu_A(y), \quad \nu_{\underline{E}(A)}(x) = \bigvee_{y \in [x]_E} \nu_A(y), \quad \omega_{\underline{E}(A)}(x) = \bigvee_{y \in [x]_E} \omega_A(y),$$

and

$$\mu_{\overline{E}(A)}(x) = \bigvee_{y \in [x]_E} \mu_A(y), \quad \nu_{\overline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \nu_A(y), \quad \omega_{\overline{E}(A)}(x) = \bigwedge_{y \in [x]_E} \omega_A(y).$$

Now we define $\hat{R} \in \text{SVNR}(U)$ as follows

$$\hat{R} = \{ \langle (u, v), \mu_{\hat{R}}(u, v), \nu_{\hat{R}}(u, v), \omega_{\hat{R}}(u, v) \rangle \mid (u, v) \in U \times U \},$$

where

$$\begin{aligned} \mu_{\hat{R}}(u, v) &= \begin{cases} 1 & v \in [u]_R, \\ 0 & \text{otherwise,} \end{cases} \\ \omega_{\hat{R}}(u, v) &= 1 - \mu_{\hat{R}}(u, v), \quad \nu_{\hat{R}}(u, v) = \mu_{\hat{R}}(u, v). \end{aligned} \quad (2.14)$$

It is clear that

$$\begin{aligned} \bigwedge_{v \in [u]_E} \mu_A(v) &= \min \{ \mu_A(v) \mid v \in [u]_E \}, \\ &= \min \{ \mu_A(v) \vee \omega_{\hat{R}}(u, v) \mid v \in U \}, \\ &\quad \text{since } \mu_{\hat{R}}(u, v) = 1 \text{ if } v \in [u]_E \\ &= \bigwedge_{v \in U} (\omega_{\hat{R}}(u, v) \vee \mu_A(v)), \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \bigvee_{v \in [u]_E} \omega_A(v) &= \max \{ \omega_A(v) \mid v \in [u]_E \}, \\ &= \max \{ \omega_A(v) \wedge \mu_{\hat{R}}(u, v) \mid v \in U \}, \\ &\quad \text{since } \mu_{\hat{R}}(u, v) = 1 \text{ if } v \in [u]_E \\ &= \bigvee_{v \in U} (\mu_{\hat{R}}(u, v) \wedge \omega_A(v)). \end{aligned} \quad (2.16)$$

According to the above definitions, we define S-neutrosophic $G = (U, U, T, S)$ as follows:

The mapping $T : U \rightarrow \text{SVNS}(U)$ is of the form

$$T(u) = \{ \langle v, \mu_{\hat{R}}(u, v), \nu_{\hat{R}}(u, v), \omega_{\hat{R}}(u, v) \rangle \mid v \in U \} \quad (2.17)$$

and S is defined as $S = (S^{(1)}, S^{(2)}, S^{(3)})$ such that for every $A \in SVN\mathcal{S}(U)$ and $u \in U$,

$$\begin{aligned} S^{(1)}(T(u), A) &= \bigwedge_{v \in U} (\omega_{\hat{R}}(u, v) \vee \mu_A(v)) = \bigwedge_{v \in [u]_E} \mu_A(v), \\ S^{(2)}(T(u), A) &= \bigvee_{v \in U} (\nu_{\hat{R}}(u, v) \vee \nu_A(v)) \\ &= \bigvee_{v \in U} (\mu_{\hat{R}}(u, v) \vee \nu_A(v)) \\ &= \bigvee_{v \in [u]_E} \nu_A(v), \\ S^{(3)}(T(u), A) &= \bigvee_{v \in U} (\mu_{\hat{R}}(u, v) \vee \omega_A(v)) = \bigvee_{v \in [u]_E} \omega_A(v). \end{aligned} \quad (2.18)$$

It is clear

$$\begin{aligned} \sim S^{(2)}(T(u), A^c) &= \bigwedge_{v \in U} ((1 - \nu_{\hat{R}}(u, v)) \vee \sim \nu_{A^c}(v)) \\ &= \bigwedge_{v \in U} (\omega_{\hat{R}}(u, v) \vee \nu_A(v)) \\ &= \bigwedge_{v \in [u]_E} \nu_A(v). \end{aligned}$$

Therefore

$$\begin{aligned} \underline{E}(A) &= \{ \langle u, \mu_{\underline{E}(A)}(u), \nu_{\underline{E}(A)}(u), \omega_{\underline{E}(A)}(x) \rangle \mid x \in U \} \\ &= \{ \langle u, \bigwedge_{v \in [u]_E} \mu_A(v), \bigvee_{v \in [u]_E} \nu_A(v), \bigvee_{v \in [u]_E} \omega_A(v) \rangle \mid x \in U \} \\ &= \{ \langle u, S^{(1)}(T(u), A), S^{(2)}(T(u), A), S^{(3)}(T(u), A) \rangle \mid x \in U \} \\ &= \underline{SVNG}(A) \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \overline{E}(A) &= \{ \langle u, \mu_{\overline{E}(A)}(u), \nu_{\overline{E}(A)}(u), \omega_{\overline{E}(A)}(x) \rangle \mid x \in U \} \\ &= \{ \langle u, \bigvee_{v \in [u]_E} \mu_A(v), \bigwedge_{v \in [u]_E} \nu_A(v), \bigwedge_{v \in [u]_E} \omega_A(v) \rangle \mid x \in U \} \\ &= \{ \langle u, S^{(3)}(T(u), A^c), \sim S^{(2)}(T(u), A^c), S^{(1)}(T(u), A^c) \rangle \mid x \in U \} \\ &= \overline{SVNG}(A) \end{aligned} \quad (2.20)$$

So the RNSs model is expressed as a neutrosophic S-approximation space. $\square$

Proposition 2.7. *Let $SVNG = (U, W, T, S)$ be an SVN-S-space and $X \in \mathcal{SVNS}(W)$. Then, the followings hold:*

- (1) $\square \overline{SVNG}(X) = (\diamond \underline{SVNG}(X^c))^c$,
- (2) $\square \underline{SVNG}(X) = (\diamond \overline{SVNG}(X^c))^c$,
- (3) $\diamond \overline{SVNG}(X) = (\square \underline{SVNG}(X^c))^c$,
- (4) $\diamond \underline{SVNG}(X) = (\square \overline{SVNG}(X^c))^c$.

Proof. We just prove (1). The others can be proved in a similar way. By the definition of the upper approximation for an SVN- S -space, we have

$$\begin{aligned} \overline{\text{SVNG}}(X) \\ = \{ \langle x, S^{(3)}(T(x), X^c), 1 - S^{(2)}(T(x), X^c), S^{(1)}(T(x), X^c) \rangle \mid x \in U \}. \end{aligned} \quad (2.21)$$

After applying its necessity, we arrive at

$$\begin{aligned} \Box \overline{\text{SVNG}}(X) \\ = \{ \langle x, S^{(3)}(T(x), X^c), 1 - S^{(2)}(T(x), X^c), 1 - S^{(3)}(T(x), X^c) \rangle \mid x \in U \}. \end{aligned} \quad (2.22)$$

Similarly,

$$\begin{aligned} \Diamond \underline{\text{SVNG}}(X^c) \\ = \{ \langle x, 1 - S^{(3)}(T(x), X^c), S^{(2)}(T(x), X^c), S^{(3)}(T(x), X^c) \rangle \mid x \in U \}. \end{aligned} \quad (2.23)$$

After applying the complement operation, we have

$$\begin{aligned} (\Diamond \underline{\text{SVNG}}(X^c))^c \\ = \{ \langle x, S^{(3)}(T(x), X^c), 1 - S^{(2)}(T(x), X^c), 1 - S^{(3)}(T(x), X^c) \rangle \mid x \in U \}, \end{aligned} \quad (2.24)$$

which concludes the proof, i.e. $\Box \overline{\text{SVNG}}(X) = (\Diamond \underline{\text{SVNG}}(X^c))^c$. $\square$

The value of $S^{(1)}(A, B)$ can be interpreted as the degree of acceptance of the set A as the set B . Moreover, the value of $S^{(2)}(A, B)$ denotes the degree of uncertainty of the set A as the set B . Finally, the value of $S^{(3)}(A, B)$ denotes the degree of rejection of the set A as the set B .

We use a threshold triple $(a, r, s) \in [0, 1]^3$ to define the lower and upper approximations of a single-valued neutrosophic set as a crisp set with the following interpretations: The parameter $a \in [0, 1]$ denotes an acceptance accuracy threshold, $r \in [0, 1]$ denotes the rejection accuracy threshold, and $s \in [0, 1]$ denotes the uncertainty accuracy threshold. A sample application of these threshold values in conjunction with a decider mapping in an SVN- S -space is given in Table 1.

Using these interpretations, we can modify the lower and upper approximation operations in Definition 2.2 using these thresholds. This is considered next.

Definition 2.8. Let $\text{SVNG} = (U, W, T, S)$ be an SVN S -approximation space, $(a, s, r) \in [0, 1]^3$ a set of thresholds and $X \in \mathcal{SVNS}(W)$. Then, the lower and the upper approximations of X with respect to G and (a, s, r)

TABLE 1. The interpretations of a , r , and s as the acceptance and the rejection and the uncertainty thresholds for $x \in U$ and $X \in \mathcal{SVNS}(W)$.

Status	Interpretation
$S^{(1)}(T(x), X) \geq a$	The element x can be accepted as the set X with accuracy of at least a
$S^{(1)}(T(x), X) < a$	The element x cannot be accepted as the set X with accuracy of at least a
$S^{(2)}(T(x), X) \geq s$	The uncertainty in any decision made by S for an element x with respect to a set X is at least s
$S^{(2)}(T(x), X) < s$	The uncertainty in any decision made by S for an element x with respect to a set X is less than s , i.e. bounded by s
$S^{(3)}(T(x), X) \geq r$	The element x can be rejected as the set X with accuracy of at least r
$S^{(3)}(T(x), X) < r$	The element x cannot be rejected as the set X with accuracy of at least r

are defined as

$$\begin{aligned} & \underline{\text{SVNG}}_{(a,s,r)}(X) \\ &= \{x \in U \mid S^{(1)}(T(x), X) \geq a, S^{(2)}(T(x), X) < s, S^{(3)}(T(x), X) < r\} \end{aligned} \quad (2.25)$$

$$\begin{aligned} & \overline{\text{SVNG}}_{(a,s,r)}(X) \\ &= \{x \in U \mid S^{(1)}(T(x), X^c) < a, S^{(2)}(T(x), X^c) \geq s, S^{(3)}(T(x), X^c) \geq r\}. \end{aligned} \quad (2.26)$$

In other words, the lower approximation of $X \in \mathcal{SVNS}(W)$ in Definition 2.8 consists of elements $x \in U$ which can be accepted as X with accuracy of at least a and cannot be rejected with accuracy of at least r and the uncertainty in both decisions is bounded by s .

Similarly, The upper approximation of $X \in \mathcal{SVNS}(W)$ contains elements $x \in U$ that cannot be accepted as X^c with accuracy of at least a and can be rejected as X^c with accuracy of at least r and the decision has unbounded uncertainty with respect to s . Note that the lower and the upper approximations of a set X can be defined as follows, too:

$$\underline{\text{SVNG}}_{(a,s)}(X) = \{x \in U \mid S^{(1)}(T(x), X) \geq a \wedge S^{(2)}(T(x), X) < s\}, \quad (2.27)$$

and

$$\overline{\text{SVNG}}_{(s,r)}(X) = \{x \in U \mid (1 - S^{(2)}(T(x), X^c)) < s \wedge S^{(3)}(T(x), X^c) \geq r\}. \quad (2.28)$$

Then, we would have

$$\underline{\text{SVNG}}_{(a,s,r)}(X) = \underline{\text{SVNG}}_{(a,s)}(X) \cap (\overline{\text{SVNG}}_{(s,r)}(X^c))^c \quad (2.29)$$

and

$$\overline{\text{SVNG}}_{(a,s,r)}(X) = (\underline{\text{SVNG}}_{(a,s)}(X^c))^c \cap \overline{\text{SVNG}}_{(s,r)}(X) \quad (2.30)$$

These two equalities can be easily obtained from the definitions. Similarly, it is easy to see that $\underline{\text{SVNG}}_{(a,s)}(X) = \underline{\text{SVNG}}_{(a,s,1)}(X)$ and

$$\overline{\text{SVNG}}_{(s,r)}(X) = \overline{\text{SVNG}}_{(1,s,r)}(X)$$

for $0 < a, s, r < 1$. Hence, these two operators can be obtained from each other whenever $(a, s, r) \in [0, 1]^3$. However, we stick to the one in the Definition 2.8, because it makes the notation much simpler in the sequel.

3. MONOTONE SVN S-APPROXIMATION SPACES

In searching for certain decision mappings in S -approximation spaces which ascertains some necessary properties for the lower and upper approximation operators, the notion of partial monotonicity was proposed in [17]. A modified version of this notion for SVN S -approximation spaces can be stated as follows.

Definition 3.1 (Monotone SVN S -approximation Space). Let

$$\text{SVNG} = (U, W, T, S)$$

be an SVN S -approximation space. Then, G is called partial monotone if $A \subseteq B$ implies $S^{(1)}(T(x), A) \leq S^{(1)}(T(x), B)$, $S^{(2)}(T(x), A) \geq S^{(2)}(T(x), B)$ and $S^{(3)}(T(x), A) \geq S^{(3)}(T(x), B)$ for every $x \in U$ and $A, B \subseteq \mathcal{SVNS}(W)$.

Proposition 3.2. Let $\text{SVNG} = (U, W, T, S)$ be a partial monotone SVN S -approximation space and $X, Y \in \mathcal{SVNS}(W)$. Then,

- (1) $X \subseteq Y$ implies $\overline{\text{SVNG}}(X) \subseteq \overline{\text{SVNG}}(Y)$,
- (2) $X \subseteq Y$ implies $\underline{\text{SVNG}}(X) \subseteq \underline{\text{SVNG}}(Y)$,
- (3) $\overline{\text{SVNG}}(X \cup Y) \supseteq \overline{\text{SVNG}}(X) \cup \overline{\text{SVNG}}(Y)$,
- (4) $\overline{\text{SVNG}}(X \cap Y) \subseteq \overline{\text{SVNG}}(X) \cap \overline{\text{SVNG}}(Y)$,
- (5) $\underline{\text{SVNG}}(X \cup Y) \supseteq \underline{\text{SVNG}}(X) \cup \underline{\text{SVNG}}(Y)$,
- (6) $\underline{\text{SVNG}}(X \cap Y) \subseteq \underline{\text{SVNG}}(X) \cap \underline{\text{SVNG}}(Y)$.

Proof. The proof is as follows:

(1) Given $X \subseteq Y$, then $Y^c \subseteq X^c$. Hence, by monotonicity of SVN $\mathcal{S}$ we have $S^{(1)}(T(x), Y^c) \leq S^{(1)}(T(x), X^c)$, $S^{(2)}(T(x), Y^c) \geq S^{(2)}(T(x), X^c)$ and $S^{(3)}(T(x), Y^c) \geq S^{(3)}(T(x), X^c)$. So, $\overline{\text{SVNG}}(X) \subseteq \overline{\text{SVNG}}(Y)$.

(2) Since SVNG is monotone, then $X \subseteq Y$ implies

$$S^{(1)}(T(x), X) \leq S^{(1)}(T(x), Y), S^{(2)}(T(x), Y) \geq S^{(2)}(T(x), X)$$

and $S^{(3)}(T(x), Y) \geq S^{(3)}(T(x), X)$. Therefore, by the definition of the lower approximation operator, we have $\underline{\text{SVNG}}(X) \subseteq \underline{\text{SVNG}}(Y)$.

(3) By $X \subseteq X \cup Y$ and property (1), we have $\overline{\text{SVNG}}(X) \subseteq \overline{\text{SVNG}}(X \cup Y)$. Similarly, we can obtain $\overline{\text{SVNG}}(Y) \subseteq \overline{\text{SVNG}}(X \cup Y)$ which concludes the proof.

(4) $X \cap Y \subseteq X$ and property (1) imply $\overline{\text{SVNG}}(X \cap Y) \subseteq \overline{\text{SVNG}}(X)$. Similarly, we can obtain $\overline{\text{SVNG}}(X \cap Y) \subseteq \overline{\text{SVNG}}(Y)$ which concludes the proof.

(5) $X \subseteq X \cup Y$ and property (2) imply $\underline{\text{SVNG}}(X) \subseteq \underline{\text{SVNG}}(X \cup Y)$. Similarly, we can obtain $\underline{\text{SVNG}}(Y) \subseteq \underline{\text{SVNG}}(X \cup Y)$ which concludes the proof.

(6) $X \cap Y \subseteq X$ and property (2) imply $\underline{\text{SVNG}}(X \cap Y) \subseteq \underline{\text{SVNG}}(X)$. Similarly, we can obtain $\underline{\text{SVNG}}(X \cap Y) \subseteq \underline{\text{SVNG}}(Y)$ which concludes the proof. $\square$

Proposition 3.3. Let $\text{SVNG} = (U, W, T, S)$ be a monotone SVN $\mathcal{S}$ -approximation space, (a, s, r) a triplet of thresholds and $X, Y \in \text{SVNS}(W)$. Then,

- (1) $X \subseteq Y$ implies $\overline{\text{SVNG}}_{(a,s,r)}(X) \subseteq \overline{\text{SVNG}}_{(a,s,r)}(Y)$,
- (2) $X \subseteq Y$ implies $\underline{\text{SVNG}}_{(a,s,r)}(X) \subseteq \underline{\text{SVNG}}_{(a,s,r)}(Y)$,
- (3) $\overline{\text{SVNG}}_{(a,s,r)}(X \cup Y) \supseteq \overline{\text{SVNG}}_{(a,s,r)}(X) \cup \overline{\text{SVNG}}_{(a,s,r)}(Y)$,
- (4) $\overline{\text{SVNG}}_{(a,s,r)}(X \cap Y) \subseteq \overline{\text{SVNG}}_{(a,s,r)}(X) \cap \overline{\text{SVNG}}_{(a,s,r)}(Y)$,
- (5) $\underline{\text{SVNG}}_{(a,s,r)}(X \cup Y) \supseteq \underline{\text{SVNG}}_{(a,s,r)}(X) \cup \underline{\text{SVNG}}_{(a,s,r)}(Y)$,
- (6) $\underline{\text{SVNG}}_{(a,s,r)}(X \cap Y) \subseteq \underline{\text{SVNG}}_{(a,s,r)}(X) \cap \underline{\text{SVNG}}_{(a,s,r)}(Y)$.

Proof. The proof is as follows:

(1) Given $X \subseteq Y$, then we have $Y^c \subseteq X^c$. Hence, the monotonicity of SVNG implies $S^{(1)}(T(x), Y^c) \leq S^{(1)}(T(x), X^c)$, $S^{(2)}(T(x), Y^c) \geq S^{(2)}(T(x), X^c)$, and $S^{(3)}(T(x), Y^c) \geq S^{(3)}(T(x), X^c)$.

Assume that $x \in \overline{\text{SVNG}}_{(a,s,r)}(X)$, then we have

$$S^{(1)}(T(x), X^c) < a, S^{(2)}(T(x), Y^c) \geq s,$$

and $S^{(3)}(T(x), Y^c) \geq r$ which implies

$$\begin{aligned} S^{(1)}(T(x), Y^c) &\leq S^{(1)}(T(x), X^c) < a, \\ S^{(2)}(T(x), Y^c) &\geq S^{(2)}(T(x), X^c) \geq s, \\ S^{(3)}(T(x), Y^c) &\geq S^{(3)}(T(x), X^c) \geq r. \end{aligned}$$

Thus, we have, $x \in \overline{\text{SVNG}_{(a,s,r)}}(Y)$.

(2) Since SVNG is monotone, then $X \subseteq Y$ implies

$$S^{(1)}(T(x), X) \leq S^{(1)}(T(x), Y), \quad S^{(2)}(T(x), X) \geq S^{(2)}(T(x), Y),$$

and $S^{(3)}(T(x), X) \geq S^{(3)}(T(x), Y)$. Assume that $x \in \overline{\text{SVNG}_{(a,s,r)}}(X)$. Hence, we have $S^{(1)}(T(x), X) \geq a$, $S^{(2)}(T(x), X) < s$, and $S^{(3)}(T(x), X) < r$. Therefore, $a \leq S^{(1)}(T(x), X) \leq S^{(1)}(T(x), Y)$, $s > S^{(2)}(T(x), X) \geq S^{(2)}(T(x), Y)$, and $r > S^{(3)}(T(x), X) \geq S^{(3)}(T(x), Y)$ which implies that

$$x \in \overline{\text{SVNG}_{(a,s,r)}}(Y).$$

(3) By $X \subseteq X \cup Y$ and property (1), we have

$$\overline{\text{SVNG}_{(a,s,r)}}(X) \subseteq \overline{\text{SVNG}_{(a,s,r)}}(X \cup Y).$$

Similarly, we can obtain $\overline{\text{SVNG}_{(a,s,r)}}(Y) \subseteq \overline{\text{SVNG}_{(a,s,r)}}(X \cup Y)$ which concludes the proof.

(4) By $X \cap Y \subseteq X$ and property (1), we have

$$\overline{\text{SVNG}_{(a,s,r)}}(X \cap Y) \subseteq \overline{\text{SVNG}_{(a,s,r)}}(X).$$

Similarly, we can obtain $\overline{\text{SVNG}_{(a,s,r)}}(X \cap Y) \subseteq \overline{\text{SVNG}_{(a,s,r)}}(Y)$, which concludes the proof.

(5) By $X \subseteq X \cup Y$ and property (2), we have

$$\underline{\text{SVNG}_{(a,s,r)}}(X) \subseteq \underline{\text{SVNG}_{(a,s,r)}}(X \cup Y).$$

Similarly, we can obtain $\underline{\text{SVNG}_{(a,s,r)}}(Y) \subseteq \underline{\text{SVNG}_{(a,s,r)}}(X \cup Y)$, which concludes the proof.

(6) By $X \cap Y \subseteq X$ and property (2), we have

$$\underline{\text{SVNG}_{(a,s,r)}}(X \cap Y) \subseteq \underline{\text{SVNG}_{(a,s,r)}}(X).$$

Similarly, we can obtain $\underline{\text{SVNG}_{(a,s,r)}}(X \cap Y) \subseteq \underline{\text{SVNG}_{(a,s,r)}}(Y)$, which concludes the proof. $\square$

4. COMPLEMENT COMPATIBLE SVN S -APPROXIMATION SPACES

The notion of complement compatibility was introduced in [17] in order to make the lower approximation of a set included into the upper approximation of that set. For the same reason, we will introduce the notion of complement compatibility for single-valued neutrosophic S -approximation spaces as follows.

Definition 4.1. Complement Compatible SVN S -approximation Space Let $SVNG = (U, W, T, S)$ be an SVN S -approximation space. Then, $SVNG$ is called complement compatible if $S^{(1)}(T(x), A) \leq S^{(3)}(T(x), A^c)$ and $S^{(2)}(T(x), A) \geq \sim S^{(2)}(T(x), A^c)$ for every $x \in U$ and $A \in \mathcal{SVNS}(W)$.

Proposition 4.2. Let $SVNG = (U, W, T, S)$ be a complement compatible SVN S -approximation space and $X \in \mathcal{SVNS}(W)$. Then, $\underline{SVNG}(X) \subseteq \overline{SVNG}(X)$.

Proof. The proof is clear from the definition of the complement compatibility. $\square$

5. CONCLUSION AND FUTURE INTERESTS

We introduced neutrosophic sets into S -approximation spaces (SVNSs) and defined the corresponding lower and upper approximation operators. Furthermore, we investigated the properties of these operators for SVNSs in general, with particular attention to monotonicity, as well as their behavior under the complement compatibility condition.

Future research directions in this field are multifaceted. One promising avenue involves defining three-way decision regions and applying these frameworks to medical image processing. Another potential direction, as suggested by one of the reviewers, is to develop an S -approximation space based on the Dezert-Smarandache Theory (DSmT). Additionally, the study of algebraic structures capable of modeling neutrosophic S -approximations, and the investigation of how approximation operators induce congruence relations and quotient structures, also constitute important areas for further exploration.

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SINGLE-VALUED NEUTROSOPHIC S-APPROXIMATION SPACES

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فضاهای S -تقریب تک مقدار نوتروسوفیک

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در این مقاله، مفهوم فضاهای S -تقریب نوتروسوفیک تک مقدار (SVNSs) بررسی شده و ویژگی‌های بنیادی آن‌ها مورد مطالعه قرار می‌گیرد. نشان داده می‌شود که این فضاها چارچوبی یکپارچه برای نمایش دو مدل شناخته شده از مجموعه‌های ناهموار نوتروسوفیک تک مقدار فراهم می‌کنند. افزون بر این، ویژگی‌های عملگرهای تقریب پایین و تقریب بالا که در یک فضای SVNS تعریف شده‌اند، از منظر دو شرط اساسی بررسی می‌شوند: (۱) یکنواختی و (۲) سازگاری با عمل متمم.

کلمات کلیدی: فضاهای S -تقریب، مجموعه‌های ناهموار، مجموعه‌های نوتروسوفیک تک مقدار.