

## ÉTALÉ SPACES OF RESIDUATED LATTICES

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**ABSTRACT.** This paper investigates étalé spaces associated with residuated lattices. We introduce the concept of étalé spaces of residuated lattices over a topological space and establish their structural framework. Furthermore, we present a method for transferring an étalé space of residuated lattices from one topological space to another via a continuous map, thereby establishing a functorial connection between their associated categories.

### 1. INTRODUCTION

This paper delves into the exploration of étalé spaces associated with residuated lattices. The foundational developments of étalé spaces trace back to the Séminaire Henri Cartan (1948 - 1964). In collaboration with eminent mathematicians such as Jean-Pierre Serre and Jean Leray, Henri Cartan played a pivotal role in pioneering innovative concepts and techniques in topology and algebraic geometry throughout this seminar. To be precise, it can be said that the notion of an “*étalé space*” or “*Espace étalé*” emanating from the realm of algebraic geometry, was born by Henri Cartan, commonly ascribed to his collaborative efforts with Michel Lazard, exactly in Exposé 14 of this seminar, under the term “faisceau” [4]. According to [15, p. 7], Lazard’s faisceau on a regular topological space  $X$  is defined by a map  $p$  from a set  $F$  to  $X$ , satisfying two conditions:

- (1) for each  $x \in X$ ,  $p^{-1}(x) = F_x$ , is a  $K$ -module;
- (2)  $F$  has a (non-separated) topology such that the algebraic operations of  $F$  are continuous, and  $p$  is a local homeomorphism.

Roger Godement, a notable participant in Henri Cartan’s Séminaire, is credited with introducing the term “Espace étalé” instead of “faisceau” in his influential work [13]. His elegant contribution encompasses a comprehensive survey of étalé spaces. Godement’s exposition played a pivotal role in establishing the formalization and examination of étalé spaces, exerting a substantial influence on the evolution of algebraic geometry.

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The notion of étalé spaces has been extensively studied during past decades, see e.g., [16, 26, 1, 21, 3], and [25]. For more historical notes about étalé spaces, the interested reader can be referred to [15, 9, 22].

Over the years, étalé spaces have evolved into a fundamental component of modern algebraic geometry, playing a pivotal role in the formulation of schemes - an essential concept in Grothendieck's reformulation of algebraic geometry. Presently, the theory of étalé spaces remains a dynamic and fertile area of research, establishing crucial connections between algebraic geometry and topology, and catalyzing advancements across diverse mathematical disciplines.

The influence of étalé spaces has extended beyond their original domain in algebraic geometry, reaching into various branches of mathematics, including number theory and representation theory. Furthermore, the integration of étalé spaces into Fuzzy Sheaf Theory has proven to be particularly fruitful, offering valuable insights into challenges related to spatial reasoning, knowledge representation, and fuzzy image analysis.

Inspired by the foundational notion of étalé spaces, numerous authors have proposed analogous concepts under different names for various structures throughout the years. Examples include sheaves of lattices [2], sheaves of rings,  $R$ -modules, and abelian groups [17], sheaf spaces of universal algebras [6], sheaves of chains over Boolean spaces [5], sheaves of normal lattices [12], sheaves of MV-algebras [10], sheaves of BL-algebras [7], sheaves of Banach spaces [18], among others. This diversity of applications reflects the enduring impact and versatility of the étalé spaces concept across various mathematical structures and frameworks.

The above discussion motivates us to introduce and investigate the notion of étalé spaces of residuated lattices in a similar fashion to étalé spaces of rings and modules. This motivation is to find a common pattern for some facts about étalé spaces of rings and étalé spaces of residuated lattices which occur in a much similar way in both cases of rings and of residuated lattices. Also, as known, étalé spaces arise from the theory of sheaves, a foundational framework that has significantly advanced our understanding of cohomology and the topology of algebraic varieties. While the majority of the aforementioned explorations delve into étalé spaces through the lens of sheaves, our approach takes a different approach by avoiding their use.

This paper is organized into three sections as follows: In Section 2, we revisit essential definitions, properties, and results related to residuated lattices, laying the groundwork for subsequent discussions. Section 3 delves

into the concept of étalé spaces of residuated lattices over a given topological space, recalling specific definitions of étalé spaces of sets that will be utilized in our exploration. Finally, we provide a method for transferring an étalé space of residuated lattices from one topological space to another, utilizing a continuous map.

## 2. RESIDUATED LATTICES

An algebra  $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow, 0, 1)$  is called a *residuated lattice* provided that  $\ell(\mathfrak{A}) = (A; \vee, \wedge, 0, 1)$  is a bounded lattice,  $(A; \odot, 1)$  is a commutative monoid, and  $(\odot, \rightarrow)$  is an adjoint pair. A residuated lattice  $\mathfrak{A}$  is called *non-degenerate* if  $0 \neq 1$ . For a residuated lattice  $\mathfrak{A}$ , and  $a \in A$ , we put  $\neg a := a \rightarrow 0$ , and  $a^n := a \odot \cdots \odot a$  ( $n$  times) for every integer  $n$ . The class of residuated lattices is equational, and so forms a variety. For a survey of residuated lattices, the interested reader is referred to [11].

**Example 2.1.** Let  $A_3 = \{0, a, 1\}$  be a lattice whose Hasse diagram is given by Figure 1. A routine verification confirms that  $\mathfrak{A}_3 = (A_3; \vee, \wedge, \odot, \rightarrow, 0, 1)$  is a residuated lattice in which the commutative operation “ $\odot$ ” is given by Table 1, and the operation “ $\rightarrow$ ” is given by  $x \rightarrow y = \bigvee \{z \in A_3 \mid x \odot z \leq y\}$ , for any  $x, y \in A_3$ .

| $\odot$ | 0 | a | 1 |
|---------|---|---|---|
| 0       | 0 | 0 | 0 |
| a       | 0 | 0 | a |
| 1       | 0 | a | 1 |

TABLE 1. Cayley table for “ $\odot$ ” of  $\mathfrak{A}_3$

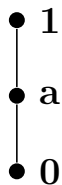
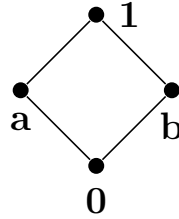


FIGURE 1. Hasse diagram of  $\mathfrak{A}_3$

**Example 2.2.** Let  $A_4 = \{0, a, b, 1\}$  be a lattice whose Hasse diagram is given by Figure 2. A routine verification confirms that  $\mathfrak{A}_4 = (A_4; \vee, \wedge, \odot, \rightarrow, 0, 1)$  is a residuated lattice in which the commutative operation “ $\odot$ ” is given by Table 2, and the operation “ $\rightarrow$ ” is given by  $x \rightarrow y = \bigvee \{z \in A_4 \mid x \odot z \leq y\}$ , for any  $x, y \in A_4$ .

| $\odot$ | 0 | a | b | 1 |
|---------|---|---|---|---|
| 0       | 0 | 0 | 0 | 0 |
| a       | 0 | a | 0 | a |
| b       | 0 | 0 | b | b |
| 1       | 0 | a | b | 1 |

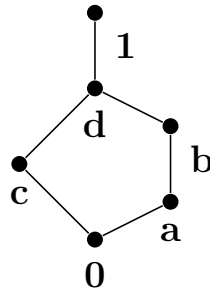
TABLE 2. Cayley table for “ $\odot$ ” of  $\mathfrak{A}_4$ FIGURE 2. Hasse diagram of  $\mathfrak{A}_4$ 

**Example 2.3.** Let  $A_6 = \{0, a, b, c, d, 1\}$  be a lattice whose Hasse diagram is given by Figure 3. A routine verification confirms that

$$\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$$

is a residuated lattice in which the commutative operation “ $\odot$ ” is given by Table 3, and the operation “ $\rightarrow$ ” is given by  $x \rightarrow y = \bigvee \{z \in A_6 \mid x \odot z \leq y\}$ , for any  $x, y \in A_6$ .

| $\odot$ | 0 | a | b | c | d | 1 |
|---------|---|---|---|---|---|---|
| 0       | 0 | 0 | 0 | 0 | 0 | 0 |
| a       | 0 | a | a | 0 | a | a |
| b       | 0 | a | a | 0 | a | b |
| c       | 0 | 0 | 0 | c | c | c |
| d       | 0 | a | a | c | d | d |
| 1       | 0 | a | b | c | d | 1 |

TABLE 3. Cayley table for “ $\odot$ ” of  $\mathfrak{A}_6$ FIGURE 3. Hasse diagram of  $\mathfrak{A}_6$

**Example 2.4.** Let  $A_8 = \{0, a, b, c, d, e, f, 1\}$  be a lattice whose Hasse diagram is given by Figure 4. A routine verification confirms that

$$\mathfrak{A}_8 = (A_8; \vee, \wedge, \odot, \rightarrow, 0, 1)$$

is a residuated lattice in which the commutative operation “ $\odot$ ” is given by Table 4, and the operation “ $\rightarrow$ ” is given by  $x \rightarrow y = \bigvee \{z \in A_8 \mid x \odot z \leq y\}$ , for any  $x, y \in A_8$ .

| $\odot$ | 0 | a | b | c | d | e | f | 1 |
|---------|---|---|---|---|---|---|---|---|
| 0       | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a       | 0 | a | 0 | a | a | a | a | a |
| b       | 0 | 0 | b | 0 | b | b | b | b |
| c       | 0 | a | 0 | c | a | c | a | c |
| d       | 0 | a | b | a | d | d | d | d |
| e       | 0 | a | b | c | d | e | d | e |
| f       | 0 | a | b | a | d | d | f | f |
| 1       | 0 | a | b | c | d | e | f | 1 |

TABLE 4. Cayley table for “ $\odot$ ” of  $\mathfrak{A}_8$

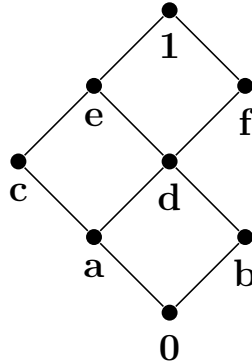


FIGURE 4. Hasse diagram of  $\mathfrak{A}_8$

**Example 2.5.** Let  $n$  be a fixed natural number. As noted by [27, p. 11, Example 2], the structure

$$_n = ([0, 1]; \max, \min, \odot_n, \rightarrow_n, 0, 1)$$

forms a residuated lattice, referred to as the *generalized Łukasiewicz structure*, which explicitly depends on the parameter  $n$ . In this structure, the commutative operation  $\odot_n$  is defined as

$$x \odot_n y = (\max\{0, x^n + y^n - 1\})^{\frac{1}{n}},$$

and the implication operation  $\rightarrow_n$  is given by

$$x \rightarrow_n y = \min \left\{ 1, (1 - x^n + y^n)^{\frac{1}{n}} \right\},$$

for all  $x, y \in [0, 1]$ .

This structure serves as a foundational model in Łukasiewicz fuzzy logic, where the operations  $\odot_n$  and  $\rightarrow_n$  represent conjunction and implication, respectively, capturing the behavior of truth values in many-valued logical systems. The dependence on  $n$  allows for a family of logical systems with varying degrees of sensitivity in their conjunction and implication operations.

Let  $\mathfrak{A}$  be a residuated lattice. A non-void subset  $F$  of  $A$  is called a *filter* of  $\mathfrak{A}$  provided that  $x, y \in F$  implies  $x \odot y \in F$ , and  $x \vee y \in F$ , for any  $x \in F$  and  $y \in A$ . The set of filters of  $\mathfrak{A}$  is denoted by  $\mathcal{F}(\mathfrak{A})$ . A filter  $F$  of  $\mathfrak{A}$  is called *proper* if  $F \neq A$ . Clearly,  $F$  is proper iff  $0 \notin F$ . For any subset  $X$  of  $A$ , the *filter of  $\mathfrak{A}$  generated by  $X$*  is denoted by  $\mathcal{F}(X)$ . For each  $x \in A$ , the filter generated by  $\{x\}$  is denoted by  $\mathcal{F}(x)$  and said to be *principal*. The set of principal filters is denoted by  $\mathcal{PF}(\mathfrak{A})$ . Following [14, §5.7], a join-complete lattice  $\mathfrak{A}$ , is called a *frame* if it satisfies the join infinite distributive law (JID), i.e., for any  $a \in A$  and  $S \subseteq A$ ,  $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$ . According to [11],  $(\mathcal{F}(\mathfrak{A}); \cap, \bigvee, \mathbf{1}, A)$  is a frame in which  $\bigvee \mathcal{F} = \mathcal{F}(\bigcup \mathcal{F})$ , for any  $\mathcal{F} \subseteq \mathcal{F}(\mathfrak{A})$ . With any filter  $F$  of a residuated lattice  $\mathfrak{A}$ , we can associate a binary relation  $\equiv_F$  on  $A$  as follows;  $x \equiv_F y$  if and only if  $x \rightarrow y, y \rightarrow x \in F$ , for any  $x, y \in A$ . The binary relation  $\equiv_F$  is a congruence on  $\mathfrak{A}$ , and called *the congruence induced by  $F$  on  $\mathfrak{A}$* . As usual, the set of all congruences on  $\mathfrak{A}$  shall be denoted by  $Con(\mathfrak{A})$ . It is well-known that  $\phi : \mathcal{F}(\mathfrak{A}) \rightarrow Con(\mathfrak{A})$ , defined by  $\phi(F) = \equiv_F$ , is a lattice isomorphism, and this implies that  $\mathcal{RL}$  is an ideal determined variety. For a filter  $F$  of a residuated lattice  $\mathfrak{A}$ , the quotient set  $A / \equiv_F$  with the natural operations becomes a residuated lattice which is denoted by  $\mathfrak{A} / F$ . Also, for any  $a \in A$ , the equivalence classes  $a / \equiv_F$  is denoted by  $a / F$ .

**Example 2.6.** Consider the residuated lattice  $\mathfrak{A}_4$  from Example 2.2, the residuated lattice  $\mathfrak{A}_6$  from Example 2.3, and the residuated lattice  $\mathfrak{A}_8$  from Example 2.4. The sets of their filters are presented in Table 5.

Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be residuated lattices. A map  $f : A \rightarrow B$  is called a *morphism of residuated lattices*, in symbols  $f : \mathfrak{A} \rightarrow \mathfrak{B}$ , if it preserves the fundamental operations. If  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  is a morphism of residuated lattices, we put  $coker(f) = f^{\leftarrow}(1)$ . One can see that  $f$  is injective if and only if  $coker(f) = \{1\}$ .

| Filters          |                                                                                                                                                             |
|------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathfrak{A}_4$ | $F_1 = \{1\}, F_2 = \{a, 1\}, F_3 = \{b, 1\}, F_4 = A_4$                                                                                                    |
| $\mathfrak{A}_6$ | $F_1 = \{1\}, F_2 = \{a, b, d, 1\}, F_3 = \{c, d, 1\}, F_4 = \{d, 1\}, F_5 = A_6$                                                                           |
| $\mathfrak{A}_8$ | $F_1 = \{1\}, F_2 = \{a, c, d, e, f, 1\}, F_3 = \{c, e, 1\}, F_4 = \{b, d, e, f, 1\},$<br>$F_5 = \{d, e, f, 1\}, F_6 = \{e, 1\}, F_7 = \{f, 1\}, F_8 = A_8$ |

TABLE 5. The sets of filters of  $\mathfrak{A}_4$ ,  $\mathfrak{A}_6$ , and  $\mathfrak{A}_8$ 

**Example 2.7.** Consider the residuated lattice  $\mathfrak{A}_4$  from Example 2.2 and the residuated lattice  $\mathfrak{A}_6$  from Example 2.3. One can see that the map  $f : A_6 \rightarrow A_4$ , given by  $f(0) = 0$ ,  $f(a) = f(b) = a$ ,  $f(c) = b$  and  $f(d) = f(1) = 1$  is a morphism of residuated lattices.

A proper filter in a residuated lattice  $\mathfrak{A}$  is termed *maximal* if it is a maximal element within the set of all proper filters of  $\mathfrak{A}$ . The collection of maximal filters of  $\mathfrak{A}$  is denoted by  $Max(\mathfrak{A})$ . Additionally, a proper filter  $\mathfrak{p}$  in  $\mathfrak{A}$  is *prime* if  $x \vee y \in \mathfrak{p}$  implies  $x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$ , for any  $x, y \in A$ . The set of prime filters of  $\mathfrak{A}$  is denoted by  $Spec(\mathfrak{A})$ . Given the distributive lattice property of  $\mathcal{F}(\mathfrak{A})$ , it follows that  $Max(\mathfrak{A}) \subseteq Spec(\mathfrak{A})$ . Zorn's lemma establishes that any proper filter is contained within a maximal filter and, consequently, within a prime filter. Minimal prime filters, identified as minimal elements within the set of prime filters, form the set  $Min(\mathfrak{A})$ . For a detailed exploration of prime filters in a residuated lattice, refer to [23].

**Example 2.8.** Consider the residuated lattice  $\mathfrak{A}_4$  from Example 2.2, the residuated lattice  $\mathfrak{A}_6$  from Example 2.3, and the residuated lattice  $\mathfrak{A}_8$  from Example 2.4. The sets of their maximal, prime, and minimal prime filters are presented in Table 7.

| Prime filters    |                 |                       |
|------------------|-----------------|-----------------------|
|                  | Maximal filters | Minimal prime filters |
| $\mathfrak{A}_4$ | $F_2, F_3$      | $F_2, F_3$            |
| $\mathfrak{A}_6$ | $F_2, F_3$      | $F_1$                 |
| $\mathfrak{A}_8$ | $F_2, F_4$      | $F_3, F_7$            |

TABLE 6. The sets of maximal, prime, and minimal prime filters of  $\mathfrak{A}_4$ ,  $\mathfrak{A}_6$ , and  $\mathfrak{A}_8$ 

Let  $\mathfrak{A}$  be a residuated lattice, and  $\Pi$  a collection of prime filters of  $\mathfrak{A}$ . For a subset  $\pi$  of  $\Pi$ , we define  $k(\pi) = \bigcap \pi$ . Additionally, for a subset  $X$  of  $A$ , we set  $h_\Pi(X) = \{P \in \Pi \mid X \subseteq P\}$  and  $d_\Pi(X) = \Pi \setminus h_\Pi(X)$ .

The collection  $\Pi$  can be endowed with various topological structures depending on how the family  $\{h_\Pi(x) \mid x \in A\}$  is interpreted. If this family is taken as a closed subbasis, the resulting topology is called the *hull-kernel topology*, denoted by  $\tau_h$ , and the corresponding topological space is denoted by  $\Pi_h$ . On the other hand, if the same family is considered as an open subbasis, the resulting topology is referred to as the *dual hull-kernel topology*, denoted by  $\tau_d$ , and the corresponding topological space is denoted by  $\Pi_d$ . Furthermore, the topology generated by the union  $\tau_h \cup \tau_d$  on  $\text{Spec}(\mathfrak{A})$  is called the *patch topology*, denoted by  $\tau_p$ , and the corresponding topological space is denoted by  $\text{Spec}_p(\mathfrak{A})$ . For a detailed discussion of the hull-kernel, dual hull-kernel, and patch topologies on a residuated lattice, we refer to [24].

To clarify, we provide some examples of topological spaces associated with residuated lattices.

**Example 2.9.** Consider the residuated lattice  $\mathfrak{A}_4$  from Example 2.2, the residuated lattice  $\mathfrak{A}_6$  from Example 2.3, and the residuated lattice  $\mathfrak{A}_8$  from Example 2.4. Table 7 presents some topological spaces related to these structures.

|                                 | Topological spaces                              |
|---------------------------------|-------------------------------------------------|
| $\text{Spec}_h(\mathfrak{A}_4)$ | $\{\emptyset, \{F_2\}, \{F_3\}, \{F_2, F_3\}\}$ |
| $\text{Max}_d(\mathfrak{A}_6)$  | $\{\emptyset, \{F_2\}, \{F_3\}, \{F_2, F_3\}\}$ |
| $\text{Min}_p(\mathfrak{A}_8)$  | $\{\emptyset, \{F_3\}, \{F_7\}, \{F_3, F_7\}\}$ |

TABLE 7. Some topological spaces associated with  $\mathfrak{A}_4$ ,  $\mathfrak{A}_6$ , and  $\mathfrak{A}_8$

### 3. ÉTALÉ SPACES OF RESIDUATED LATTICES

This section introduces and investigates the notion of étalé spaces of residuated lattices.

**3.1. Étale Spaces of Sets.** In this subsection, we recall specific definitions of étalé spaces of sets that will be used in our exploration. For the sake of completeness, we prove some properties of étalé spaces of sets.

Following [8, §4.4], a map  $f : \mathcal{T} \rightarrow \mathcal{B}$  is called a *local homeomorphism* if, for every point in  $\mathcal{T}$ , there exists a neighborhood  $U$  containing this point such that the restriction  $f|_U$  is a homeomorphism onto an open set.

**Lemma 3.1.** *Every local homeomorphism is continuous.*

*Proof.* Let  $f : \mathcal{T} \rightarrow \mathcal{B}$  be a local homeomorphism and  $V$  an open subset of  $\mathcal{B}$ . Consider  $t \in f^{-1}(V)$ . There exists a neighborhood  $U$  of  $t$  such that  $f|_U$  is

a homeomorphism. This shows that  $(f|_U)^\leftarrow(V \cap f(U))$  is a neighborhood of  $t$  in  $f^\leftarrow(V)$ .  $\square$

**Lemma 3.2.** *Every local homeomorphism is open.*

*Proof.* Let  $f : \mathcal{T} \rightarrow \mathcal{B}$  be a local homeomorphism. Assume that  $W$  is an open subset of  $\mathcal{T}$ . Consider  $b \in f(W)$ . Pick  $t \in f^\leftarrow(b)$ . There exists a neighborhood  $U$  of  $t$  such that  $f|_U$  is a homeomorphism. This shows that  $f|_W(U \cap W)$  is a neighborhood of  $b$  in  $f(W)$ .  $\square$

Let  $f : \mathcal{T} \rightarrow Y$  be a map where  $\mathcal{T}$  is a topological space. We say  $f$  is *locally injective* if for every  $t \in \mathcal{T}$ , there exists a neighborhood  $U \subset \mathcal{T}$  of  $t$  such that the restriction  $f|_U : U \rightarrow Y$  is injective.

**Lemma 3.3.** *Every local homeomorphism is locally injective.*

*Proof.* The proof is immediate.  $\square$

**Theorem 3.4.** *A map is a local homeomorphism if and only if it is continuous, open, and locally injective. In particular, a bijective local homeomorphism is a homeomorphism.*

*Proof.* It is an immediate consequence of lemmas 3.1, 3.2 & 3.3.  $\square$

**Proposition 3.5.** *Let  $f : \mathcal{T} \rightarrow \mathcal{B}$  be a local homeomorphism. The family  $\{V \mid V \text{ is open in } \mathcal{T}, f|_V : V \rightarrow \mathcal{B} \text{ is a homeomorphism}\}$  is a basis for the topology of  $\mathcal{T}$ .*

*Proof.* Let  $U$  be an open set in  $\mathcal{T}$ . Consider  $x \in U$ . There exists a neighborhood  $W$  of  $x$  such that  $f|_W : W \rightarrow \mathcal{B}$  is a homeomorphism. Let  $V = U \cap W$ . It is easy to check that  $f|_V : V \rightarrow \mathcal{B}$  is a homeomorphism and  $x \in V \subseteq U$ .  $\square$

**Lemma 3.6.** (1) *The restriction of a local homeomorphism to a topological subspace is a local homeomorphism;*  
 (2) *The composition of two local homeomorphisms is a local homeomorphism.*

*Proof.* It is evident.  $\square$

Let  $\mathcal{C}$  be a category and  $c$  an object of  $\mathcal{C}$ . Recall that the *slice category* (an especial type of comma category [20, §6 of Chapter II]) of  $\mathcal{C}$  over  $c$ , denoted by  $\mathcal{C}/c$ , is the category whose objects are all pairs  $(a, f)$ , where  $a$  is an object of  $\mathcal{C}$  and  $f$  is a morphism from  $a$  to  $c$ , and whose morphisms from  $(a, f)$  to

$(b, g)$  given by morphisms  $h$  of  $\mathcal{C}$  that commutes the following diagram.

$$\begin{array}{ccc} a & \xrightarrow{h} & b \\ & \searrow f \quad \swarrow g & \\ & c & \end{array}$$

If  $\mathcal{C}/c$  is a slice category,  $c$  is known as the *base space* of the category  $\mathcal{C}/c$ .

In the sequel, **Top** refers to the category of topological spaces with continuous maps, while **Top<sub>ℓ</sub>** denotes the category of topological spaces with local homeomorphisms.

Following [26, §2.3], for a given space  $\mathcal{B}$ , the objects of the slice category **Étale**( $\mathcal{B}$ ) := **Top<sub>ℓ</sub>**/ $\mathcal{B}$  are called *étale spaces of sets* over  $\mathcal{B}$ , and the morphisms are *morphisms of étale spaces* over  $\mathcal{B}$ . An étale space  $(\mathcal{T}, \pi)$  over  $\mathcal{B}$  consists of:

- A *total space*  $\mathcal{T}$ ,
- An *étale projection*  $\pi: \mathcal{T} \rightarrow \mathcal{B}$ .

For each point  $b \in \mathcal{B}$ , the fiber  $\pi^{-1}(b)$  is called the *stalk of  $\mathcal{T}$  at  $b$*  and is denoted by  $\mathcal{T}_b$ .

In the following discussion, an étalé space  $(\mathcal{T}, \pi)$  of sets over  $\mathcal{B}$  will be denoted by  $\mathcal{T} \downarrow^{\pi} \mathcal{B}$  and will simply be referred to as an étalé space. Additionally, a morphism of étalé spaces of sets  $h: (\mathcal{T}, \pi) \rightarrow (\mathcal{S}, \phi)$  over  $\mathcal{B}$  will be denoted by  $h: \mathcal{T} \downarrow^{\pi} \mathcal{B} \rightarrow \mathcal{S} \downarrow^{\phi} \mathcal{B}$  and referred to as a morphism of étalé spaces. When the base space and the étalé projection are clear from context, they will typically be omitted for brevity.

The next proposition is a restatement of Lemma 3.5 from [26, §2], with a more detailed proof.

**Proposition 3.7.** *Let  $\mathcal{T} \downarrow^{\pi} \mathcal{B}$  and  $\mathcal{S} \downarrow^{\phi} \mathcal{B}$  be two étalé spaces. If  $h: \mathcal{T} \rightarrow \mathcal{S}$  is a map such that  $\phi h = \pi$ , the following are equivalent:*

- (1)  *$h$  is a local homeomorphism;*
- (2)  *$h$  is open;*
- (3)  *$h$  is continuous.*

*Proof.* (1)  $\Rightarrow$  (2): It is evident by Lemma 3.2.

(2)  $\Rightarrow$  (3): Applying Proposition 3.5, let  $V$  be a basic open set in  $\mathcal{S}$  such that  $\phi|_V$  is a homeomorphism. Let  $x \in h^{\leftarrow}(V)$ . So  $\pi(x) \in \phi(V)$ . One can find a neighborhood  $U$  of  $x$  such that  $\pi|_U$  is a homeomorphism and  $\pi(U) \subseteq \phi(V)$ . Let  $W = h(U) \cap \phi|_V^{\leftarrow}(\pi(U))$ . Consider  $t \in \pi|_U^{\leftarrow}(\phi|_V(W))$ . There exist  $w \in W$

and  $u \in U$  such that  $\pi(u) = \phi(h(u)) = \phi(w) = \pi(t)$ , and this implies that  $u = t$ . This states that  $t \in h^{\leftarrow}(V)$ .

(3) $\Rightarrow$ (1): Let  $t \in \mathcal{T}$ . There exist neighborhoods  $U$  and  $V$ , respectively, of  $t$  and  $h(t)$  such that  $\pi|_U$  and  $\phi|_V$  are homeomorphisms. By continuity of  $h$ ,  $W = U \cap h^{\leftarrow}(V)$  is a neighborhood of  $t$ . Let  $h(t_1) = h(t_2)$ , for some  $t_1, t_2 \in W$ . So we have  $\pi(t_1) = \phi(h(t_1)) = \phi(h(t_2)) = \pi(t_2)$  and this states that  $t_1 = t_2$ , since  $\pi|_U$  is injective. Let  $O$  be an open subset in  $W$ . So we have  $h(O) = (\phi|_V)^{\leftarrow}(\phi(h(O))) = (\phi|_V)^{\leftarrow}(\pi(h(O)))$  and this shows that  $h(O)$  is open.  $\square$

**Definition 3.8.** Let  $\mathcal{T} \xrightarrow{\pi} \mathcal{B}$  be an étalé space. A continuous map  $\sigma : X \rightarrow \mathcal{T}$ , where  $X$  is a subspace of  $\mathcal{B}$ , is referred to as a *section of  $\mathcal{T}$  over  $X$*  if  $\pi\sigma = 1$ . The set of all sections of  $\mathcal{T}$  over  $X$  is denoted by  $\Gamma(X, \mathcal{T})$ . Elements of  $\Gamma(\mathcal{B}, \mathcal{T})$  are termed *global sections* of  $\mathcal{T}$ .

*Remark 3.9.* The nonemptiness of the set of global sections of an étalé space ensures the surjectivity of its étalé projection.

**Lemma 3.10.** Let  $\mathcal{T} \xrightarrow{\pi} \mathcal{B}$  be an étalé space, and  $t \in \mathcal{T}$ . There exist a neighborhood  $U$  of  $\pi(t)$  and a section  $\sigma$  of  $\mathcal{T}$  over  $U$  such that  $\sigma\pi(t) = t$ .

*Proof.* There exists a neighborhood  $V$  of  $t$  such that  $\pi|_V$  is a homeomorphism. Take  $U = \pi(V)$  and define  $\sigma : U \rightarrow \mathcal{T}$  by  $\sigma(x) = (\pi|_V)^{\leftarrow}(x)$ , for all  $x \in U$ .  $\square$

**Lemma 3.11.** Let  $\mathcal{T} \xrightarrow{\pi} \mathcal{B}$  be an étalé space,  $\sigma$  a section of  $\mathcal{T}$  over an open subset of  $\mathcal{B}$ ,  $x$  an element of the domain  $\sigma$ , and  $V$  a neighborhood of  $\sigma(x)$  such that  $\pi|_V$  is a homeomorphism. There exists a neighborhood  $U$  of  $x$  such that  $\sigma(U) \subseteq V$  and  $\sigma|_U = (\pi|_V)^{\leftarrow}|_U$ .

*Proof.* By hypothesis  $U = \sigma^{\leftarrow}(V)$  is a neighborhood of  $x$  in  $\mathcal{B}$ . Obviously,  $\sigma(U) \subseteq V$ . Also, we have

$$(\pi|_V)^{\leftarrow}|_U(x) = (\pi|_V)^{\leftarrow}|_U(\pi\sigma(x)) = (\pi|_V)^{\leftarrow}(\pi\sigma(x)) = \sigma(x),$$

for any  $x \in U$ .  $\square$

If  $X, Y \subseteq \mathcal{B}$ , and  $\sigma \in \Gamma(X, \mathcal{T})$  and  $\tau \in \Gamma(Y, \mathcal{T})$ , the *equalizer* of  $\sigma|_{X \cap Y}$  and  $\tau|_{X \cap Y}$  in the category **Top** is denoted, for simplicity, by  $\text{Eq}(\sigma, \tau)$ . This equalizer is defined as the set  $\{b \in X \cap Y \mid \sigma(b) = \tau(b)\}$  equipped with the subspace topology. By a slight abuse of language, we refer to it as the equalizer of  $\sigma$  and  $\tau$ .

**Proposition 3.12.** *Let  $\mathcal{T} \downarrow^\pi \mathcal{B}$  be an étalé space. The following hold:*

- (1) *If  $U$  and  $V$  are open sets in  $\mathcal{B}$ ,  $\sigma \in \Gamma(U, \mathcal{T})$  and  $\tau \in \Gamma(V, \mathcal{T})$ , then  $\text{Eq}(\sigma, \tau)$  is open;*
- (2) *If  $\mathcal{T}$  is Hausdorff,  $U$  and  $V$  are open sets in  $\mathcal{B}$ ,  $\sigma \in \Gamma(U, \mathcal{T})$  and  $\tau \in \Gamma(V, \mathcal{T})$ , then  $\text{Eq}(\sigma, \tau)$  is clopen;*
- (3) *Each section of  $\mathcal{T}$ , defined over an open set in  $\mathcal{B}$ , is open.*

*Proof.* (1): Let  $b \in \text{Eq}(\sigma, \tau)$ . Assume that  $W$  is a neighborhood of  $\sigma(b)$  such that  $\pi|_W$  is a homeomorphism. Applying Lemma 3.11, there exist neighborhoods  $U_0$  and  $V_0$  of  $b$  such that  $\sigma(U_0), \tau(V_0) \subseteq W$ . One can see that  $U_0 \cap V_0$  is a neighborhood of  $b$  contained in  $\text{Eq}(\sigma, \tau)$ .

(2): It follows by (1) and [28, Theorem 13.13].

(3): Let  $U$  be an open set in  $\mathcal{B}$  and  $\sigma \in \Gamma(U, \mathcal{T})$ . Consider  $t \in \sigma(U)$ . So  $t = \sigma(b)$ , for some  $b \in U$ . Let  $W$  be a neighborhood  $t$  such that  $\pi|_W$  is a homeomorphism. By Lemma 3.11, there exists a neighborhood  $V$  of  $b$  such that  $\sigma(V) \subseteq W$  and  $\sigma|_V = (\pi|_W)^\leftarrow|_V$ . So  $\sigma(V) = (\pi|_W)^\leftarrow(V)$  is a neighborhood of  $t$  contained in  $\sigma(U)$ .  $\square$

**Proposition 3.13.** *Let  $\mathcal{T} \downarrow^\pi \mathcal{B}$  be an étalé space. The family*

$$\{\sigma(U) \mid U \text{ is open in } \mathcal{B}, \sigma \in \Gamma(U, \mathcal{T})\}$$

*is a basis for the topology of  $\mathcal{T}$ .*

*Proof.* Let  $V$  be an open set in  $\mathcal{T}$  such that  $\pi|_V$  is a homeomorphism. Take  $U = \pi(V)$  and define  $\sigma : U \rightarrow \mathcal{T}$  by  $\sigma(x) = (\pi|_V)^\leftarrow(x)$ , for all  $x \in U$ . It is evident that  $\sigma \in \Gamma(U, \mathcal{T})$  and  $V = \sigma(U)$ . So

$$\begin{aligned} & \{V \mid V \text{ is Open in } \mathcal{T}, \pi|_V \text{ is a homeomorphism}\} \\ & \subseteq \{\sigma(U) \mid U \text{ is Open in } \mathcal{B}, \sigma \in \Gamma(U, \mathcal{T})\} \end{aligned}$$

and so the result holds due to Proposition 3.5.  $\square$

Let  $X$  be a set,  $\{Y_i\}_{i \in I}$  be a collection of topological spaces, and  $\{f_i : Y_i \rightarrow X\}_{i \in I}$  be a collection of maps. The *final topology* induced by  $\{f_i\}_{i \in I}$  on  $X$  is the finest topology on  $X$  which makes  $f_i : Y_i \rightarrow X$  continuous for all  $i \in I$ . It is obvious that a subset  $U$  of  $X$  is open in the final topology if and only if  $f_i^\leftarrow(U)$  is open in  $Y_i$ , for each  $i \in I$ .

**Proposition 3.14.** *Let  $\mathcal{T} \downarrow^\pi \mathcal{B}$  be an étalé space. The following hold:*

- (1) *The topology of  $\mathcal{T}$  coincides with the final topology induced by the set of sections of  $\mathcal{T}$ ;*

(2) *The topology induced on each stalk of  $\mathcal{T}$  is the discrete topology.*

*Proof.* (1): Let  $V \subseteq \mathcal{T}$ . Suppose first that  $V$  is open in the topology of  $\mathcal{T}$ . Then for every section  $\sigma$  of  $\mathcal{T}$ , the preimage  $\sigma^{\leftarrow}(V)$  is open in the domain of  $\sigma$ , so  $V$  is open in the final topology.

Conversely, assume  $V$  is open in the final topology. Let  $x \in V$ . By Lemma 3.10, there exists a section  $\sigma$  such that  $\sigma(\pi(x)) = x$ . Then, by Proposition 3.12(3), the set  $\sigma(\sigma^{\leftarrow}(V))$  is a neighborhood of  $x$  contained in  $V$ . Hence,  $V$  is open in the topology of  $\mathcal{T}$ .

(2): Let  $x \in \mathcal{B}$ . By Lemma 3.10, for any  $y \in \mathcal{T}_x$  there exists a neighborhood  $U$  of  $\pi(y)$  and a section  $\sigma$  of  $\mathcal{T}$  over  $U$  such that  $\sigma(\pi(y)) = y$ . One can then observe that  $\sigma(U) \cap \mathcal{T}_x = \{y\}$ , which implies that the singleton  $\{y\}$  is open in  $\mathcal{T}_x$ . Thus, the induced topology on each stalk is discrete.  $\square$

**3.2. Étale Spaces of Residuated Lattices.** In this subsection, we introduce the concept of étale spaces of residuated lattices and the morphisms between them. We demonstrate that for a given space  $\mathcal{B}$ , the collection of étale spaces of residuated lattices over  $\mathcal{B}$  along with the morphisms between them forms a subcategory of **Étale**( $\mathcal{B}$ ).

**Definition 3.15.** An étale space  $\mathcal{T} \overset{\pi}{\downarrow} \mathcal{B}$  is called an *étale space of residuated lattices* if the following conditions hold:

- (1) For each point  $b \in \mathcal{B}$ , the stalk  $\mathcal{T}_b$  is a residuated lattice. The operations of the residuated lattice  $\mathcal{T}_b$  are indexed by  $b$ .
- (2) For any subset  $X \subseteq \mathcal{B}$ , the set of sections  $\Gamma(X, \mathcal{T})$  forms a residuated lattice, with operations defined pointwise as follows:

$$\begin{aligned} \vee_X : \Gamma(X, \mathcal{T}) \times \Gamma(X, \mathcal{T}) &\rightarrow \Gamma(X, \mathcal{T}) \\ (\sigma, \rho) &\mapsto (x \mapsto \sigma(x) \vee_x \rho(x)) \end{aligned}$$

$$\begin{aligned} \wedge_X : \Gamma(X, \mathcal{T}) \times \Gamma(X, \mathcal{T}) &\rightarrow \Gamma(X, \mathcal{T}) \\ (\sigma, \rho) &\mapsto (x \mapsto \sigma(x) \wedge_x \rho(x)) \end{aligned}$$

$$\begin{aligned} \odot_X : \Gamma(X, \mathcal{T}) \times \Gamma(X, \mathcal{T}) &\rightarrow \Gamma(X, \mathcal{T}) \\ (\sigma, \rho) &\mapsto (x \mapsto \sigma(x) \odot_x \rho(x)) \end{aligned}$$

$$\begin{aligned} \rightarrow_X : \Gamma(X, \mathcal{T}) \times \Gamma(X, \mathcal{T}) &\rightarrow \Gamma(X, \mathcal{T}) \\ (\sigma, \rho) &\mapsto (x \mapsto \sigma(x) \rightarrow_x \rho(x)) \end{aligned}$$

$$\begin{aligned} 0_X : X &\rightarrow \mathcal{T} \\ x &\mapsto 0_x \end{aligned}$$

$$\begin{aligned} 1_X : X &\rightarrow \mathcal{T} \\ x &\mapsto 1_x \end{aligned}$$

To clarify the discussion, we now present some examples of étalé spaces of residuated lattices over a topological space.

**Example 3.16.** Consider the residuated lattice  $\mathfrak{A}_4$  from Example 2.2, the topological space  $\text{Spec}_h(\mathfrak{A}_4)$  from Example 2.9, and the Boolean algebra  $\mathfrak{A}_2 = \{0, 1\}$  with the usual operations.

Let  $\mathcal{T}$  be the disjoint union  $\mathfrak{A}_2 \coprod \mathfrak{A}_2$ , and denote its elements by

$$\mathcal{T} = \{0_1, 1_1\} \cup \{0_2, 1_2\},$$

where  $\{0_1, 1_1\}$  and  $\{0_2, 1_2\}$  are distinct copies of  $\mathfrak{A}_2$ . Equip  $\mathcal{T}$  with the discrete topology.

Define a map  $\pi : \mathcal{T} \rightarrow \text{Spec}(\mathfrak{A}_4)$  by

$$\pi(0_1) = \pi(1_1) = F_2, \quad \pi(0_2) = \pi(1_2) = F_3,$$

where  $F_2$  and  $F_3$  are prime filters in  $\text{Spec}(\mathfrak{A}_4)$ . Then the fiber over  $F_2$  is

$$\mathcal{T}_{F_2} = \pi^{-1}(F_2) = \{0_1, 1_1\} \cong \mathfrak{A}_2,$$

and similarly, the fiber over  $F_3$  is

$$\mathcal{T}_{F_3} = \pi^{-1}(F_3) = \{0_2, 1_2\} \cong \mathfrak{A}_2.$$

Each fiber  $\mathcal{T}_b$  for  $b \in \{F_2, F_3\}$  is isomorphic to the Boolean algebra  $\mathfrak{A}_2$ , and hence is a residuated lattice with operations defined pointwise (inherited from  $\mathfrak{A}_2$ ).

The map  $\pi : \mathcal{T} \rightarrow \text{Spec}(\mathfrak{A}_4)$  is continuous, since the discrete topology on  $\mathcal{T}$  ensures that every point is open, and the preimage of any open set in  $\text{Spec}_h(\mathfrak{A}_4)$  is open. Furthermore,  $\pi$  is a local homeomorphism: each point in  $\mathcal{T}$  has an open neighborhood (a singleton) that is mapped homeomorphically onto its image in  $\text{Spec}_h(\mathfrak{A}_4)$ .

Thus,  $\mathcal{T} \overset{\pi}{\downarrow} \text{Spec}(\mathfrak{A}_4)$  is an étalé space of residuated lattices, where the stalk over each point is isomorphic to  $\mathfrak{A}_2$ , and the structure on sections is defined pointwise.

**Example 3.17.** Consider the residuated lattice  $\mathfrak{A}_3$  from Example 2.1, the residuated lattice  $\mathfrak{A}_6$  from Example 2.3, and the topological space  $\text{Max}_d(\mathfrak{A}_6)$  from Example 2.9. Let  $\mathcal{T}$  be the disjoint union  $\mathfrak{A}_2 \coprod \mathfrak{A}_3 = \{0_1, 1_1, 0_2, a_2, 1_2\}$ , where  $\{0_1, 1_1\}$  is a copy of  $\mathfrak{A}_2$  and  $\{0_2, a_2, 1_2\}$  is a copy of  $\mathfrak{A}_3$ . Equip  $\mathcal{T}$  with the discrete topology.

Define a map  $\pi : \mathcal{T} \rightarrow \text{Max}_d(\mathfrak{A}_6)$  by:

$$\pi(0_1) = \pi(1_1) = F_2, \quad \pi(0_2) = \pi(a_2) = \pi(1_2) = F_3,$$

where  $F_2$  and  $F_3$  are maximal filters in  $\text{Max}_d(\mathfrak{A}_6)$ .

The fibers (stalks) of  $\pi$  are given by:

$$\mathcal{T}_{F_2} = \pi^{-1}(F_2) = \{0_1, 1_1\} \cong \mathfrak{A}_2, \quad \mathcal{T}_{F_3} = \pi^{-1}(F_3) = \{0_2, a_2, 1_2\} \cong \mathfrak{A}_3.$$

Each stalk  $\mathcal{T}_b$  inherits the structure of a residuated lattice from its corresponding component ( $\mathfrak{A}_2$  or  $\mathfrak{A}_3$ ), with operations defined pointwise and indexed by the point  $b \in \text{Max}_d(\mathfrak{A}_6)$ .

The topology on  $\mathcal{T}$  is discrete, and the map  $\pi$  is continuous since the preimage of every open set in  $\text{Max}_d(\mathfrak{A}_6)$  is open in the discrete topology. Moreover,  $\pi$  is a local homeomorphism: each singleton  $\{x\} \subseteq \mathcal{T}$  is open, and  $\pi$  restricts to a homeomorphism on each singleton fiber component.

Hence, the structure  $\mathcal{T} \xrightarrow{\pi} \text{Max}_d(\mathfrak{A}_6)$  defines an étalé space of residuated lattices, where each stalk carries the algebraic structure of either  $\mathfrak{A}_2$  or  $\mathfrak{A}_3$ , and the étalé condition is satisfied via the local homeomorphism property of  $\pi$ .

**Example 3.18.** Consider the residuated lattice  $\mathfrak{A}_8$  from Example 2.4, and the topological space  $\text{Min}_p(\mathfrak{A}_8)$  from Example 2.9. Let  $\mathcal{T}$  be the disjoint union  $\mathfrak{A}_3 \coprod \mathfrak{A}_4 = \{0_1, a_1, 1_1, 0_2, a_2, b_2, 1_2\}$ , where  $\{0_1, a_1, 1_1\}$  is a copy of  $\mathfrak{A}_3$  and  $\{0_2, a_2, b_2, 1_2\}$  is a copy of  $\mathfrak{A}_4$ . Equip  $\mathcal{T}$  with the discrete topology.

Define a map  $\pi : \mathcal{T} \rightarrow \text{Min}_p(\mathfrak{A}_8)$  by:

$$\pi(0_1) = \pi(a_1) = \pi(1_1) = F_3, \quad \pi(0_2) = \pi(a_2) = \pi(b_2) = \pi(1_2) = F_7,$$

where  $F_3$  and  $F_7$  are prime filters in  $\text{Min}_p(\mathfrak{A}_8)$ .

The fiber (stalk) over  $F_3$  is given by:

$$\mathcal{T}_{F_3} = \pi^{-1}(F_3) = \{0_1, a_1, 1_1\} \cong \mathfrak{A}_3,$$

and the fiber over  $F_7$  is:

$$\mathcal{T}_{F_7} = \pi^{-1}(F_7) = \{0_2, a_2, b_2, 1_2\} \cong \mathfrak{A}_4.$$

Each stalk  $\mathcal{T}_b$  is a residuated lattice (either  $\mathfrak{A}_3$  or  $\mathfrak{A}_4$ ) with operations inherited from its corresponding component, defined pointwise for each  $b \in \text{Min}_p(\mathfrak{A}_8)$ . The topology on  $\mathcal{T}$  is discrete, so every singleton is open. Since  $\pi$  maps points to their assigned prime filters, and the preimage of an open set in  $\text{Min}_p(\mathfrak{A}_8)$  is open in  $\mathcal{T}$ ,  $\pi$  is continuous.

Moreover,  $\pi$  is a local homeomorphism: every point in  $\mathcal{T}$  has a neighborhood (a singleton) on which  $\pi$  restricts to a homeomorphism onto its image.

Therefore, the structure  $\mathcal{T} \overset{\pi}{\downarrow} \text{Min}_p(\mathfrak{A}_8)$  is an étalé space of residuated lattices, with each stalk capturing the algebraic behavior of either  $\mathfrak{A}_3$  or  $\mathfrak{A}_4$ .

Let  $\mathcal{T} \overset{\pi}{\downarrow} \mathcal{B}$  be an étalé space of residuated lattices. Define

$$\mathcal{T}^{(2)} = \{(a_1, a_2) \in \mathcal{T}^2 \mid \pi(a_1) = \pi(a_2)\},$$

and equip it with the subspace topology inherited from the product topology on  $\mathcal{T}^2$ . If  $\diamond$  is a binary fundamental operation of residuated lattices, and  $\diamond_b$  denotes the corresponding operation on the stalk  $\mathcal{T}_b$  for each  $b \in \mathcal{B}$ , then we define a map

$$\overset{\mathcal{T}}{\diamond} : \mathcal{T}^{(2)} \rightarrow \mathcal{T}, \quad \text{by} \quad \overset{\mathcal{T}}{\diamond}(a_1, a_2) := \diamond_b(a_1, a_2),$$

where  $\pi(a_1) = \pi(a_2) = b$ . Likewise, if  $\diamond$  is a nullary function symbol and  $\diamond_b$  is the corresponding constant in the stalk  $\mathcal{T}_b$ , we define

$$\overset{\mathcal{T}}{\diamond} : \mathcal{B} \rightarrow \mathcal{T}, \quad \text{by} \quad \overset{\mathcal{T}}{\diamond}(b) := \diamond_b.$$

In what follows, we will omit the superscript  $\mathcal{T}$  when the context is clear.

The next result, inspired by a similar statement for universal algebras [6, Lemma 1.2], provides necessary and sufficient conditions for an étalé space to be an étalé space of residuated lattices.

**Proposition 3.19.** *Let  $\mathcal{T} \overset{\pi}{\downarrow} \mathcal{B}$  be an étalé space such that each stalk  $\mathcal{T}_b$  is a residuated lattice. The following conditions are equivalent:*

- (1)  *$\mathcal{T}$  is an étalé space of residuated lattices over the topological space  $\mathcal{B}$ ;*
- (2) *For every open subset  $U \subseteq \mathcal{B}$ , the set  $\Gamma(U, \mathcal{T})$  of sections over  $U$  forms a residuated lattice under pointwise operations;*
- (3) *The operations  $\vee, \wedge, \odot, \rightarrow, 0$ , and  $1$  are continuous.*

*Proof.* (1) $\Rightarrow$ (2): This implication follows directly from Definition 3.15.

(2) $\Rightarrow$ (3): Let  $U \subseteq \mathcal{B}$  be open, and let  $\sigma \in \Gamma(U, \mathcal{T})$ . By Proposition 3.13, the image  $\sigma(U)$  is a basic open set in  $\mathcal{T}$ . Consider  $(x, y) \in \vee^{-1}(\sigma(U))$ . Then

$$x \vee_b y = \sigma(b), \quad \text{where } \pi(x) = \pi(y) = b \in U.$$

By Lemma 3.10, there exist neighborhoods  $W$  of  $b$  and sections  $\sigma_x, \sigma_y \in \Gamma(W, \mathcal{T})$  such that  $\sigma_x(b) = x$  and  $\sigma_y(b) = y$ . Then the section  $\sigma_x \vee \sigma_y$  satisfies  $(\sigma_x \vee \sigma_y)(b) = \sigma(b)$ , and by Proposition 3.12(1), the set

$$V = \text{Eq}(\sigma, \sigma_x \vee \sigma_y)$$

is a neighborhood of  $b$ . Hence,

$$(x, y) \in \sigma_x(V) \times \sigma_y(V) \cap \mathcal{T}^{(2)} \subseteq \vee^{-1}(\sigma(U)),$$

showing that  $\vee$  is continuous. The continuity of  $\wedge$ ,  $\odot$ , and  $\rightarrow$  follows analogously. The continuity of  $0$  and  $1$  is evident, as they are defined pointwise by  $b \mapsto 0_b$  and  $b \mapsto 1_b$ .

**(3)  $\Rightarrow$  (1):** Let  $X \subseteq \mathcal{B}$  and  $\sigma, \tau \in \Gamma(X, \mathcal{T})$ . For any binary fundamental operation  $\diamond$  of residuated lattices, note that the pointwise operation  $\diamond_X(\sigma, \tau)$  is given by

$$\diamond_X(\sigma, \tau) = \diamond \circ (\sigma, \tau),$$

which is continuous since both  $\sigma$  and  $\tau$  are continuous and  $\diamond$  is continuous by assumption. Thus  $\diamond_X(\sigma, \tau) \in \Gamma(X, \mathcal{T})$ . For each nullary operation symbol  $\diamond$ , define the section  $\diamond_X$  by restriction:

$$\diamond_X = \diamond|_X.$$

Then  $(\Gamma(X, \mathcal{T}); \vee_X, \wedge_X, \odot_X, \rightarrow_X, 0_X, 1_X)$  forms a residuated lattice under pointwise operations, and therefore  $\mathcal{T}$  satisfies Definition 3.15.  $\square$

**Definition 3.20.** Let  $\mathcal{T} \xrightarrow{\pi} \mathcal{B}$  and  $\mathcal{S} \xrightarrow{\phi} \mathcal{B}$  be two étalé spaces of residuated lattices. A morphism of étalé spaces  $h : \mathcal{T} \rightarrow \mathcal{S}$  is called a *morphism of étalé spaces of residuated lattices* if, for every point  $b \in \mathcal{B}$ , the restriction  $h|_{\mathcal{T}_b}$  - denoted by  $h_b$  - is a morphism of residuated lattices.

*Remark 3.21.* Let  $\mathcal{T} \xrightarrow{\pi} \mathcal{B}$  and  $\mathcal{S} \xrightarrow{\phi} \mathcal{B}$  be two étalé spaces of residuated lattices. To define a morphism of étalé spaces of residuated lattices from  $\mathcal{T}$  to  $\mathcal{S}$ , it suffices to specify a family  $\{h_b : \mathcal{T}_b \rightarrow \mathcal{S}_b\}_{b \in \mathcal{B}}$  of residuated lattice morphisms. The global morphism  $h$  is then defined by

$$h(t) := h_{\pi(t)}(t), \quad \text{for all } t \in \mathcal{T}.$$

Moreover, the morphism  $h$  is injective (respectively, surjective) if and only if each  $h_b$  is injective (respectively, surjective) for all  $b \in \mathcal{B}$ .

**Theorem 3.22.** *For a given topological space  $\mathcal{B}$ , the class of étalé spaces of residuated lattices over  $\mathcal{B}$ , together with the class of morphisms of étalé spaces of residuated lattices, forms a subcategory of  $\mathbf{Etale}(\mathcal{B})$ . This subcategory is denoted by  $\mathbf{RL-Etale}(\mathcal{B})$ .*

*Proof.* The result follows directly from the fact that identities and compositions of morphisms of étalé spaces of residuated lattices over  $\mathcal{B}$  are again morphisms of étalé spaces over  $\mathcal{B}$ . Hence, the required category axioms are satisfied.  $\square$

**3.3. Pullbacks of Étale Spaces of Residuated Lattices.** So far, we have only considered morphisms of étale spaces of residuated lattices defined over the same base space. In this section, we present a method for transferring an étale space of residuated lattices from one topological space to another via a continuous map.

Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map, and let  $\mathcal{S} \overset{\phi}{\downarrow} \mathcal{C}$  be an étale space. The *pullback of  $\mathcal{S}$  along  $f$* , denoted  $f^*\mathcal{S}$ , is the topological space defined as

$$f^*\mathcal{S} := \{(b, s) \in \mathcal{B} \times \mathcal{S} \mid f(b) = \phi(s)\},$$

equipped with the subspace topology induced from the product topology on  $\mathcal{B} \times \mathcal{S}$ . The canonical projection onto the first coordinate

$$f^*\phi : f^*\mathcal{S} \rightarrow \mathcal{B}, \quad (b, s) \mapsto b$$

is called the *pullback projection*.

**Proposition 3.23.** *Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map.*

- (1) *The pullback of an étale space over  $\mathcal{C}$  along  $f$  is an étale space over  $\mathcal{B}$ .*
- (2) *The pullback of a morphism of étale spaces over  $\mathcal{C}$  along  $f$  is a morphism of étale spaces over  $\mathcal{B}$ .*

*Proof.* (1): Let  $\mathcal{S} \overset{\phi}{\downarrow} \mathcal{C}$  be an étale space. Consider the following pullback square in **Top**:

$$\begin{array}{ccc} f^*\mathcal{S} & \xrightarrow{f'} & \mathcal{S} \\ f^*\phi \downarrow & & \downarrow \phi \\ \mathcal{B} & \xrightarrow{f} & \mathcal{C} \end{array}$$

Let  $(b, s) \in f^*\mathcal{S}$ . Since  $\phi$  is a local homeomorphism, there exists an open neighborhood  $U \subseteq \mathcal{S}$  of  $s$  such that  $\phi|_U : U \rightarrow \phi(U)$  is a homeomorphism. Then the set

$$W := (V \times U) \cap f^*\mathcal{S}, \quad \text{where } V = f^{-1}(\phi(U)),$$

is an open neighborhood of  $(b, s)$  in  $f^*\mathcal{S}$ , and the restriction  $f^*\phi|_W : W \rightarrow V$  is a homeomorphism. Therefore,  $f^*\phi$  is a local homeomorphism, and

$$f^*\mathcal{S} \overset{f^*\phi}{\downarrow} \mathcal{B}$$

is an étale space.

(2): Let  $h : \mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{C} \rightarrow \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{C}$  be a morphism of étalé spaces. Consider the induced pullbacks:

$$\begin{array}{ccc} f^*\mathcal{S}_1 & \xrightarrow{f_1} & \mathcal{S}_1 \\ f^*\phi_1 \downarrow & & \downarrow \phi_1 \\ \mathcal{B} & \xrightarrow{f} & \mathcal{C} \end{array} \quad \begin{array}{ccc} f^*\mathcal{S}_2 & \xrightarrow{f_2} & \mathcal{S}_2 \\ f^*\phi_2 \downarrow & & \downarrow \phi_2 \\ \mathcal{B} & \xrightarrow{f} & \mathcal{C} \end{array}$$

Define  $f^*h : f^*\mathcal{S}_1 \rightarrow f^*\mathcal{S}_2$  by

$$f^*h(b, s) := (b, h(s)).$$

This map is well-defined because  $f(b) = \phi_1(s) \Rightarrow f(b) = \phi_2(h(s))$ , due to the commutativity  $\phi_2 \circ h = \phi_1$ . It is continuous as a restriction of the map  $\mathcal{B} \times \mathcal{S}_1 \rightarrow \mathcal{B} \times \mathcal{S}_2$  given by  $(b, s) \mapsto (b, h(s))$ . Since  $h$  is a morphism of étalé spaces,  $f^*h$  respects the pullback projections. Therefore,  $f^*h$  is a morphism of étalé spaces over  $\mathcal{B}$ .  $\square$

**Definition 3.24.** Let  $\mathcal{S} \xrightarrow{\phi} \mathcal{C}$  be an étalé, and let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map. The étalé space  $f^*\mathcal{S} \xrightarrow{f^*\phi} \mathcal{B}$  is called the *inverse image* (or *pullback*) of  $\mathcal{S} \xrightarrow{\phi} \mathcal{C}$  along  $f$ .

**Theorem 3.25.** Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map. Then pulling back along  $f$  defines a functor

$$f^* : \mathbf{Etale}(\mathcal{C}) \rightarrow \mathbf{Etale}(\mathcal{B}),$$

which maps each étalé space over  $\mathcal{C}$  to its inverse image over  $\mathcal{B}$ , and each morphism of étalé spaces over  $\mathcal{C}$  to its corresponding pullback morphism over  $\mathcal{B}$ .

*Proof.* This result follows directly from Proposition 3.23, which ensures that the pullback of an étalé and of an étalé morphism remains an étalé and a morphism, respectively. See also (author?) [19, p. 35].  $\square$

**Proposition 3.26.** Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map, and let  $\mathcal{S} \xrightarrow{\phi} \mathcal{C}$  be an étalé space of residuated lattices. Then the pair  $(f^*\mathcal{S}, f^*\phi)$  forms an étalé space of residuated lattices over  $\mathcal{B}$ .

*Proof.* By Proposition 3.23(1),  $(f^*\mathcal{S}, f^*\phi)$  is an étalé space over  $\mathcal{B}$ .

For each  $b \in \mathcal{B}$ , the stalk  $(f^*\mathcal{S})_b = \{b\} \times \mathcal{S}_{f(b)}$  inherits the structure of a residuated lattice from  $\mathcal{S}_{f(b)}$  in a natural way.

Let  $\diamond$  be a binary fundamental operation (e.g.,  $\vee, \wedge, \odot, \rightarrow$ ). Define the domain of the lifted operation as:

$$f^*\mathcal{S}^{(2)} := \{((b, s_1), (b, s_2)) \mid b \in \mathcal{B}, s_1, s_2 \in \mathcal{S}_{f(b)}\}.$$

We define the lifted operation pointwise by:

$$f^*\mathcal{S}_{\diamond}((b, s_1), (b, s_2)) := (b, \mathcal{S}_{\diamond}(s_1, s_2)).$$

This can be expressed as the composition:

$$f^*\mathcal{S}_{\diamond} = (f^*\phi \circ \pi_1, \mathcal{S}_{\diamond} \circ (\pi_2 \times \pi_2)),$$

where  $\pi_1, \pi_2$  denote the natural projections from  $f^*\mathcal{S}^{(2)}$ . Since all components are continuous, the map  $f^*\mathcal{S}_{\diamond}$  is continuous.

For nullary operations  $\diamond \in \{0, 1\}$ , we define:

$$f^*\mathcal{S}_{\diamond}(b) := (b, \mathcal{S}_{\diamond}(f(b))).$$

This map is also continuous, being a composition of continuous maps.

Hence,  $(f^*\mathcal{S}, f^*\phi)$  is an étalé space of residuated lattices.  $\square$

**Proposition 3.27.** *Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map, and let  $h : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  be a morphism of étalé spaces of residuated lattices over  $\mathcal{C}$ . Then the map*

$$f^*h : f^*\mathcal{S}_1 \rightarrow f^*\mathcal{S}_2, \quad (b, s) \mapsto (b, h(s)),$$

*is a morphism of étalé spaces of residuated lattices over  $\mathcal{B}$ .*

*Proof.* By Proposition 3.23(2), the map  $f^*h$  is a morphism of étalé spaces over  $\mathcal{B}$ . It remains to show that  $f^*h$  preserves the residuated lattice structure on each stalk.

For any  $b \in \mathcal{B}$ , the induced map on stalks is given by

$$f^*h_b : (b, s) \mapsto (b, h_{f(b)}(s)),$$

where  $h_{f(b)} : \mathcal{S}_{1,f(b)} \rightarrow \mathcal{S}_{2,f(b)}$  is a morphism of residuated lattices. Since the stalk  $(f^*\mathcal{S}_i)_b = \{b\} \times \mathcal{S}_{i,f(b)}$  for  $i = 1, 2$ , it follows that  $f^*h_b$  is also a morphism of residuated lattices.

Therefore,  $f^*h$  is a morphism of étalé spaces of residuated lattices over  $\mathcal{B}$ .  $\square$

**Theorem 3.28.** *Let  $f : \mathcal{B} \rightarrow \mathcal{C}$  be a continuous map. Then the assignment  $f^*$  defines a functor*

$$f^* : \mathbf{RL-Etale}(\mathcal{C}) \longrightarrow \mathbf{RL-Etale}(\mathcal{B}),$$

which maps each étalé space of residuated lattices over  $\mathcal{C}$  to its pullback over  $\mathcal{B}$ , and each morphism of such spaces to its corresponding pullback morphism.

*Proof.* This follows directly from Propositions 3.26 and 3.27, which establish that:

- the pullback of an étalé space of residuated lattices is again an étalé space of residuated lattices; and
- the pullback of a morphism between such spaces is again a morphism in the category.

These together ensure that  $f^*$  is indeed a well-defined functor.  $\square$

## CONCLUSION

This paper establishes the foundational theory of étalé spaces for residuated lattices, merging algebraic and topological perspectives. The authors introduce a structural framework where stalks over a topological space carry residuated lattice structures, with sections forming pointwise operations. Key contributions include explicit examples (e.g., spaces over prime/maximal filters) and a functorial pullback construction enabling the transfer of étalé spaces along continuous maps. The category **RL-Etale**( $\mathcal{B}$ ) is shown to be well-defined, with operations preserved under pullbacks. These results extend étalé space theory to non-classical algebraic structures, offering new tools for studying residuated lattices in topological contexts.

Future work should investigate the dual notion of *pushouts*, complementing the pullback construction. Pushouts would allow “pushing forward” étalé spaces along base space morphisms, facilitating the gluing of local data or extension of structures. This could deepen categorical insights into colimits of étalé spaces and strengthen connections to sheaf theory. Additionally, applications in fuzzy logic and spatial reasoning may benefit from these dual constructions, further bridging algebraic and geometric methodologies in residuated lattice theory.

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ETAL'E SPACES OF RESIDUATED LATTICES

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فضاهای ساقه‌ای شبکه‌های مانده‌دار

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این مقاله به بررسی فضاهای ساقه‌ای مرتبط با شبکه‌های مانده‌دار می‌پردازد. ما مفهوم فضاهای ساقه‌ای شبکه‌های مانده‌دار را روی یک فضای توپولوژیکی معرفی کرده و چارچوب ساختاری آن‌ها را ایجاد می‌کنیم. علاوه بر این، برای انتقال یک فضای ساقه‌ای شبکه‌های مانده‌دار از یک فضای توپولوژیکی به فضای توپولوژیکی دیگر از طریق یک نگاشت پیوسته، روشی را ارائه می‌دهیم که نتیجه‌ی آن یک ارتباط عملکردی بین رسته‌های مرتبط با آن‌ها می‌باشد.

کلمات کلیدی: بافه شبکه‌های مانده‌دار، کلاف شبکه‌های مانده‌دار، فضای ساقه‌ای شبکه‌های مانده‌دار.