

L-ALGEBRAS WITH HYBRID STRUCTURES

A. Paad

ABSTRACT. In this paper, the notion of hybrid structure on L-algebras is introduced, along with some related properties. Moreover, the concept of hybrid ideals in L-algebras is presented, and several related results are discussed. In the following, the quotient L-algebras that respect a hybrid ideal of an L-algebra are studied. Finally, it is proved that there is a one-to-one correspondence between the set of all congruence relations and the set of all hybrid ideals, given a few additional conditions.

1. INTRODUCTION

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [7]. Many examples show that L-algebras are very useful. Since L-algebras have applications in various fields, such as number theory [9], group theory [8, 11], and lattice theory [10], their study remains a topic of current interest. Yang and Rump [12] characterized pseudo-MV-algebras and Bosbachs non-commutative bricks as L-algebras. Wu and Yang [14] proved that orthomodular lattices constitute a special class of L-algebras in distinct ways. Additionally, other mathematicians have examined the relationship between basic algebras and L-algebras. Wu et al. [13] demonstrated that every lattice-ordered effect algebra possesses an underlying L-algebra structure. Moreover, non-classical logic has emerged as a formal and effective tool in computer science for handling uncertain information. From a logical point of view, various ideals and congruence relations correspond to different sets of provable formulae. That is, ideal theory and congruence relations occupy significant roles in the study of non-classical logical algebras. Following the introduction of fuzzy sets by Zadeh [15], the theory has been applied in various domains of computer and management sciences, including artificial intelligence, expert systems, operations research, pattern recognition, robotics, control engineering, and more. Molodtsov [5] introduced the concept of soft sets to address uncertainties while avoiding difficulties associated with

MSC(2020): 03G25, 03B50, 08A72, 03E72, 11Y50.

Keywords: L-Algebra; Hybrid subalgebra; Hybrid ideal; Relative congruence.

Received: 23 June 2024, Accepted: 26 May 2025.

conventional theoretical approaches. The notion of hybrid structure was introduced by Jun, Song, Muhiuddin, and Paad [2, 6] as a parallel framework combining fuzzy and soft sets. This concept was initially applied to a set of parameters in a universal set and was later extended to BCK/BCI algebras and linear spaces. Inspired by this development, we explore hybrid structures within L-algebras, introducing the concepts of hybrid subalgebras of L-algebras and examining some of their properties.

This paper is organized as follows: In Section 2, the basic definitions, properties, and theorems of L-algebras and hybrid structures are reviewed. In Section 3, the notion of hybrid structure on L-algebras is defined, and some of its properties are provided. In Section 4, the concept of hybrid (pre)ideals in L-algebras is introduced, along with some related results. Subsequently, quotient L-algebras respecting a hybrid ideal of a KL-algebra are presented, and it is shown that this quotient structure forms an L-algebra for any hybrid ideal. Finally, it is proved that there is a one-to-one correspondence between the set of all congruence relations and the set of all hybrid ideals, given a few additional conditions.

2. PRELIMINARIES

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1. [1] An L-algebra is an algebraic structure $(X; \rightarrow, 1)$ of type $(2, 0)$ satisfying

(L1) $x \rightarrow x = x \rightarrow 1 = 1$, and $1 \rightarrow x = x$,

(L2) $(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$,

(L3) if $x \rightarrow y = y \rightarrow x = 1$, then $x = y$, for any $x, y, z \in X$.

Condition (L1) states that 1 is a logical unit, while (L2) is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. By (L3), the relation $x \leq y$ if and only if $x \rightarrow y = 1$, defines a partial order for any L-algebra X . If X admits a smallest element 0, then X is called a bounded L-algebra. Let $(A, \rightarrow, 1)$ and $(B, \rightarrow, 1)$ be two L-algebras. A map $f : A \rightarrow B$ is called an L-homomorphism if $f(x \rightarrow y) = f(x) \rightarrow f(y)$, for all $x, y \in A$.

Definition 2.2. [7] An L-algebra $(X; \rightarrow, 1)$ which satisfies the following condition

$$x \rightarrow (y \rightarrow x) = 1, \quad (K)$$

for any $x, y \in L$, is called a KL-algebra. A CL-algebra is an L-algebra $(X; \rightarrow, 1)$ such that

$$(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1, \quad (C)$$

Moreover, if X satisfies condition (C), then

$$x \rightarrow ((x \rightarrow y) \rightarrow y) = 1, \quad (D).$$

Definition 2.3. [3] Assume $(X, \rightarrow, 1)$ is an L-algebra. The nonempty set $H \subseteq X$ is called a subalgebra of X if, for any $x, y \in H$, $x \rightarrow y \in H$.

Definition 2.4. [7] A subset I of an L-algebra $(X; \rightarrow, 1)$ is called an ideal of X if it satisfies the following conditions for all $x, y \in X$,

(I1) $1 \in I$,

(I2) if $x \in I$ and $x \rightarrow y \in I$, then $y \in I$,

(I3) if $x \in I$, then $(x \rightarrow y) \rightarrow y \in I$,

(I4) if $x \in I$, then $y \rightarrow x \in I$ and $y \rightarrow (x \rightarrow y) \in I$.

Clearly, $\{1\}$ and L are two trivial ideals of L-algebra. An ideal I of X is called a proper ideal if $I \neq X$. The set of all ideals of L is denoted by $\mathbf{I}(X)$.

In the following, we shall use conventions I for the unit interval, X for the set of parameters and $\mathbf{P}(U)$ for the power set of an initial universe set U .

Definition 2.5. [2] A hybrid structure in X over U is defined to be a mapping

$$\tilde{\varphi}_\lambda := (\tilde{\varphi}, \lambda) : X \longrightarrow \mathbf{P}(U) \times I, x \longrightarrow (\tilde{\varphi}(x), \lambda(x))$$

where $\tilde{\varphi} : X \longrightarrow \mathbf{P}(U)$ and $\lambda : X \longrightarrow I$ are mappings.

Moreover, the sets $\tilde{\varphi}_\lambda[\alpha, t] := \{x \in X \mid \tilde{\varphi}(x) \supseteq \alpha, \lambda(x) \leq t\}$ is called $[\alpha, t]$ -hybrid cut of $\tilde{\varphi}_\lambda$.

Let $\beta : X \rightarrow Y$ be a mapping from a set X to a set Y . For a hybrid structure $\tilde{\varphi}_\nu$ in Y over U , consider a hybrid structure $\beta^{-1}(\tilde{\varphi}_\nu) := (\beta^{-1}(\tilde{\varphi}), \beta^{-1}(\nu))$ in X over U where $\beta^{-1}(\tilde{\varphi})(x) = \tilde{\varphi}(\beta(x))$ and $\beta^{-1}(\nu)(x) = \nu(\beta(x))$, for all $x \in X$. We say that $\beta^{-1}(\tilde{\varphi}_\nu)$ is the hybrid preimage of $\tilde{\varphi}_\nu$ under β . For a hybrid structure $\tilde{\psi}_\lambda$ in X over U , the hybrid image of $\tilde{\psi}_\lambda$ under β is defined to be a hybrid structure $\beta(\tilde{\psi}_\lambda) := (\beta(\tilde{\psi}), \beta(\lambda))$ in Y over U where

$$\beta(\tilde{\psi})(y) = \begin{cases} \bigcup_{x \in \beta^{-1}(y)} \psi(x) & \text{if } \beta^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\beta(\lambda)(y) = \begin{cases} \bigwedge_{x \in \beta^{-1}(y)} \lambda(x) & \text{if } \beta^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Note. From now on, in this paper, we assume $(X; \rightarrow, 1)$ is an L-algebra unless stated otherwise.

3. HYBRID STRUCTURES ON L-ALGBRAS

In this section, hybrid structures on L-algebras are introduced, and some related results are provided.

Definition 3.1. Let X be an L-algebra. Then a hybrid structure $\tilde{\varphi}_\lambda$ in X over U is called a *hybrid subalgebra* of X over U if the following condition holds for all $x, y \in X$:

$$(HL) \quad \tilde{\varphi}(x \rightarrow y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y), \quad \lambda(x \rightarrow y) \leq \lambda(x) \vee \lambda(y).$$

Example 3.2. [4] Let $X = \{0, a, b, 1\}$ be a chain where $0 < a < b < 1$. Define the operation $\rightarrow$ on X as follows:

Table. 1: Implication

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then $(X, \rightarrow, 0, 1)$ is a bounded L-algebra. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in X over U which is given by Table. 2:

Table. 2: Hybrid structure

L	$\tilde{\varphi}$	λ
0	$\{u_1, u_3, u_4\}$	0.5
a	$\{u_3, u_4\}$	0.4
b	$\{u_2, u_3, u_4\}$	0.3
1	U	0.2

Then, through direct calculations, we verify that $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U .

Lemma 3.1. Let $\tilde{\varphi}_\lambda$ be a hybrid subalgebra of X over U . Then for all $x \in X$, $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$ and $\lambda(1) \leq \lambda(x)$.

Proof. Since by (L1), $x \rightarrow x = 1$, we get that

$$\tilde{\varphi}(1) = \tilde{\varphi}(x \rightarrow x) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x) = \tilde{\varphi}(x)$$

and $\lambda(1) = \lambda(x \rightarrow x) \leq \lambda(x) \vee \lambda(x) = \lambda(x)$. □

Theorem 3.2. *Let X be an L -algebra and $\tilde{\varphi}_\lambda$ be a hybrid structure in X over U . Then the following statements are equivalent:*

- (i) $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U .
- (ii) For any $\alpha \in \mathbf{P}(\mathbf{U})$ and $t \in I$ the nonempty sets

$$\tilde{\varphi}_\lambda(\alpha) := \{x \in X \mid \alpha \subseteq \tilde{\varphi}(x)\}$$

and $\tilde{\varphi}_\lambda(t) := \{x \in X \mid \lambda(x) \leq t\}$ are subalgebras of X .

Proof. (i) $\Rightarrow$ (ii): Assume that $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U and $\alpha \in \mathbf{P}(\mathbf{U})$, $t \in I$ such that $\tilde{\varphi}_\lambda(\alpha) \neq \emptyset$ and $\tilde{\varphi}_\lambda(t) \neq \emptyset$. If $x, y \in \tilde{\varphi}_\lambda(\alpha) \cap \tilde{\varphi}_\lambda(t)$, then $\alpha \subseteq \tilde{\varphi}(x), \tilde{\varphi}(y)$ and $\lambda(x), \lambda(y) \leq t$. Now, by Definition 3.1,

$$\tilde{\varphi}(x \rightarrow y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$$

and $\lambda(x \rightarrow y) \leq \lambda(x) \vee \lambda(y)$ and since $\tilde{\varphi}(x) \cap \tilde{\varphi}(y) \supseteq \alpha$ and $\lambda(x) \vee \lambda(y) \leq t$, we obtain $\tilde{\varphi}(x \rightarrow y) \supseteq \alpha$ and $\lambda(x \rightarrow y) \leq t$. Thus, $x \rightarrow y \in \tilde{\varphi}_\lambda(\alpha) \cap \tilde{\varphi}_\lambda(t)$.

(ii) $\Rightarrow$ (i): Let $\tilde{\varphi}_\lambda(\alpha), \tilde{\varphi}_\lambda(t)$ be subalgebras of X and $x, y \in L$ be such that $\tilde{\varphi}(x) = \alpha_x$ and $\tilde{\varphi}(y) = \alpha_y$. Then by taking $\alpha = \alpha_x \cap \alpha_y$, we have $x, y \in \tilde{\varphi}_\lambda(\alpha)$. Indeed, $\alpha = \alpha_x \cap \alpha_y = \tilde{\varphi}(x) \cap \tilde{\varphi}(y) \subseteq \tilde{\varphi}(x), \tilde{\varphi}(y)$ and so $\alpha \subseteq \tilde{\varphi}(x), \tilde{\varphi}(y)$. Hence, $x, y \in \tilde{\varphi}_\lambda(\alpha)$ and so $x \rightarrow y \in \tilde{\varphi}_\lambda(\alpha)$. Thus, $\alpha \subseteq \tilde{\varphi}(x \rightarrow y)$ and so $\tilde{\varphi}(x \rightarrow y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$. Moreover, let $t = \lambda(x) \vee \lambda(y)$, for $x, y \in X$. Then $\lambda(x), \lambda(y) \leq t$ and so $x, y \in \tilde{\varphi}_\lambda(t)$. Hence, $x \rightarrow y \in \tilde{\varphi}_\lambda(t)$ and so $\lambda(x \rightarrow y) \leq t = \lambda(x) \vee \lambda(y)$. Therefore, $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U . $\square$

Proposition 3.3. *Let $\xi : X_1 \rightarrow X_2$ be an L -homomorphism and $\tilde{\varphi}_\nu$ a hybrid subalgebra of X_2 over U . Then $\xi^{-1}(\tilde{\varphi}_\nu)$ is a hybrid subalgebra of X_1 over U .*

Proof. Let $x, y \in L$ and $\tilde{\varphi}_\nu$ a hybrid subalgebra of X_2 . Then

$$\begin{aligned} \xi^{-1}(\tilde{\varphi})(x \rightarrow y) &= \tilde{\varphi}(\xi(x \rightarrow y)) \\ &= \tilde{\varphi}(\xi(x) \rightarrow \xi(y)) \\ &\supseteq \tilde{\varphi}(\xi(x)) \cap \tilde{\varphi}(\xi(y)) \\ &= \xi^{-1}(\tilde{\varphi})(x) \cap \xi^{-1}(\tilde{\varphi})(y). \end{aligned}$$

And

$$\begin{aligned} \xi^{-1}(\nu)(x \rightarrow y) &= \nu(\xi(x \rightarrow y)) \\ &= \nu(\xi(x) \rightarrow \xi(y)) \\ &\leq \nu(\xi(x)) \vee \nu(\xi(y)) \\ &= \xi^{-1}(\nu)(x) \vee \xi^{-1}(\nu)(y). \end{aligned}$$

Therefore, $\xi^{-1}(\tilde{\varphi}_\nu)$ is a hybrid subalgebra of X_1 over U . $\square$

Proposition 3.4. *Let $\xi : X_1 \rightarrow X_2$ be an L-epimorphism. For every hybrid structure $\tilde{\psi}_\lambda$ in X_2 over U , if $\xi^{-1}(\tilde{\psi}_\lambda)$ is a hybrid subalgebra of X_1 over U , then $\tilde{\psi}_\lambda$ is a hybrid subalgebra of X_2 over U .*

Proof. Let $\xi : X_1 \rightarrow X_2$ be an L-epimorphism and $\xi^{-1}(\tilde{\psi}_\lambda)$ be hybrid subalgebra of X_1 over U , where a $\tilde{\psi}_\lambda$ hybrid structure in X_2 over U and $a, b \in X_2$. Then there exist $x, y \in X_1$ such that $\xi(x) = a$ and $\xi(y) = b$. Now,

$$\begin{aligned}\tilde{\psi}(a \rightarrow b) &= \tilde{\psi}(\xi(x) \rightarrow \xi(y)) \\ &= \tilde{\psi}(\xi(x \rightarrow y)) \\ &= \xi^{-1}(\tilde{\psi})(x \rightarrow y) \\ &\supseteq \xi^{-1}(\tilde{\psi})(x) \cap \xi^{-1}(\tilde{\psi})(y) \\ &= \tilde{\psi}(\xi(x)) \cap \tilde{\psi}(\xi(y)) \\ &= \tilde{\psi}(a) \cap \tilde{\psi}(b).\end{aligned}$$

And

$$\begin{aligned}\lambda(a \rightarrow b) &= \lambda(\xi(x) \rightarrow \xi(y)) \\ &= \lambda(\xi(x \rightarrow y)) \\ &= \xi^{-1}(\lambda)(x \rightarrow y) \\ &\leq \xi^{-1}(\lambda)(x) \vee \xi^{-1}(\lambda)(y) \\ &= \lambda(\xi(x)) \vee \lambda(\xi(y)) \\ &= \lambda(a) \vee \lambda(b).\end{aligned}$$

Therefore, $\tilde{\psi}_\lambda$ is a hybrid subalgebra of X_2 over U . □

4. HYBRID IDEALS IN L-ALGEBRAS

In this section, we introduce the concept of hybrid ideals in L-algebras and present some related results.

Definition 4.1. Let X be an L-algebra. Then a hybrid structure $\tilde{\varphi}_\lambda$ in X over U is called a *hybrid ideal* of X over U if it satisfies the followings for any $x, y \in X$:

- (HI0) $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$, $\lambda(1) \leq \lambda(x)$,
- (HI1) $\tilde{\varphi}(y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \rightarrow y)$, $\lambda(y) \leq \lambda(x) \vee \lambda(x \rightarrow y)$.
- (HI2) $\tilde{\varphi}((x \rightarrow y) \rightarrow y) \supseteq \tilde{\varphi}(x)$, $\lambda((x \rightarrow y) \rightarrow y) \leq \lambda(x)$.
- (HI3) $\tilde{\varphi}(y \rightarrow x) \cap \tilde{\varphi}(y \rightarrow (x \rightarrow y)) \supseteq \tilde{\varphi}(x)$,
 $\lambda(y \rightarrow x) \vee \lambda(y \rightarrow (x \rightarrow y)) \leq \lambda(x)$.

Denote **HID**(X) the set of all hybrid ideals of X over U . Note that, by (HI3), every hybrid ideal of X over U is also a hybrid subalgebra of X over U .

Example 4.2. Let X be the L-algebra as in Example 4.3 and

$$U = \{u_1, u_2, u_3, u_4, u_5\}$$

be an initial universe set and $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in X over U which is given by Table. 3:

Table. 3: Hybrid structure

L	$\tilde{\varphi}$	λ
0	$\{u_3, u_4\}$	0.5
a	$\{u_3, u_4\}$	0.5
b	$\{u_1, u_2, u_3, u_4\}$	0.4
1	U	0.3

Then by direct calculations, we verify that $\tilde{\varphi}_\lambda$ is a hybrid ideal of X over U .

Example 4.3. [4] Let $X = \{a, b, c, 1\}$ be a poset with $a, c \leq b \leq 1$. Define the operation $\rightarrow$ on X as follows:

Table. 4: Implication

$\rightarrow$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Then $(X, \rightarrow, 0, 1)$ is a bounded L-algebra. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in X over U which is given by Table. 2:

Table. 5: Hybrid structure

L	$\tilde{\varphi}$	λ
a	$\{u_1, u_3, u_5\}$	0.8
b	$\{u_2, u_3\}$	0.6
c	$\{u_3, u_4, u_5\}$	0.4
1	U	0.2

Then by direct calculations, we verify that $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U and is not a hybrid ideal of X over U , because

$$\{u_2, u_3\} = \tilde{\varphi}(b) \not\supseteq \tilde{\varphi}(c) \cap \tilde{\varphi}(c \rightarrow b) = \{u_3, u_4, u_5\}$$

and $0.6 = \lambda(b) \not\leq \lambda(c) \vee \lambda(c \rightarrow b) = 0.4$.

Proposition 4.1. *Every hybrid ideal of X is a hybrid subalgebra of X .*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of X over U and $x, y \in X$. Then by (HI3), $\tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(x \rightarrow (y \rightarrow x)) \supseteq \tilde{\varphi}(y)$ and since

$$\tilde{\varphi}(x \rightarrow y) \supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(x \rightarrow (y \rightarrow x))$$

and $\tilde{\varphi}(y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$, we conclude that $\tilde{\varphi}(x \rightarrow y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$. Moreover, since (HI3) states that (HI3), $\lambda(x \rightarrow y) \vee \lambda(x \rightarrow (y \rightarrow x)) \leq \lambda(y)$ and since $\lambda(x \rightarrow y) \leq \lambda(x \rightarrow y) \vee \lambda(x \rightarrow (y \rightarrow x))$ and $\lambda(y) \leq \lambda(x) \vee \lambda(y)$, we conclude that $\lambda(x \rightarrow y) \leq \lambda(x) \vee \lambda(y)$. Therefore, $\tilde{\varphi}_\lambda$ is a hybrid subalgebra of X over U . $\square$

Proposition 4.2. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of X over U . Then for any $x, y, z \in X$:*

- (i) *if $x \leq y$, then $\tilde{\varphi}(x) \subseteq \tilde{\varphi}(y)$ and $\lambda(y) \leq \lambda(x)$.*
- (ii) *If $x \leq y \rightarrow z$, then $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$ and $\lambda(z) \leq \lambda(x) \vee \lambda(y)$.*
- (iii) *If $x \rightarrow (y \rightarrow z) = 1$, then $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$ and $\lambda(z) \leq \lambda(x) \vee \lambda(y)$.*
- (iv) *$\tilde{\varphi}(x \rightarrow z) \supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(y \rightarrow z)$ and $\lambda(x \rightarrow z) \leq \lambda(x \rightarrow y) \vee \lambda(y \rightarrow z)$, where X is a KL-algebra.*
- (v) *$\tilde{\varphi}(y \rightarrow z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}((x \rightarrow y) \rightarrow z)$ and*

$$\lambda(y \rightarrow z) \leq \lambda(x) \vee \lambda((x \rightarrow y) \rightarrow z),$$

where X is a KL-algebra.

Proof. (i) Since $x \leq y$, we have $x \rightarrow y = 1$. By (HI0) and (HI1), we obtain:

$$\tilde{\varphi}(y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(x) \cap \tilde{\varphi}(1) = \tilde{\varphi}(x)$$

Additionally,

$$\lambda(y) \leq \lambda(x) \vee \lambda(x \rightarrow y) = \lambda(x) \vee \lambda(1) = \lambda(x).$$

(ii) Suppose $x \leq y \rightarrow z$, for $x, y, z \in X$. By (i), $\tilde{\varphi}(y \rightarrow z) \supseteq \tilde{\varphi}(x)$ and $\lambda(y \rightarrow z) \leq \lambda(x)$. Moreover, by using (HI1), $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(y \rightarrow z) \cap \tilde{\varphi}(y)$ and $\lambda(z) \leq \lambda(y \rightarrow z) \vee \lambda(y)$,

Substituting from above, we conclude $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$ and

$$\lambda(z) \leq \lambda(x) \vee \lambda(y).$$

(iii) It is clear by (ii).

(iv) Since X is a KL-algebra, we have $y \rightarrow z \leq (y \rightarrow x) \rightarrow (y \rightarrow z)$ and so by (i), we obtain:

$$\tilde{\varphi}((y \rightarrow x) \rightarrow (y \rightarrow z)) \supseteq \tilde{\varphi}(y \rightarrow z)$$

$$\lambda((y \rightarrow x) \rightarrow (y \rightarrow z)) \leq \lambda(y \rightarrow z)$$

Moreover, by (L2),

$$\tilde{\varphi}((x \rightarrow y) \rightarrow (x \rightarrow z)) = \tilde{\varphi}((y \rightarrow x) \rightarrow (y \rightarrow z))$$

$$\lambda((x \rightarrow y) \rightarrow (x \rightarrow z)) = \lambda((y \rightarrow x) \rightarrow (y \rightarrow z))$$

Thus, we conclude:

$$\tilde{\varphi}((x \rightarrow y) \rightarrow (x \rightarrow z)) \supseteq \tilde{\varphi}(y \rightarrow z)$$

$$\lambda((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \lambda(y \rightarrow z)$$

Furthermore,

$$\begin{aligned} \tilde{\varphi}(x \rightarrow z) &\supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}((x \rightarrow y) \rightarrow (x \rightarrow z)) \\ &\supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(y \rightarrow z) \end{aligned}$$

Finally,

$$\begin{aligned} \lambda(x \rightarrow z) &\leq \lambda(x \rightarrow y) \vee \lambda((x \rightarrow y) \rightarrow (x \rightarrow z)) \\ &\leq \lambda(x \rightarrow y) \vee \lambda(y \rightarrow z) \end{aligned}$$

(v) By (L2), we have

$$(y \rightarrow (x \rightarrow y)) \rightarrow (y \rightarrow z) = ((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z),$$

we get that

$$\tilde{\varphi}((y \rightarrow (x \rightarrow y)) \rightarrow (y \rightarrow z)) = \tilde{\varphi}(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z))$$

$$\lambda((y \rightarrow (x \rightarrow y)) \rightarrow (y \rightarrow z)) = \lambda(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z))$$

Moreover, Since X is a KL-algebra, we have:

$$\tilde{\varphi}(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z)) \supseteq \tilde{\varphi}((x \rightarrow y) \rightarrow z)$$

$$\lambda(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z)) \leq \lambda((x \rightarrow y) \rightarrow z)$$

Additionally,

$$\tilde{\varphi}(y \rightarrow (x \rightarrow y)) \supseteq \tilde{\varphi}(x)$$

$$\lambda(y \rightarrow (x \rightarrow y)) \leq \lambda(x)$$

Now, by (HI2), we conclude that

$$\begin{aligned} \tilde{\varphi}(y \rightarrow z) &\supseteq \tilde{\varphi}(y \rightarrow (x \rightarrow y)) \cap \tilde{\varphi}((y \rightarrow (x \rightarrow y)) \rightarrow (y \rightarrow z)) \\ &= \tilde{\varphi}(y \rightarrow (x \rightarrow y)) \cap \tilde{\varphi}(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z)) \\ &\supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}((x \rightarrow y) \rightarrow z) \end{aligned}$$

Thus, combining these results, we obtain:

$$\begin{aligned}\lambda(y \rightarrow z) &\leq \lambda(y \rightarrow (x \rightarrow y)) \vee \lambda((y \rightarrow (x \rightarrow y)) \rightarrow (y \rightarrow z)) \\ &= \lambda(y \rightarrow (x \rightarrow y)) \vee \lambda(((x \rightarrow y) \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow z)) \\ &\leq \lambda(x) \vee \lambda((x \rightarrow y) \rightarrow z)\end{aligned}$$

□

Remark 4.4. Let X be an L-algebra and $\tilde{\varphi}_\lambda$ be a hybrid ideal of X over U . Then

(i) if X satisfies condition (K), then for any $x, y \in X$, $x \rightarrow (y \rightarrow x) = 1$ and so by (HI1), we obtain $\tilde{\varphi}(x) \subseteq \tilde{\varphi}(1) = \tilde{\varphi}(x \rightarrow (y \rightarrow x))$ and

$$\lambda(x) \geq \lambda(1) = \lambda(x \rightarrow (y \rightarrow x)).$$

Moreover, since $x \leq (y \rightarrow x)$, by condition(K), applying Proposition 4.2(i) yields:

$\tilde{\varphi}(x) \subseteq \tilde{\varphi}(y \rightarrow x)$ and $\lambda(x) \geq \lambda(y \rightarrow x)$. Thus,

$$\tilde{\varphi}(y \rightarrow x) \cap \tilde{\varphi}(y \rightarrow (x \rightarrow y)) \supseteq \tilde{\varphi}(x), \lambda(y \rightarrow x) \vee \lambda(y \rightarrow (x \rightarrow y)) \leq \lambda(x)$$

and so (HI3) holds. Hence, (HI3) holds, and (HI3) can be omitted.

(ii) If X satisfies condition (D), then $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$ by (HI0) and (HI1), we get that

$$\tilde{\varphi}((x \rightarrow y) \rightarrow y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \rightarrow ((x \rightarrow y) \rightarrow y)) = \tilde{\varphi}(x) \cap \tilde{\varphi}(1) = \tilde{\varphi}(x)$$

Additionally,

$$\lambda((x \rightarrow y) \rightarrow y) \leq \lambda(x) \vee \lambda(x \rightarrow ((x \rightarrow y) \rightarrow y)) = \lambda(x) \vee \lambda(1) = \lambda(x).$$

(iii) If X satisfies condition (C), then condition (D) and (K) hold. Consequently, by (i) and (ii), (HI2) and (HI3) can be omitted.

Proposition 4.3. Let $(X; \rightarrow, 1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid structure of X over U such that satisfies in (HI0). Then the following statements are equivalent:

- (i) $\tilde{\varphi}_\lambda$ is a hybrid ideal of X over U ,
- (ii) $\tilde{\varphi}(x \rightarrow z) \supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(y \rightarrow z)$ and $\lambda(x \rightarrow z) \leq \lambda(x \rightarrow y) \vee \lambda(y \rightarrow z)$, for any $x, y, z \in X$,
- (iii) If $x \leq y \rightarrow z$, then $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$ and $\lambda(z) \leq \lambda(x) \vee \lambda(y)$, for any $x, y, z \in X$.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow directly from Proposition 4.2(ii) and (iv).

(ii) $\Rightarrow$ (i): Since X is a CKL-algebra, by Remark 4.4, we have (HI2) and (HI3). Now, we prove (HI1). Let $x, y \in X$. Then by (L1), $x = 1 \rightarrow x$ and $y = 1 \rightarrow y$ and so by assumption

$$\tilde{\varphi}(y) = \tilde{\varphi}(1 \rightarrow y) \supseteq \tilde{\varphi}(1 \rightarrow x) \cap \tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(x) \cap \tilde{\varphi}(x \rightarrow y)$$

$$\lambda(y) = \lambda(1 \rightarrow y) \leq \lambda(1 \rightarrow x) \vee \lambda(x \rightarrow y) = \lambda(x) \vee \lambda(x \rightarrow y).$$

Hence, $\tilde{\varphi}_\lambda$ satisfies in (HI1) and so is a hybrid ideal of X over U .

(iii) $\Rightarrow$ (i): Since X is a CKL-algebra, we only need to prove (HI1). Since $x \rightarrow y \leq x \rightarrow y$, for any $x, y \in X$, by (iii) we get that $\tilde{\varphi}(y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \rightarrow y)$ and $\lambda(y) \leq \lambda(x) \vee \lambda(x \rightarrow y)$. Thus, $\tilde{\varphi}_\lambda$ is a hybrid ideal of X over U . $\square$

Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of X over U . Define the binary relation $\equiv_{\tilde{\varphi}_\lambda}$ on X as follows:

$$x \equiv_{\tilde{\varphi}_\lambda} y \text{ if and only if } \tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(y \rightarrow x) = \tilde{\varphi}(1), \\ \lambda(x \rightarrow y) = \lambda(y \rightarrow x) = \lambda(1)$$

Theorem 4.4. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of KL-algebra X over U . Then $\equiv_{\tilde{\varphi}_\lambda}$ is a congruence relation on L .*

Proof. Let $x, y, z \in X$. Then by (L1), $\tilde{\varphi}(x \rightarrow x) = \tilde{\varphi}(1)$ and $\lambda(x \rightarrow x) = \lambda(1)$ and so $x \equiv_{\tilde{\varphi}_\lambda} x$. Hence, $\equiv_{\tilde{\varphi}_\lambda}$ is a reflexive relation on X . By Definition $\equiv_{\tilde{\varphi}_\lambda}$, it is clear that $\equiv_{\tilde{\varphi}_\lambda}$ is a symmetric relation on L . Now, let $x \equiv_{\tilde{\varphi}_\lambda} y$ and $y \equiv_{\tilde{\varphi}_\lambda} z$. Then $\tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(y \rightarrow x) = \tilde{\varphi}(1)$, $\lambda(x \rightarrow y) = \lambda(y \rightarrow x) = \lambda(1)$ and $\tilde{\varphi}(y \rightarrow z) = \tilde{\varphi}(z \rightarrow y) = \tilde{\varphi}(1)$, $\lambda(y \rightarrow z) = \lambda(z \rightarrow y) = \lambda(1)$. Now, Proposition 4.2(iv),

$$\tilde{\varphi}(x \rightarrow z) \supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(y \rightarrow z) = \tilde{\varphi}(1) \cap \tilde{\varphi}(1) = \tilde{\varphi}(1)$$

$$\tilde{\varphi}(z \rightarrow x) \supseteq \tilde{\varphi}(z \rightarrow y) \cap \tilde{\varphi}(y \rightarrow x) = \tilde{\varphi}(1) \cap \tilde{\varphi}(1) = \tilde{\varphi}(1)$$

Moreover,

$$\lambda(x \rightarrow z) \leq \lambda(x \rightarrow y) \vee \lambda(y \rightarrow z) = \lambda(1) \vee \lambda(1) = \lambda(1)$$

$$\lambda(z \rightarrow x) \leq \lambda(z \rightarrow y) \vee \lambda(y \rightarrow x) = \lambda(1) \vee \lambda(1) = \lambda(1)$$

Hence, by (HI0),

$$\tilde{\varphi}(x \rightarrow z) = \tilde{\varphi}(z \rightarrow x) = \tilde{\varphi}(1), \lambda(x \rightarrow z) = \lambda(z \rightarrow x) = \lambda(1)$$

And so $x \equiv_{\tilde{\varphi}_\lambda} z$. Therefore, $\equiv_{\tilde{\varphi}_\lambda}$ is a transitive relation on X and so $\equiv_{\tilde{\varphi}_\lambda}$ is an equivalence relation on X .

Now, we prove that $\equiv_{\tilde{\varphi}_\lambda}$ is compatible with $\rightarrow$. Assume that $x \equiv_{\tilde{\varphi}_\lambda} y$, for $x, y, z \in X$, then $\tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(y \rightarrow x) = \tilde{\varphi}(1)$ and

$$\lambda(x \rightarrow y) = \lambda(y \rightarrow x) = \lambda(1)$$

and since by condition (K), $x \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y)$, by Proposition 4.2(i) and (L1), we conclude that

$$\tilde{\varphi}((z \rightarrow x) \rightarrow (z \rightarrow y)) = \tilde{\varphi}((x \rightarrow z) \rightarrow (x \rightarrow y)) \supseteq \tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(1)$$

And

$$\lambda((z \rightarrow x) \rightarrow (z \rightarrow y)) = ((x \rightarrow z) \rightarrow (x \rightarrow y)) \leq \lambda(x \rightarrow y) = \lambda(1).$$

Hence, by (HI0), $\tilde{\varphi}((z \rightarrow x) \rightarrow (z \rightarrow y)) = \tilde{\varphi}(1)$ and

$$\lambda((z \rightarrow x) \rightarrow (z \rightarrow y)) = \lambda(1).$$

Similarly, we obtain $\tilde{\varphi}((z \rightarrow y) \rightarrow (z \rightarrow x)) = \tilde{\varphi}(1)$ and

$$\lambda((z \rightarrow y) \rightarrow (z \rightarrow x)) = \lambda(1).$$

Thus, $z \rightarrow x \equiv_{\tilde{\varphi}_\lambda} z \rightarrow y$. Now, we prove that $x \rightarrow z \equiv_{\tilde{\varphi}_\lambda} y \rightarrow z$. Since $x \equiv_{\tilde{\varphi}_\lambda} y$, we have $\tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(y \rightarrow x) = \tilde{\varphi}(1)$ and $\lambda(x \rightarrow y) = \lambda(y \rightarrow x) = \lambda(1)$. By (L1) and (HI2),

$$\begin{aligned} \tilde{\varphi}(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) &= \tilde{\varphi}(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \\ &\supseteq \tilde{\varphi}(y \rightarrow x) \\ &= \tilde{\varphi}(1) \end{aligned}$$

And

$$\begin{aligned} \lambda(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) &= \lambda(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \\ &\leq \lambda(y \rightarrow x) \\ &= \lambda(1) \end{aligned}$$

Hence, by (HI0), we conclude that

$$\tilde{\varphi}(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) = \tilde{\varphi}(1)$$

and $\lambda(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) = \lambda(1)$. Now, since by (HI1), we have:

$$\begin{aligned} \tilde{\varphi}((x \rightarrow y) \rightarrow (y \rightarrow z)) &\supseteq \tilde{\varphi}(x \rightarrow y) \cap \tilde{\varphi}(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) \\ &= \tilde{\varphi}(1) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}(1) \end{aligned}$$

And

$$\begin{aligned} \lambda((x \rightarrow y) \rightarrow (y \rightarrow z)) &\supseteq \lambda(x \rightarrow y) \vee \lambda(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow z)) \\ &= \lambda(1) \vee \lambda(1) \\ &= \lambda(1) \end{aligned}$$

Hence, by (HI0), we conclude that $\tilde{\varphi}((x \rightarrow z) \rightarrow (y \rightarrow z)) = \tilde{\varphi}(1)$ and $\lambda((x \rightarrow z) \rightarrow (y \rightarrow z)) = \lambda(1)$. Moreover, since by (L1) and (HI2),

$$\begin{aligned}\tilde{\varphi}(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) &= \tilde{\varphi}(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow z)) \\ &\supseteq \tilde{\varphi}(x \rightarrow y) \\ &= \tilde{\varphi}(1)\end{aligned}$$

And

$$\begin{aligned}\lambda(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) &= \lambda(((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow z)) \\ &\leq \lambda(x \rightarrow y) \\ &= \lambda(1)\end{aligned}$$

Hence, by (HI0), we conclude that

$$\tilde{\varphi}(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) = \tilde{\varphi}(1)$$

and $\lambda(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) = \lambda(1)$. Now, since by (HI1), we have:

$$\begin{aligned}\tilde{\varphi}((y \rightarrow z) \rightarrow (x \rightarrow z)) &\supseteq \tilde{\varphi}(y \rightarrow x) \cap \tilde{\varphi}(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) \\ &= \tilde{\varphi}(1) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}(1)\end{aligned}$$

And

$$\begin{aligned}\lambda((y \rightarrow z) \rightarrow (x \rightarrow z)) &\supseteq \lambda(y \rightarrow x) \vee \lambda(((y \rightarrow x) \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)) \\ &= \lambda(1) \vee \lambda(1) \\ &= \lambda(1)\end{aligned}$$

Hence, by (HI0), we conclude that $\tilde{\varphi}((y \rightarrow z) \rightarrow (x \rightarrow z)) = \tilde{\varphi}(1)$ and $\lambda((y \rightarrow z) \rightarrow (x \rightarrow z)) = \lambda(1)$. Therefore, $x \rightarrow z \equiv_{\tilde{\varphi}_\lambda} y \rightarrow z$ and so $\equiv_{\tilde{\varphi}_\lambda}$ is a congruence relation on X . $\square$

Let $x \in X$ and $\tilde{\varphi}_\lambda$ be a hybrid ideal of X over U . Then we denote the equivalence x w.r.t $\equiv_{\tilde{\varphi}_\lambda}$ by $[x]_{\tilde{\varphi}_\lambda}$ ($[x]$ for short). Furthermore, we define quotient algebra $\frac{X}{\tilde{\varphi}_\lambda} := \{[x] \mid x \in L\}$ and the operation $\Rightarrow$, for $[x], [y] \in \frac{X}{\tilde{\varphi}_\lambda}$ as follow:

$$[x] \Rightarrow [y] := [x \rightarrow y]$$

The top element is $[1]$ and $[x] \leq [y]$ if and only if $[x] \Rightarrow [y] = [1]$ if and only if $[x \rightarrow y] = [1]$ if and only if $x \rightarrow y \equiv_{\tilde{\varphi}_\lambda} 1$ if and only if

$$\tilde{\varphi}(1 \rightarrow (x \rightarrow y)) = \tilde{\varphi}((x \rightarrow y) \rightarrow 1) = \tilde{\varphi}(1)$$

and

$$\lambda(1 \rightarrow (x \rightarrow y)) = \lambda((x \rightarrow y) \rightarrow 1) = \lambda(1)$$

if and only if

$$\tilde{\varphi}(x \rightarrow y) = \tilde{\varphi}(1) \text{ and } \lambda(x \rightarrow y) = \lambda(1).$$

Theorem 4.5. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of KL-algebra X over U . Then $(\frac{X}{\tilde{\varphi}_\lambda}, \Rightarrow, [1])$ is an L-algebra which is called quotient L-algebra respect to $\tilde{\varphi}_\lambda$.*

Proof. It followos from Theorem 4.4. □

Proposition 4.6. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of KL-algebra X over U . Then*

(i) $[1]$ ($[1]_{\tilde{\varphi}_\lambda}$) *is an ideal of* X .

(ii) *If θ is a congruence on E , then $[1]_\theta$ is an ideal of* X .

Proof. (i) It is clear that $1 \in [1]$. Let $x, x \rightarrow y \in [1]$, for $x, y \in X$. Then $x \equiv_{\tilde{\varphi}_\lambda} 1$ and $x \rightarrow y \equiv_{\tilde{\varphi}_\lambda} 1$. By Theorem 4.4, $x \rightarrow y \equiv_{\tilde{\varphi}_\lambda} 1 \rightarrow y$ and so by transitivity of $\equiv_{\tilde{\varphi}_\lambda}$, we conclude that $1 \rightarrow y \equiv_{\tilde{\varphi}_\lambda} 1$ and since by (L1), $1 \rightarrow y = y$, we get that $y \equiv_{\tilde{\varphi}_\lambda} 1$ and so $1 \in [1]$. Now, let $x \in [1]$. Then $x \equiv_{\tilde{\varphi}_\lambda} 1$ and so by Theorem 4.4, $x \rightarrow y \equiv_{\tilde{\varphi}_\lambda} 1 \rightarrow y = y$. Hence,

$$(x \rightarrow y) \rightarrow y \equiv_{\tilde{\varphi}_\lambda} y \rightarrow y = 1$$

and so $(x \rightarrow y) \rightarrow y \in [1]$. Moreover, if $x \in [1]$, then by Theorem 4.4,

$$y \rightarrow (x \rightarrow y) \equiv_{\tilde{\varphi}_\lambda} y \rightarrow y = 1$$

and $y \rightarrow x \equiv_{\tilde{\varphi}_\lambda} y \rightarrow 1 = 1$ and so $y \rightarrow (x \rightarrow y) \in [1]$ and $y \rightarrow x \in [1]$. Therefore, $[1]$ is an ideal of X .

(ii) The proof is similar to proof (i). □

For congruence relation θ on X , we define the hybrid structure $\tilde{\Psi}_{\lambda, \theta}$ in X over U as follow:

$$\tilde{\Psi}_{\lambda, \theta} := (\tilde{\Psi}_\theta, \lambda_\theta) : L \longrightarrow \mathbf{P}(\mathbf{U}) \times I$$

$$\tilde{\Psi}_\theta(x) := \begin{cases} U & \text{if } x \in [1]_\theta, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\lambda_\theta(x) := \begin{cases} 0 & \text{if } x \in [1]_\theta, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 4.7. *Let θ be a congruence on KL-algebra X . Then $\tilde{\Psi}_{\lambda, \theta}$ is a hybrid ideal of X over U .*

Proof. Let θ be a congruence on X . Then by Proposition 4.6(ii), $[1]_\theta$ is an ideal of X and so $1 \in [1]_\theta$. Hence, $\tilde{\Psi}_\theta(1) = U \supseteq \tilde{\Psi}_\theta(x)$ and $\lambda_\theta(1) = 0 \leq \lambda_\theta(x)$, for any $x \in X$. Hence $\tilde{\Psi}_{\lambda,\theta}$ satisfies in (HI0). Now, let $x, y \in X$. If $x, x \rightarrow y \in [1]_\theta$, then by Proposition 4.6(i), $y \in [1]_\theta$ and so

$$\tilde{\Psi}_\theta(y) = U \supseteq \tilde{\Psi}_\theta(x) \cap \tilde{\Psi}_\theta(x \rightarrow y)$$

and $\lambda_\theta(y) = 0 \leq \lambda_\theta(x) \vee \lambda_\theta(x \rightarrow y)$. If $x \notin [1]_\theta$ or $x \rightarrow y \notin [1]_\theta$, then $\tilde{\Psi}_\theta(x) = \emptyset$ and $\lambda_\theta(x) = 1$ or $\tilde{\Psi}_\theta(x \rightarrow y) = \emptyset$ and $\lambda_\theta(x \rightarrow y) = 1$. Hence, $\tilde{\Psi}_\theta(y) \supseteq \emptyset = \tilde{\Psi}_\theta(x) \cap \tilde{\Psi}_\theta(x \rightarrow y)$ and $\lambda_\theta(y) \leq 1 = \lambda_\theta(x) \vee \lambda_\theta(x \rightarrow y)$ and so $\tilde{\Psi}_{\lambda,\theta}$ satisfies in (HI1). Now, for (HI2), suppose that $x \in [1]_\theta$, then by Proposition 4.6(i), $(x \rightarrow y) \rightarrow y \in [1]_\theta$ and so

$$\tilde{\Psi}_\theta((x \rightarrow y) \rightarrow y) = U \supseteq U = \tilde{\Psi}_\theta(x)$$

and $\lambda_\theta((x \rightarrow y) \rightarrow y) = 0 \leq 0 = \lambda_\theta(x)$ and if $x \notin [1]_\theta$, then by definition $\tilde{\Psi}_{\lambda,\theta}$, $\tilde{\Psi}_\theta((x \rightarrow y) \rightarrow y) \supseteq \emptyset = \tilde{\Psi}_\theta(x)$ and $\lambda_\theta((x \rightarrow y) \rightarrow y) \leq 1 = \lambda_\theta(x)$. Hence, $\tilde{\Psi}_{\lambda,\theta}$ satisfies in (HI2). Moreover, for (HI1), again if $x \in [1]_\theta$, then by Proposition 4.6(i), $y \rightarrow x \in [1]_\theta$ and $y \rightarrow (x \rightarrow y) \in [1]_\theta$ and so

$$\tilde{\Psi}_\theta(y \rightarrow x) \cap \tilde{\Psi}_\theta(y \rightarrow (x \rightarrow y)) = U \supseteq U = \tilde{\Psi}_\theta(x)$$

$$\lambda_\theta(y \rightarrow x) \vee \lambda_\theta(y \rightarrow (x \rightarrow y)) = 0 \leq 0 = \lambda_\theta(x)$$

And $x \notin [1]_\theta$, then by definition $\tilde{\Psi}_{\lambda,\theta}$,

$$\tilde{\Psi}_\theta(y \rightarrow x) \cap \tilde{\Psi}_\theta(y \rightarrow (x \rightarrow y)) \supseteq \emptyset = \tilde{\Psi}_\theta(x)$$

$$\lambda_\theta(y \rightarrow x) \vee \lambda_\theta(y \rightarrow (x \rightarrow y)) \leq 1 = \lambda_\theta(x)$$

And so (HI3) holds. Therefore, $\tilde{\Psi}_{\lambda,\theta}$ is a hybrid ideal of X over U . $\square$

Theorem 4.8. Let $\mathbf{C}(\mathbf{X})$ be set of all congruence relations on KL-algebra X . Then

(i) the mapping $\alpha : \mathbf{HID}(\mathbf{X}) \longrightarrow \mathbf{C}(\mathbf{X})$ which is defined by $\alpha(\tilde{\varphi}_\lambda) = \equiv_{\tilde{\varphi}_\lambda}$ is onto.

(ii) The mapping $\beta : \mathbf{C}(\mathbf{X}) \longrightarrow \mathbf{HID}(\mathbf{X})$ which is defined by $\beta(\theta) = \tilde{\Psi}_{\lambda,\theta}$ is on-to-one.

(iii) $\alpha \circ \beta = id_{\mathbf{C}(\mathbf{X})}$.

(iv) There exists an on-to-one correspondence between $\mathbf{C}(\mathbf{X})$ and $\mathbf{HID}^*(\mathbf{X})$, where

$$\begin{aligned} \mathbf{HID}^*(\mathbf{X}) = \{ \tilde{\varphi}_\lambda \in \mathbf{HID}(\mathbf{X}) \mid \tilde{\varphi}(1) = U, \lambda(1) = 0 \text{ and } \tilde{\varphi}(x) = \emptyset, \\ \text{when } \tilde{\varphi}(x) \subsetneq U, \lambda(x) = 1, \text{ when } \lambda(x) \neq 0 \}. \end{aligned}$$

Proof. (i) Consider the mapping $\alpha : \mathbf{HID}(\mathbf{X}) \longrightarrow \mathbf{C}(\mathbf{X})$ by $\alpha(\tilde{\varphi}_\lambda) \equiv \tilde{\varphi}_\lambda$. Firstly, we prove that α is well define. Let $\tilde{\varphi}_\lambda = \tilde{\psi}_\nu$. Then $\tilde{\varphi}(x) = \tilde{\psi}(x)$ and $\lambda(x) = \nu(x)$, for any $x \in X$. Hence, $(a, b) \in \equiv_{\tilde{\varphi}_\lambda}$ if and only if

$$\tilde{\varphi}(a \rightarrow b) = \tilde{\varphi}(b \rightarrow a) = \tilde{\varphi}(1)$$

and $\lambda(a \rightarrow b) = \lambda(b \rightarrow a) = \lambda(1)$ if and only if

$$\tilde{\psi}(a \rightarrow b) = \tilde{\psi}(b \rightarrow a) = \tilde{\psi}(1)$$

and $\nu(a \rightarrow b) = \nu(b \rightarrow a) = \nu(1)$ if and only if $(a, b) \in \equiv_{\tilde{\psi}_\nu}$. Thus, $\equiv_{\tilde{\varphi}_\lambda} = \equiv_{\tilde{\psi}_\nu}$ and so $\alpha(\tilde{\varphi}_\lambda) = \alpha(\tilde{\psi}_\nu)$. Therefore, α is a well define mapping. If $\theta \in \mathbf{C}(\mathbf{X})$, then by Theorem 4.7, $\tilde{\Psi}_{\lambda, \theta}$ is a hybrid ideal of X over U and so $\tilde{\Psi}_{\lambda, \theta} \in \mathbf{HID}(\mathbf{X})$. Now, we prove that $\alpha(\tilde{\varphi}_{\lambda, \theta}) = \theta$. In fact we must prove that $\equiv_{\tilde{\Psi}_{\lambda, \theta}} = \theta$. Let $u, v \in X$. Then $(u, v) \in \equiv_{\tilde{\Psi}_{\lambda, \theta}}$ if and only if

$$\tilde{\Psi}_\theta(u \rightarrow v) = \tilde{\Psi}_\theta(v \rightarrow u) = \tilde{\Psi}_\theta(1)$$

and $\lambda_\theta(u \rightarrow v) = \lambda_\theta(v \rightarrow u) = \lambda_\theta(1)$ and since $\theta \in \mathbf{C}(\mathbf{X})$, by Proposition 4.6(i), we get that $[1]_\theta$ is an ideal of X and so $1 \in [1]_\theta$. Hence,

$$\tilde{\Psi}_\theta(u \rightarrow v) = \tilde{\Psi}_\theta(v \rightarrow u) = \tilde{\Psi}_\theta(1) = U$$

and $\lambda_\theta(u \rightarrow v) = \lambda_\theta(v \rightarrow u) = \lambda_\theta(1) = 0$ and so $u \rightarrow v \in [1]_\theta$ and $v \rightarrow u \in [1]_\theta$. Thus, $[u]_\theta \supseteq [v]_\theta = [1]_\theta$ and $[v]_\theta \supseteq [u]_\theta = [1]_\theta$ and so $[u]_\theta \leq [v]_\theta$ and $[v]_\theta \leq [u]_\theta$. Therefore, $[u]_\theta = [v]_\theta$ and so $(u, v) \in \theta$. Hence, $\equiv_{\tilde{\varphi}_{\lambda, \theta}} \subseteq \theta$. Conversely, let $(z, w) \in \theta$. Then $[z]_\theta = [w]_\theta$ and so $[z \rightarrow w]_\theta = [z]_\theta \supseteq [w]_\theta = [1]_\theta$. Hence, $z \rightarrow w \in [1]_\theta$ and so $\tilde{\Psi}_\theta(z \rightarrow w) = U$ and $\lambda_\theta(z \rightarrow w) = 0$ and since $1 \in [1]_\theta$, we conclude that $\tilde{\Psi}_\theta(1) = U$ and $\lambda_\theta(1) = 0$. Thus, $\tilde{\varphi}_\theta(z \rightarrow w) = \tilde{\varphi}_\theta(1)$ and $\lambda_\theta(z \rightarrow w) = \lambda_\theta(1)$. Similarly, $\tilde{\varphi}_\theta(w \rightarrow z) = \tilde{\varphi}_\theta(1)$ and $\lambda_\theta(w \rightarrow z) = \lambda_\theta(1)$ and so $z \equiv_{\tilde{\varphi}_{\lambda, \theta}} w$. Hence, $(z, w) \in \equiv_{\tilde{\varphi}_{\lambda, \theta}}$ and so $\theta \subseteq \equiv_{\tilde{\varphi}_{\lambda, \theta}}$. Therefore, $\equiv_{\tilde{\varphi}_{\lambda, \theta}} = \theta$ and so α is onto mapping.

(ii) Consider the mapping $\beta : \mathbf{C}(\mathbf{X}) \longrightarrow \mathbf{HID}(\mathbf{X})$ by $\beta(\theta) = \tilde{\Psi}_{\lambda, \theta}$. Firstly, we show that β is well define. If $\theta_1 = \theta_2$, then $[1]_{\theta_1} = [1]_{\theta_2}$ and so $\tilde{\Psi}_{\theta_1}(x) = \tilde{\Psi}_{\theta_2}(x)$ and $\lambda_{\theta_1}(x) = \lambda_{\theta_2}(x)$, for any $x \in X$ and so $\tilde{\Psi}_{\lambda, \theta_1} = \tilde{\Psi}_{\lambda, \theta_2}$. Therefore, $\beta(\theta_1) = \beta(\theta_2)$ and so β is well define mapping. Now, we prove β is an on-to-one mapping. Let $\beta(\theta_1) = \beta(\theta_2)$, for $\theta_1, \theta_2 \in \mathbf{C}(\mathbf{X})$ and $(c, d) \in \theta_1$. Then $[c]_{\theta_1} = [d]_{\theta_1}$ and so $[d \rightarrow c]_\theta = [c \rightarrow d]_\theta = [1]_{\theta_1}$. Hence,

$$d \rightarrow c, c \rightarrow d \in [1]_{\theta_1}$$

and so $\tilde{\Psi}_{\theta_1}(c \rightarrow d) = \tilde{\Psi}_{\theta_1}(d \rightarrow c) = U$ and since $\beta(\theta_1) = \beta(\theta_2)$, we conclude that $\tilde{\Psi}_{\lambda, \theta_1} = \tilde{\Psi}_{\lambda, \theta_2}$ and so $\tilde{\Psi}_{\theta_1}(c \rightarrow d) = \tilde{\Psi}_{\theta_2}(c \rightarrow d)$ and

$$\tilde{\Psi}_{\theta_1}(d \rightarrow c) = \tilde{\Psi}_{\theta_2}(d \rightarrow c).$$

Hence, $\tilde{\Psi}_{\theta_2}(c \rightarrow d) = U$ and $\tilde{\Psi}_{\theta_2}(d \rightarrow c) = U$ and so $d \rightarrow c, c \rightarrow d \in [1]_{\theta_2}$. Thus, $[c]_{\theta_2} \Rightarrow [d]_{\theta_2} = [c \rightarrow d]_{\theta_2} = [1]_{\theta_2}$ and $[d]_{\theta_2} \Rightarrow [c]_{\theta_2} = [d \rightarrow c]_{\theta_2} = [1]_{\theta_2}$ and so $[c]_{\theta_2} \leq [d]_{\theta_2}$ and $[d]_{\theta_2} \leq [c]_{\theta_2}$. Hence, $[c]_{\theta_2} = [d]_{\theta_2}$ and so $(c, d) \in \theta_2$. Hence, $\theta_1 \subseteq \theta_2$ and by similarly, $\theta_2 \subseteq \theta_1$. Therefore, $\theta_1 = \theta_2$ and so β is an on-to-one mapping.

(iii) For any $\theta \in \mathbf{C}(\mathbf{X})$, we have

$$\alpha \circ \beta(\theta) = \alpha(\beta(\theta)) = \alpha(\tilde{\Psi}_{\lambda, \theta}) = \equiv_{\tilde{\Psi}_{\lambda, \theta}} = \theta$$

Hence, $\alpha \circ \beta = id_{\mathbf{C}(\mathbf{X})}$.

(iv) We define mapping $g^* : \mathbf{C}(\mathbf{X}) \longrightarrow \mathbf{HID}^*(\mathbf{X})$, by $g^*(\theta) = \beta(\theta)$, where

$$\begin{aligned} \mathbf{HID}^*(\mathbf{X}) &= \{\tilde{\varphi}_\lambda \in \mathbf{HID}(\mathbf{X}) \mid \tilde{\varphi}(1) = U, \lambda(1) = 0 \text{ and } \tilde{\varphi}(x) = \emptyset, \\ &\text{when } \tilde{\varphi}(x) \subsetneq U, \lambda(x) = 1, \text{ when } \lambda(x) \neq 0\}. \end{aligned}$$

By (ii), g^* is an one-to-one mapping. Now, we prove that g^* is onto. Let $\tilde{\varphi}_\lambda \in \mathbf{HID}^*(\mathbf{X})$. Then by Theorem 4.4, $\equiv_{\tilde{\varphi}_\lambda}$ is a congruence relation on X and so $\equiv_{\tilde{\varphi}_\lambda} \in \mathbf{C}(\mathbf{X})$. Moreover, we show that $g^*(\equiv_{\tilde{\varphi}_\lambda}) = \tilde{\varphi}_\lambda$. By definition g^* we have $g^*(\equiv_{\tilde{\varphi}_\lambda}) = \tilde{\Psi}_{\lambda, \equiv_{\tilde{\varphi}_\lambda}} = (\tilde{\Psi}_{\equiv_{\tilde{\varphi}_\lambda}}, \lambda_{\equiv_{\tilde{\varphi}_\lambda}})$.

Now, if $1 \in [1]_{\equiv_{\tilde{\varphi}_\lambda}}$, then $x \equiv_{\tilde{\varphi}_\lambda} 1$ and so

$$\tilde{\varphi}(1 \rightarrow x) = \tilde{\varphi}(1), \lambda(1 \rightarrow x) = \lambda(x) = \lambda(1)$$

and so by (L1), $\tilde{\varphi}(x) = \tilde{\varphi}(1)$, $\lambda(x) = \lambda(1)$ and since $\tilde{\varphi}_\lambda \in \mathbf{HID}^*(\mathbf{X})$, we get that $\tilde{\varphi}(1) = U$, $\lambda(1) = 0$ and so $\tilde{\varphi}(x) = U$, $\lambda(x) = 0$. Moreover, since $1 \in [1]_{\equiv_{\tilde{\varphi}_\lambda}}$, we get that $\tilde{\Psi}_{\equiv_{\tilde{\varphi}_\lambda}}(x) = U$, $\lambda_{\equiv_{\tilde{\varphi}_\lambda}}(x) = 0$. Hence, $\tilde{\Psi}_{\equiv_{\tilde{\varphi}_\lambda}}(x) = \tilde{\varphi}(x)$ and $\lambda_{\equiv_{\tilde{\varphi}_\lambda}}(x) = \lambda(x)$.

If $1 \notin [1]_{\equiv_{\tilde{\varphi}_\lambda}}$, then $x \not\equiv_{\tilde{\varphi}_\lambda} 1$ and so

$$\tilde{\varphi}(1 \rightarrow x) = \tilde{\varphi}(x) \neq \tilde{\varphi}(1) = U, \lambda(1 \rightarrow x) = \lambda(x) \neq \lambda(1) = 0$$

and since $\tilde{\varphi}_\lambda \in \mathbf{HID}^*(\mathbf{X})$, we get that $\tilde{\varphi}(x) = \emptyset$, $\lambda(x) = 1$ and since $1 \notin [1]_{\equiv_{\tilde{\varphi}_\lambda}}$, we get that

$$\tilde{\Psi}_{\equiv_{\tilde{\varphi}_\lambda}}(x) = \emptyset, \lambda_{\equiv_{\tilde{\varphi}_\lambda}}(x) = 1.$$

Hence, $\tilde{\Phi}_{\equiv_{\tilde{\varphi}_\lambda}}(x) = \tilde{\varphi}(x)$ and $\lambda_{\equiv_{\tilde{\varphi}_\lambda}}(x) = \lambda(x)$, for any $x \in L$. Therefore, $\tilde{\Psi}_{\lambda, \equiv_{\tilde{\varphi}_\lambda}} = \tilde{\varphi}_\lambda$ and so $g^*(\equiv_{\tilde{\varphi}_\lambda}) = \tilde{\varphi}_\lambda$. Therefore, g^* is an on-to-one correspondence between $\mathbf{C}(\mathbf{X})$ and $\mathbf{HID}^*(\mathbf{X})$. $\square$

5. CONCLUSION

The results of this paper focus on studying the notion of hybrid structures in L-algebras. By defining hybrid subalgebras within L-algebras, this structure is examined, and hybrid ideals of L-algebras are constructed. It is shown that the quotient L-algebra, which respects a hybrid ideal, is an L-algebra for any hybrid ideal. Finally, it is proved that there is a one-to-one correspondence between the set of all congruence relations and the set of all hybrid ideals, given a few additional conditions.

REFERENCES

1. V. G. Drinfeld, On some unsolved problems in quantum group theory, in: P. P. Kulish (Ed.), *Quantum Groups (Leningrad, 1990)*, Lecture Notes in Mathematics, Vol. 1510, Springer, Berlin, 1992.
2. Y. B. Jun, S. Z. Song and G. Muhiuddin, Hybrid structures and applications, *Annals of Communications in Mathematics*, **1**(1) (2018), 11–25.
3. M. A. Kologani, Some results on L-algebras, *Soft Computing*, **27** (2023), 13765–13777.
4. M. Mohseni Takallo, Block code on L-algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **4**(1) (2023), 47–60.
5. D. Molodtsov, Soft set theory-First results, *Computers and Mathematics with Applications*, **37** (1999), 19–31.
6. A. Paad, Hybrid structures on EQ-algebras, *New Mathematics and Natural Computation*, **18**(2) (2022), 365–383.
7. W. Rump, L-Algebras, self-similarity, and l-groups, *J. Algebra*, **320** (2008), 2328–2348.
8. W. Rump, Right l-groups, geometric Garside groups, and solutions of the quantum Yang-Baxter equation, *J. Algebra*, **439** (2015) 470–510.
9. W. Rump, The L-algebra of Hurwitz primes, *J. Number Theory*, **190** (2018) 394–413.
10. W. Rump, The structure group of a generalized orthomodular lattice, *Studia Logica*, **106** (2018) 85–100.
11. W. Rump, Von Neumann algebras, L-algebras, Baer*-monoids, and Garside groups, *Forum Math.*, **3** (2018) 973–995.
12. W. Rump and Y. Yang, Interval in l-groups as L-algebras, *Algebra Universalis*, **67**(2) (2012), 121–130.
13. Y. L. Wu, J. Wang and Y. C. Yang, Lattice-ordered effect algebras and L-algebras, *Fuzzy Sets and Systems*, **369** (2019), 103–113.
14. Y. L. Wu and Y. C. Yang, Orthomodular lattices as L-algebras, *Soft Computing*, **24** (2020), 14391–14400.
15. L. A. Zadeh, Fuzzy sets, *Information and control*, **8** (1965), 338–353.

Akbar Paad

Department of Mathematics, University of Bojnord, P.O. Box 9453155111, Bojnord, Iran.

Email: akbar.paad@gmail.com

L-ALGEBRAS WITH HYBRID STRUCTURES

A. Paad

L -جبرها با ساختارهای هیبریدی

اکبر پاد

گروه ریاضی، دانشکده علوم پایه، دانشگاه بجنورد، بجنورد، ایران

در این مقاله، مفهوم ساختار هیبریدی در L -جبرها معرفی شده و برخی ویژگی‌های مرتبط با آن مورد بررسی قرار گرفته‌اند. علاوه بر این، مفهوم ایده‌آل‌های هیبریدی در L -جبرها مطرح شده و نتایج متعددی در این زمینه بدست آمده است. سپس، L -جبر خارج قسمتی القا شده توسط یک ایده‌آل هیبریدی مورد مطالعه قرار گرفته است. در نهایت، ثابت شده است که بین مجموعه‌ی تمام روابط هم‌نهشتی و مجموعه‌ی تمام ایده‌آل‌های هیبریدی، تحت شرایطی، یک تناظر یک به یک برقرار است.

کلمات کلیدی: L -جبر، زیرجبر هیبریدی، ایده‌آل هیبریدی، هم‌نهشتی نسبی.