Journal of Algebraic Systems, Vol. 14(No. 4): (2026), pp 801-821.
https://doi.org/10.22044/JAS.2025.15366.1909

MINIMAL GENERATING SETS FOR PARAMETRIC POLYNOMIAL IDEALS

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Abstract. This paper generalizes Schreyer's syzygy theorem for polynomial ideals with parametric coefficients and introduces an algorithm to compute the minimal generating set of homogeneous parametric polynomial ideals that form a Gröbner basis with respect to variables. By employing comprehensive Gröbner systems, we partition the parameter space into finite subspaces and identify the minimal generating set for each subspace. The algorithms are implemented in `Maple`, and we demonstrate the main algorithm through a simple example.

1. Introduction

Determining a minimal set of generators for polynomial ideals represents an intriguing and complex challenge in computer algebra. This problem is particularly significant because it applies directly to various mathematical domains and disciplines. The intricacy arises because omitting any polynomial from the set of generators naturally leads to a smaller set, which complicates the identification of the truly minimal generating set. A notable contribution to this field is Schreyer's syzygy theorem, formulated in 1980 [25], which provides a foundational method for computing the minimal generators of homogeneous polynomial ideals. A simpler version of Schreyer's theorem appears in [4, Theorem 3.2, page 223, and Theorem 3.3, page 224]. In this paper, we build upon Schreyer's results and extend the approach to include parametric polynomial ideals, thereby enhancing the applicability of these concepts and potentially solving new problems within this context. Our goal is to explore the implications of this extension and provide a more comprehensive framework for understanding generators in homogeneous as well as parametric scenarios. For this purpose, we utilize a powerful computer algebra tool called Gröbner systems. Doing so, we first review Gröbner bases, as discussed in [1, 4].

MSC(2020): Primary: 13P10; Secondary: 13D02, 13P20.
Keywords: Minimal generating set; Gröbner systems, Parametric polynomials; Minimal generator system; Schreyer's syzygy theorem.
Received: 17 November 2024, Accepted: 9 June 2025.

Let $R = \mathbb{K}[\mathbf{x}]$ be the polynomial ring over the field $\mathbb{K}$ in variables $\mathbf{x} = x_1, \ldots, x_n$. A monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is denoted as $\mathbf{x}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ consists of non-negative integers. A monomial order $\prec$ on R is a total order on all monomials with the following properties:

1) If $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$, then $\mathbf{x}^{\alpha+\gamma} \prec \mathbf{x}^{\beta+\gamma}$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.

2) The constant monomial is the smallest; specifically, $1 \prec \mathbf{x}^\alpha$ for all $\alpha \in \mathbb{N}^n$.

A common example is the lexicographical order $\prec_{lex}$, where $\mathbf{x}^\beta \prec_{lex} \mathbf{x}^\alpha$ if the left-most nonzero entry of $\alpha - \beta$ is positive.

For $f \in R$, the leading monomial, denoted $\mathrm{LM}(f)$, is the greatest monomial with respect to $\prec$. The leading monomial ideal of $I = \langle f_1, \ldots, f_m \rangle \subset R$ is defined as $\mathrm{LM}(I) = \langle \mathrm{LM}(f) | f \in I \rangle$. A finite subset $\{g_1, \ldots, g_k\} \subset I$ is a Gröbner basis with respect to $\prec$ if $\mathrm{LM}(I) = \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_k) \rangle$.

Introduced by Bruno Buchberger in 1965 through Buchberger's algorithm [3], Gröbner bases are named after his advisor, Wolfgang Gröbner. Buchberger also developed criteria to eliminate unnecessary S-polynomials, improving the algorithm's efficiency [2]. However, the algorithm's complexity limits its practicality in some cases. In 1983, Lazard presented an algorithm for computing Gröbner bases using linear algebra techniques [19]. Faugère later introduced the F_4 and F_5 algorithms in 1999 and 2002, respectively [9, 10]. Subsequently, Gao et al. proposed the incremental G2V algorithm [11], an efficient variant of F_5, followed by the GVW algorithm [12] for computing Gröbner bases of ideals. A Gröbner basis G is called reduced if the leading coefficient of every polynomial in G is one and no monomial of any polynomial $p \in G$ lies in the ideal generated by the leading monomials of $G \setminus \{p\}$. Every nonzero ideal I has a unique reduced Gröbner basis for any fixed monomial ordering.

The first syzygy, or simply syzygy, of a sequence $F = f_1, \ldots, f_k$ is the set of k-tuples $(g_1, \ldots, g_k) \in R^k$ such that $\sum_{i=1}^k g_i \cdot f_i = 0$. Furthermore, a syzygy matrix represents the syzygies among generators of a finitely generated module over a commutative ring, capturing the essential relationships among these generators and aiding in understanding the module's structure. Schreyer's syzygy theorem, introduced by Schreyer in 1980 [25], a simpler format [4, Theorem 3.2 and Theorem 3.3], focuses on syzygies in modules and algebraic geometry. It offers an algorithm for computing syzygies in polynomial rings and aids in understanding the projective dimension of modules and solving problems like Stillman's Question [24, Problem 3.14].

To prepare for Schreyer's Theorem, let $S(g_i, g_j)$ be the S-polynomial of g_i and g_j. Assume $G = \{g_1, \ldots, g_m\}$ is a Gröbner basis for a submodule $M \subset R^s$

with respect to a fixed monomial order $\prec$. Since G is a Gröbner basis, the remainder of $S(g_i, g_j)$ when divided by G is $\mathbf{0}$. The division algorithm yields an expression: $S(g_i, g_j) = \sum_{k=1}^{m} a_{ijk} g_k$, where $a_{ijk} \in R$, and

$$\mathrm{LT}(a_{ijk} g_k) \prec \mathrm{LT}(S(g_i, g_j))$$

for all i, j, k.

Let $e_1, \ldots, e_m$ be the standard basis vectors in R^m. Define

$$m_{ij} = \mathrm{LCM}(\mathrm{LT}(g_i), \mathrm{LT}(g_j)),$$

and let $\mathbf{a}_{ij} \in R^m$ be the column vector:

$$\mathbf{a}_{ij} = a_{ij1} e_1 + a_{ij2} e_2 + \cdots + a_{ijm} e_m.$$

For pairs (i, j) where $m_{ij} \neq \mathbf{0}$, define $\mathbf{s}_{ij} \in R^m$ as:

$$\mathbf{s}_{ij} = \frac{m_{ij}}{\mathrm{LT}(g_i)} e_i - \frac{m_{ij}}{\mathrm{LT}(g_j)} e_j - \mathbf{a}_{ij}.$$

Theorem 1.1 (Schreyer's Theorem [4])**.** *Let $G = \{g_1, \ldots, g_m\} \subset R^s$ be a Gröbner basis with respect to any monomial order $\prec$ on R^s. The vectors $\mathbf{s}_{ij}$ form a Gröbner basis for the syzygy module $\mathrm{Syz}(g_1, \ldots, g_m)$ w.r.t. a TOP monomial order (terms over position) on R^m.*

According to Schreyer's theorem, the following important proposition is concluded:

Proposition 1.2. *Let $G = \{g_1, \ldots, g_m\} \subset \mathbb{K}[\mathbf{x}]$ be the homogeneous reduced Gröbner basis. Then, $g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m \rangle$ iff i^{th} row of syzygy matrix of G contains at least one nonzero constant $k \in \mathbb{K}$.*

Proof. If $g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m \rangle$, then $g_i = \sum_{j=1, j\neq i}^{m} a_j g_j$ for some $a_j \in R$. Thus, $\left(\sum_{j=1, j\neq i}^{m} a_j g_j\right) - g_i = 0$, leading to the following column matrix being a generator of $\mathrm{syz}(G)$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ -1 \\ a_{i+1} \\ \vdots \\ a_m \end{pmatrix} \in R^m.$$

Consequently, if $g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m \rangle$, then the i^{th} row of the syzygy matrix of G contains either -1 or 1.

Conversely, if

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0 \neq k \in \mathbb{K} \\ a_{i+1} \\ \vdots \\ a_m \end{pmatrix} \in R^m$$

is a column of the first syzygy matrix of G, it implies that

$$\begin{aligned} & a_1g_1 + \cdots + a_{i-1}g_{i-1} + kg_i + a_{i+1}g_{i+1} + \cdots + a_mg_m = 0 \\ & \Rightarrow -kg_i = a_1g_1 + \cdots + a_{i-1}g_{i-1} + a_{i+1}g_{i+1} + \cdots + a_mg_m \\ & \Rightarrow g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m \rangle . \end{aligned}$$

□

Numerous engineering and scientific challenges can be modeled using specific sets of polynomials with parametric coefficients. These problems often require repeated examination with different parameter values. This paper focuses on identifying minimal generating sets for homogeneous parametric polynomial ideals, a problem that has not been previously addressed in the literature to our knowledge. To accomplish this, we introduce the notion of a *minimal generator system* for homogeneous parametric polynomial ideals (which are Gröbner bases) and propose an algorithm for its computation. In this direction, we use Gröbner systems (extending Gröbner bases to polynomials with parametric coefficients).

2. Minimal Generator System

The extension of Gröbner bases is essential for tackling practical challenges in systems of polynomial equations with parameters, playing a significant role in fields such as parametric linear algebra [5, 8, 14], automated geometry theorem proving and algebraic geometry [16, 21, 22, 27], robotics and electrical networks [20, 22, 23]. These applications often require repeated analyses with different parameter values.

Gröbner systems extend Gröbner bases from polynomial ideals with numerical coefficients to those with parametric coefficients. A Gröbner system consists of a finite set of triples containing parametric constraints and a parametric polynomial set. For any specialization of the parameters, a corresponding branch satisfies the assignment, and after parameter substitution,

the specialized polynomial set forms a Gröbner basis for the parametric ideal. We briefly review parametric Gröbner systems and the related literature.

In this article, we consider $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$, where $\mathbb{K}$ is any field, $\mathbf{a} = a_1, \ldots, a_m$ is a sequence of parameters, and $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of variables. We define monomial orders $\prec_{\mathbf{x}}$ and $\prec_{\mathbf{a}}$ for the variables and parameters, respectively. These orders combine to form a new ordering on S, denoted $\prec_{\mathbf{x},\mathbf{a}}$: for any $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^m$, we have $\mathbf{x}^\alpha \mathbf{a}^\gamma \prec_{\mathbf{x},\mathbf{a}} \mathbf{x}^\beta \mathbf{a}^\delta$ if $\mathbf{x}^\alpha \prec_{\mathbf{x}} \mathbf{x}^\beta$, or if $\mathbf{x}^\alpha = \mathbf{x}^\beta$ and $\mathbf{a}^\gamma \prec_{\mathbf{a}} \mathbf{a}^\delta$.

Definition 2.1. Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a finite set, and let

$$G = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$$

be a finite triple set where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ and $G_i \subset S$. The set G is a Gröbner system for $\langle F \rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ if, for any i and homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$, the following conditions hold:

- $\sigma(G_i) \subset \mathbb{K}'[\mathbf{x}]$ is a Gröbner basis for $\sigma(\langle F \rangle) \subset \mathbb{K}'[\mathbf{x}]$ w.r.t. $\prec_{\mathbf{x}}$
- $\sigma(p) = 0;\ \forall p \in N_i$ and $\sigma(q) \neq 0;\ \forall q \in W_i$

The concept of a Gröbner system was introduced by Weispfenning in 1992, demonstrating that every parametric polynomial ideal has a finite Gröbner system along with the initial computation algorithm [27]. In 1997, M. Kalkbrenner established critical criteria on the specialization of parametric polynomial ideals and their Gröbner bases' stability under specialization [17]. In 2002, Montes developed a more efficient algorithm called DisPGB for computing Gröbner systems, building on Buchberger's algorithm [22]. In 2013, Hashemi et al. enhanced DisPGB by incorporating two Buchberger criteria to improve efficiency [15]. In 2006, Manubens and Montes refined the DisPGB algorithm with the concept of discriminant ideals [20]. Following Kalkbrenner's stability criteria, Suzuki and Sato devised an effective algorithm for Gröbner system computation, known as the Suzuki-Sato algorithm [26]. In 2010, Kapur-Sun-Wang introduced the efficient PGBMain (KSW) algorithm, inspired by Weispfenning's and Suzuki-Sato's algorithms, utilizing a minimal Dickson basis to streamline calculations and include radical membership checks for non-null parametric conditions to minimize unnecessary branches in the output [18].

More recently, Dehghani et al. have advanced the computation of Gröbner systems by examining parametric polynomial ideals and developing optimized algorithms in this domain [7, 8, 16, 13]. In 2024, Dehghani presented the efficient GES-GVW-CGS algorithm for computing these systems [6].

Example 2.2. Let $F = \{(b-a)xy - x - cy, cx^2 - 2xy\} \subset \mathbb{K}[a,b,c,x,y]$, where x, y are variables and a, b, c are parameters. A Gröbner system of F with respect to the product of monomial orderings $y \prec_{lex} x$ and $c \prec_{lex} b \prec_{lex} a$ via our `Maple` implementation of the GES-GVW-CGS algorithm is as follows:

$$\begin{cases} ([c], & [\,], & [x]) \\ ([ac-bc], & [c, c^2+2], & [c^3y^2+2cy^2, cy+x]) \\ ([c^3+2c, ac-bc], & [c], & [cy+x]) \\ ([\,], & [2c^2(a-b)], & [2acy^3-2bcy^3+c^3y^2+2cy^2, \\ & & 2acy^2-2bcy^2+c^3y+c^2x]) \end{cases}$$

This Gröbner system comprises four triples. By setting $a = b = 2$ and $c = 1$, the second branch corresponds to these values, yielding $\{3y^2, x+y\}$ as a Gröbner basis for the ideal $\langle F\rangle|_{a=b=2,c=1}$.

We now introduce and elaborate on the concept of a *minimal generator system* for homogeneous parametric polynomial ideals, along with an algorithm for computing these minimal generating sets. These ideals consist of homogeneous polynomials in terms of their variables and are significant in algebraic geometry and computer algebra due to their theoretical and practical applications. Determining a minimal generating set enhances computational efficiency and deepens our understanding of their structural properties. This insight simplifies complex problems and organizes the representation of the ideals, ultimately benefiting further applications in these mathematical fields.

Definition 2.3. Let $I \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a homogeneous parametric polynomial ideal that forms a Gröbner basis with respect to $\prec_{\mathbf{x}}$. We define $M = \{(N_i, W_i, M_i)\}_{i=1}^{\ell}$ as a *minimal generator system* of I if, for each i and any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ satisfying the parametric constraint (N_i, W_i), the set $\sigma(M_i)$ is a minimal generating set of $\sigma(I)$.

This definition allows us to extend Proposition 1.2, which can be considered as an extension of Schreyer's syzygy theorem.

Theorem 2.4. *Let $F = \{f_1, \ldots, f_t\} \subset S$ be a reduced Gröbner basis with respect to $\prec_{\mathbf{x}}$, and let the triples set $G = \{(G_i, N_i, W_i)\}_{i=1}^{\ell}$ be a Gröbner system for $\langle F\rangle$. For any $(G_i, N_i, W_i) \in G$ where $G_i = \{g_1, \ldots, g_r\}$, we have $g_j \in \langle g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r\rangle$ if and only if the j^{th} row of the syzygy matrix of G_i contains a nonzero polynomial $q \in \mathbb{K}[\mathbf{a}]$ that satisfies the conditions (N_i, W_i).*

Proof. This proposition's proof is inspired by Proposition 1.2 and follows directly from it. Specifically, let $(G_i, N_i, W_i) \in G$ be a triple of a Gröbner

system of F, where $G_i = \{g_1, \ldots, g_r\}$. Since (N_i, W_i, G_i) is a branch of a Gröbner system of F with respect to $\prec_{x,a}$, all leading coefficients of the polynomials in G are nonzero polynomials in $\mathbb{K}'[\mathbf{a}]$ according to (N_i, W_i).

If $g_j \in \langle g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r \rangle$, then $g_j = \sum_{i=1, i\neq j}^{r} p_i g_i$ for some $p_i \in S$. Thus, $\left(\sum_{i=1, i\neq j}^{r} p_i g_i\right) - g_j = 0$, which implies that the following column matrix serves as a generator of $\mathrm{syz}(G_i)$:

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{j-1} \\ -1 \\ p_{j+1} \\ \vdots \\ p_r \end{pmatrix} \in R^r$$

Consequently, if $g_j \in \langle g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r \rangle$, the j^{th} row of the syzygy matrix of G_i contains either -1 or 1.

Conversely, if

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{j-1} \\ 0 \neq p_j \in \mathbb{K}[a] \\ p_{j+1} \\ \vdots \\ p_r \end{pmatrix} \in R^r$$

is a column of the first syzygy matrix of G_i, it implies

$$p_1 g_1 + \cdots + p_{j-1} g_{j-1} + p_j g_j + p_{j+1} g_{j+1} + \cdots + p_r g_r = 0.$$

Thus, $g_j \in \langle g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r \rangle$. □

We now present the simple MGSystem algorithm for computing a minimal generator system based on the above definition and theorem. This algorithm receives as input a reduced Gröbner basis I with respect to $\prec_{\mathbf{x}}$ and returns a minimal generator system of I. Below, we employ the Remove function, which takes a set of polynomials $F \subset S$ as input and returns $F_i \subset F$ by removing the i^{th} element of F if the i^{th} row of the syzygy matrix of F involves at least one polynomial $q \in \mathbb{K}[\mathbf{a}]$. In this algorithm, the notion MGS denotes a variable that is an empty set at the beginning of the algorithm,

and in the end, it is a minimal generator system. Additionally, we use the REMOVE function which receives as input a set of polynomial $F \subset S$ and returns $F_i \subset F$ obtained by removing the i^{th} element of F if i^{th} row of syzygy matrix of F contains at least one polynomial $0 \neq q \in \mathbb{K}[\mathbf{a}]$.

Algorithm 1 MGSYSTEM

Require: $I = \langle f_1, \ldots, f_m \rangle \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ is the reduced Gröbner basis w.r.t. $\prec_{\mathbf{x}}$
Ensure: A minimal generator system of I
MGS := { }
$G := \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ a Gröbner system of I with respect to $\prec_{\mathbf{x},\mathbf{a}}$
for $(N_i, W_i, G_i) \in G$ **do**
Syz := syzygy matrix of G_i
while the j^{th} row of Syz contains a nonzero polynomial $q \in \mathbb{K}[\mathbf{a}]$ **do**
F_i:=REMOVE(G_i)
Syz := syzygy matrix of F_i
end while
MGS := MGS $\bigcup\{(N_i, W_i, F_i)\}$
end for
Return (MGS)

Theorem 2.5. *The* MGSYSTEM *algorithm terminates in a finite number of steps and correctly computes a minimal generator system.*

Proof. The termination of the MGSYSTEM algorithm is assured by the termination of Gröbner system computations, while its correctness follows from theorem 2.4. Specifically, let (N_i, W_i, G_i) represent the i^{th} branch of the Gröbner system I. During the i-th iteration of the **for**-loop, since all parameters in G_i are nonzero, we can apply theorem 2.4 to find the matrix syzygy of G_i. Thus, any F_i is a minimal generating set of I that satisfies the parametric conditions (N_i, W_i). □

3. A SIMPLE EXAMPLE

This section illustrates the algorithm steps using a simple example.

Example 3.1. Let us consider the following generators for the polynomial ideal $I \subset \mathbb{K}[a, b, c, d][x, y, z]$, which forms a Gröbner basis:

$$\left\{ \begin{array}{c} z^4, yz^3, z^2y^2, by^3 + y^2z + (d-1)z^2y, xz^3, (ac+a-2)xyz - 2z^2y + z^3a, \\ xy^2(ac+a-2) + (-a^3c - a^3 + 2a^2)xz^2 + y^2z(ac+a-2) + 2az^2y + \\ (-a^3bc - a^3c^2 - a^3b - a^3c + 2a^2b + 2a^2c - a^2)z^3, ax^2 - xy - yz \end{array} \right\}$$

Using our `Maple` implementation of the MGSystem algorithm, we first compute a Gröbner system of I based on the algorithm's structure:

$$\left\{ \begin{array}{ll}
([a],[b], & \{\ z^4, yz^3, z^2y^2, by^3+dyz^2+y^2z-yz^2, xz^3, xy+yz\ \} \\
([b,a],[1], & \{\ z^4, yz^3, dyz^2+y^2z-yz^2, xz^3, xy+yz\ \} \\
([b,ac+a-2],[a], & \left\{ \begin{array}{l} z^4, -az^3+2yz^2, adz^3-az^3+2y^2z, \\ xz^3, -cxy-cyz+2x^2-xy-yz \end{array} \right\} \\
([ac+a-2],[a,b], & \left\{ \begin{array}{l} z^4, 2yz^2-az^3, adz^3-az^3+2by^3+2y^2z, \\ xz^3, 2x^2-cxy-cyz-xy-yz \end{array} \right\} \\
([b],[a,ac+a-2], & \left\{ \begin{array}{l} z^4, yz^3, dyz^2+y^2z-yz^2, xz^3, aceyz+axyz+az^3- \\ 2xyz-2yz^2, -a^3c^2z^3-a^3bcz^3-a^3cxz^2-a^3cz^3- \\ a^3xz^2+2a^2cz^3-acdyz^2+2a^2xz^2-a^2z^3+acey^2 \\ +acyz^2-adyz^2+axy^2+3ayz^2+2dyz^2-2xy^2- \\ 2yz^2, ax^2-xy-yz \end{array} \right\} \\
([\,],[ab(ac+a-2)], & \left\{ \begin{array}{l} z^4, yz^3, y^2z^2, by^3+dyz^2+y^2z-yz^2, xz^3, aceyz+ \\ axyz+az^3-2xyz-2yz^2, -a^3bcz^3-a^3c^2z^3- \\ a^3bcz^3-a^3cxz^2-a^3cz^3-a^3xz^2+2a^2bz^3+2a^2cz^3 \\ +2a^2xz^2-a^2z^3+acey^2+acy^2z+axy^2+ay^2z+ \\ 2ayz^2-2xy^2-2y^2z, ax^2-xy-yz \end{array} \right\}
\end{array} \right\}$$

For any triple $(N_i, W_i, G_i) \in G$, we calculate a syzygy matrix for G_i. The Remove function eliminates the i^{th} element of G_i if the i^{th} row of the syzygy matrix for G_i contains at least one nonzero polynomial $q \in \mathbb{K}[\mathbf{a}]$. We then update the syzygy matrix for new G_i until no such polynomial exists. At this point, (N_i, W_i, G_i) is added to MGS, where G_i is the most recently updated version. Ultimately, we obtain the minimal generator system for I:

$$\left\{ \begin{array}{ll}
([a],[b], & \{\ z^4, yz^3, z^2y^2, by^3+dyz^2+y^2z-yz^2, xz^3, xy+yz\ \} \\
([b,a],[1], & \{\ z^4, yz^3, dyz^2+y^2z-yz^2, xz^3, xy+yz\ \} \\
([b,ac+a-2],[a], & \left\{ \begin{array}{l} -az^3+2yz^2, adz^3-az^3+2y^2z, \\ xz^3, -cxy-cyz+2x^2-xy-yz \end{array} \right\} \\
([ac+a-2],[a,b], & \left\{ \begin{array}{l} z^4, 2yz^2-az^3, adz^3-az^3+2by^3+2y^2z, \\ xz^3, 2x^2-cxy-cyz-xy-yz \end{array} \right\} \\
([b],[a,ac+a-2], & \left\{ \begin{array}{l} dyz^2+y^2z-yz^2, xz^3, \\ aceyz+axyz+az^3-2xyz-2yz^2, \\ 2a^2cz^3-a^3c^2z^3-a^3cz^3-a^3cxz^2-a^3xz^2-acdyz^2 \\ +2a^2xz^2-a^2z^3+acey^2+acyz^2+axy^2-adyz^2 \\ +3ayz^2+2dyz^2-2xy^2-2yz^2, ax^2-xy-yz \end{array} \right\} \\
([\,],[ab(ac+a-2)], & \left\{ \begin{array}{l} by^3+dyz^2+y^2z-yz^2, aceyz+axyz+az^3-2xyz- \\ 2yz^2, acey^2-a^3c^2z^3-a^3bcz^3-a^3cxz^2-a^3cz^3-a^3xz^2 \\ +2a^2bz^3+2a^2cz^3+2a^2xz^2-a^2z^3+acey^2z-a^3bcz^3 \\ +axy^2+ay^2z+2ayz^2-2xy^2-2y^2z, ax^2-xy-yz \end{array} \right\}
\end{array} \right\}$$

For instance, if $a = \frac{1}{2}, b = 2, c = -3$, and $d = -1$, the sixth branch corresponds to these parameter values (which satisfy (N_6, W_6)). Thus,

$$G_6|_{a=\frac{1}{2},b=2,c=-3,d=-1} = \{2y^3 + y^2z - 2yz^2, -3xyz + \frac{1}{2}z^3 - 2yz^2, \\ -z^3 + \frac{3}{4}xz^2 - 3xy^2 - 3y^2z + yz^2, \frac{1}{2}x^2 - xy - yz\}$$

is a minimal generating set for the ideal

$$I|_{a=\frac{1}{2},b=2,c=-3,d=-1} = [z^4, yz^3, y^2z^2, 2y^3 + y^2z - 2yz^2, xz^3, -3xyz + \frac{1}{2}z^3 - 2yz^2, \\ -z^3 + \frac{3}{4}xz^2 - 3xy^2 - 3y^2z + yz^2, \frac{1}{2}x^2 - xy - yz].$$

This means that for any $1 \leq i \leq 6$ and for each specialization σ satisfying (N_i, W_i), $\sigma(G_i)$ is a minimal generating set of $\sigma(I)$. Moreover, for the last branch, under the parametric constraint (N_6, W_6), the minimal generating set G_6 has four generators, while the ideal input I, which is a Gröbner basis, has eight elements, indicating a significant reduction in the size of the generating set of I.

4. Additional Examples and Performance Analysis

In this section, the performance of the MGSystem algorithm, implemented in `Maple`, is further demonstrated with examples computing minimal generating systems for parametric polynomial ideals. These examples (which are Gröbner bases w.r.t. variables), selected for their varying sizes and complexities in the ring $\mathcal{S} = \mathbb{K}[a, b, c, d][x, y, z, u]$, compute minimal generating systems for each ideal with respect to the ordering $u \prec_{lex} z \prec_{lex} y \prec_{lex} x$. When necessary, the monomial ordering on parameters is $d \prec_{lex} c \prec_{lex} b \prec_{lex} a$. All computations were performed on a computer with a Ryzen 6800 processor, 8 GB RAM, and Windows 10 (64-bit).

Ex.1

$$\{yz^3, (a^2b - 2ab^2 + b^3)y^2z - cz^2y, y^3(a^2 - 2ab + b^2) + cz^2y, -ayz^2 + xz^2, \\ (a - b)xy + yz, -byz + cx^2\}$$

Ex.2

$$\{z^4, yz^3, z^2y^2, by^3 + y^2z + (d - 1)z^2y, xz^3, (ac + a - 2)xyz - 2z^2y + z^3a, \\ xy^2(ac + a - 2) + (-a^3c - a^3 + 2a^2)xz^2 + y^2z(ac + a - 2) + 2az^2y \\ +(-a^3bc - a^3c^2 - a^3b - a^3c + 2a^2b + 2a^2c - a^2)z^3, ax^2 - xy - yz\}$$

Ex.3

$$\{z^4, yz^3, y^2z^2, zy^3, y^4, (a^2d^2 - abd^2 + ad^2 - bd^2 + 2a^2 - 2ab + ad \\ -bd - 2a + 2b - 2c + 2d)xz^2 + (a^2bd^2 - ab^2d^2 + a^2d^2 - b^2d^2 + 2a^2b \\ -2ab^2 + abd + ad^2 - b^2d - bd^2 + cd^2 - d^3 + 2a^2 - 4ab + ad + 2b^2$$

$$-2bc+bd-cd+d^2-2a+2b-2c+2d)y^3+(a^2cd^2-a^2d^3-a^2cd$$
$$+a^2d^2+ad^3-ad^2-cd^2+2d^3+2ad+cd-d^2-2a-3d+2)z^2y,$$
$$(a^2d^2-abd^2+ad^2-bd^2+2a^2-2ab+ad-bd-2a+2b-2c$$
$$+2d)xyz+(ad-bd)y^3+(a^3d-a^2bd-ad+bd+2)z^2y,(a^2d^3$$
$$-abd^3+ad^3-bd^3+2a^2d-2abd+ad^2-bd^2-2ad+2bd-2cd$$
$$+2d^2)xy^2+(-a^2+ab+a-b+c-d)y^3+(a^3d^2-a^2bd^2+2a^3$$
$$-2a^2b+a^2d-abd-ad^2+bd^2-4a^2+4ab-2ac+ad+bd+2a$$
$$-2b+2c-2d)y^2z+(-a^4+a^3b+a^3-a^2b+a^2c-a^2d+a^2-ab$$
$$+ad-a+b-c+2d+1)z^2y,x^2-dxy+(1-a)yz\}$$

Ex.4

$$\{z^5,yz^4,y^2z^3,zy^3+(-ac-a)yz^3+(ab-a)z^4,(ac-a+2c)y^4$$
$$+(-2a^2c^2+a^2c-3ac^2)y^2z^2+(-a^2d+a^2-ac-ad)yz^3+(a^3c^3$$
$$+a^2c^3+a^2bd-a^2b-a^2d+abc+abd-ad)z^4,acz^2+axz-y^2,(a^2c$$
$$-a^2+2ac)xy^3+(-2a^4c^2+a^4c-5a^3c^2+a^2c^3+a^3c-a^2bc-3a^2c^2$$
$$+a^2b-a^2c-3abc+a^2+ab-2ac-2bc)y^2z^2+(-a^4d-a^3c^2-a^3cd$$
$$+a^4-a^3d-3a^2c^2-a^2cd+a^3-a^2c-a^2d)yz^3+(a^5c^3+2a^4c^3-a^3c^4$$
$$+a^4bd+2a^3bc^2+a^3bcd+a^3c^3-a^4b-a^4d-a^3bc+a^3bd-a^3cd$$
$$+6a^2bc^2+a^2bcd-a^3b-a^3d+a^2bd-a^2cd+2abc^2-a^2d)z^4,(ac-a$$
$$+2c)x^2y^2+y^2z^2c^3+(-acd+ad-c^2-cd)yz^3+(-ac^4+abcd-abd$$
$$-acd+bc^2+bcd+ad-cd)z^4,(a^2+a)x^3+xay^2-y^3+acyz^2+z^3a\}$$

Ex.5

$$\{u^3,u^2z,(a-1)z^2+u^2,bu+y+z,b^2u^2+(c-1)xu+bzu,-bu^2$$
$$+(ab^2-ab-b^2+b)uz+(ad-d)xz\}$$

Ex.6

$$\{u^4,u^3z,(bc-bd-b-c+1)u^2z+(c-d-1)uz^2,(abcd-abd^2-abd)u^3$$
$$+(-b^3c+b^3d+acd-ad^2+b^3+2b^2c-b^2d-ad-2b^2-bc+b)u^2z$$
$$+(bc-bd-cd+d^2-b+d)z^3,bu+y+z,b^2u^2+(c-1)xu+bzu,$$
$$(b^2-b)uz+dxz+bz^2,(-a^2bcd+a^2bd^2+a^2bd+abcd-abd^2-abd)u^3$$
$$+(ab^3c-ab^3d-a^2cd+a^2d^2-ab^3-2ab^2c+ab^2d-b^3c+b^3d+a^2d$$
$$+2ab^2+abc+acd-ad^2+b^3+2b^2c-b^2d-ab-ad-2b^2-bc+b)u^2z$$
$$+(bc-bd-cd+d^2-b+d)x^3\}$$

Ex.7

$$\{u^4,zu^3,u^2z^2,uz^3,z^4,u^3y,zyu^2,uyz^2,yz^3,buy^2+(c-d)uz^2$$
$$+(-b+c)yz^2,y^2z^2,b^3cdzyu+(a^3b^2+a^2b^2d-a^2c^3+a^2c^2d-ac^3d$$
$$+ac^2d^2)uz^2+(a^2b^2d+ab^2d^2)y^3+(b^4c-b^3c^2)y^2z+(a^2bc^2-a^2c^3+abc^2d$$
$$-ac^3d)yz^2+abc^3z^3,(b+c)xu^2+ayz^2+du^3,(a^3bd^2+2a^2bd^3+abd^4)xzu$$
$$+ab^3cdzyu+(a^4b^2+a^3b^2d-a^3c^3+a^3c^2d-a^2b^2d^2+2a^2bcd^2-a^2c^3d$$
$$+a^2c^2d^2-2ab^2d^3+3abcd^3-b^2d^4+bcd^4)uz^2+(ab^4c-ab^3c^2)y^2z$$
$$+(a^3bc^2-a^3c^3-a^2b^3d+3a^2b^2cd-a^2bc^2d-a^2c^3d-2ab^3d^2+5ab^2cd^2$$

$-3abc^2d^2 - b^3d^3 + 2b^2cd^3 - bc^2d^3)yz^2 + a^2bc^3z^3, (a^2c - a^2d + 2acd$
$-2ad^2 + cd^2 - d^3)uz^2 + b^2cxz^2 + b^3y^2z + (-a^2b + a^2c - 2abd + 2acd$
$-bd^2 + cd^2)yz^2 + (-abc - bcd)z^3, axy + (b - c)yz + dzu, (a + d)x^2$
$+by^2 + cxz\}$

Ex.8

$\{u^3z, u^2z^2, uz^3, z^4, u^3y, (ab^2d^2 + a^2bd - 2ab^2d - a^2b + ab^2)u^2z$
$+(a^2bd - cd^3 + a^3 - a^2b + 3cd^2 - d^3 - 3cd + 3d^2 + c - 3d + 1)uyz, yz^3,$
$(-bd^3 + 3bd^2 - 3bd + b)u^2z + (a^2bd - cd^3 + a^3 - a^2b + 3cd^2 - d^3 - 3cd$
$+3d^2 + c - 3d + 1)uy^2, (a^2b^2d^2 - bcd^4 + a^3bd - a^2b^2d + 3bcd^3 - bd^4$
$-3bcd^2 + 3bd^3 + bcd - 3bd^2 + bd)u^2y + (-a^3b^3d - a^4b^2 + a^3b^3)u^2z$
$+(a^2b^2d - bcd^3 + a^3b - a^2b^2 + 3bcd^2 - bd^3 - 3bcd + 3bd^2 + bc - 3bd$
$+b)y^2z + (-a^3bd + acd^3 - a^4 + a^3b - 3acd^2 + ad^3 + 3acd - 3ad^2$
$-ac + 3ad - a)yz^2 + (-a^2bcd^2 + c^2d^4 - a^3cd + a^2bcd - a^2bd^2 - 3c^2d^3$
$+2cd^4 - a^3d + a^2bd + 3c^2d^2 - 6cd^3 + d^4 - c^2d + 6cd^2 - 3d^3 - 2cd$
$+3d^2 - d)z^3, (-a^2b^4d^2 - 2a^3b^3d + 2a^2b^4d - a^4b^2 + 2a^3b^3 - a^2b^4)u^2z$
$+(a^3b^2cd - abc^2d^3 + a^4bc - a^3b^2c + a^3b^2d + 3abc^2d^2 - 2abcd^3 + a^4b$
$-a^3b^2 - 3abc^2d + 6abcd^2 - abd^3 + abc^2 - 6abcd + 3abd^2 + 2abc - 3abd$
$+ab)y^3 + (-a^3bd + acd^3 - a^4 + a^3b - 3acd^2 + ad^3 + 3acd - 3ad^2 - ac$
$+3ad - a)yz^2 + (-a^2bcd^2 + c^2d^4 - a^3cd + a^2bcd - a^2bd^2 - 3c^2d^3 + 2cd^4$
$-a^3d + a^2bd + 3c^2d^2 - 6cd^3 + d^4 - c^2d + 6cd^2 - 3d^3 - 2cd + 3d^2 - d)z^3,$
$(d - 1)ux - ayu, xz + (c + 1)y^2 + bzu, axy^2 + dz^3, bxy + x^2 + yz\}$

Ex.9

$\{uz^3, z^4, u^3y, uyz, yz^3, uy^2, (abd - bcd)u^2y + (ab - bc)y^2z$
$+(-a^2 + ac)yz^2 + (-acd - ad)z^3, (abc - bc^2 + ab - bc)y^3$
$+(-a + c)yz^2 + (-cd - d)z^3, (d - 1)ux - ayu, xz + (c + 1)y^2,$
$(a - c)xy^2 + dz^3, bxy + x^2 + yz\}$

Ex.10

$\{u^8, u^7y, yzu^5, u^5y + u^3yz^2, uyz^3, (-b^5 + b^4c - b^4 - 2b^3c + 4b^2c^2 - bc^3$
$-4b^2c + 5bc^2 - c^3 - bc + c^2 - c)u^6y + (b^6 - 2b^5c + b^4c^2 + 4b^5 - 8b^4c$
$+4b^3c^2 + 6b^4 - 12b^3c + 6b^2c^2 + 4b^3 - 8b^2c + 4bc^2)yz^6, (b - c)uy^2 + yzu,$
$(b^4 + 6b^2c - 4bc^2 + c^3 + 4bc - c^2 + c)u^6y + (b^6 - 2b^5c + b^4c^2 + 4b^5 - 8b^4c$
$+4b^3c^2 + 6b^4 - 12b^3c + 6b^2c^2 + 4b^3 - 8b^2c + 4bc^2)y^2z^5, (-b^2 - c)yzu^4$
$+(b - c)y^3z^3 + (-b^3 + b^2c - 2b^2 + 2bc - b + c)yz^5, u^4y + u^2yz^2$
$+(1 + b)y^4z + y^3z^2, -u^4y - u^2yz^2 + (b^2 + b)y^5 - y^3z^2 + (1 + b)yz^4,$
$(b^2 - 2bc + c^2)u^4x + cu^2yz^2 + (-b^3 + 2b^2c - bc^2 - b^2 + 2bc - c^2)y^2z^3$
$+(-b^2 + 2bc - c^2)yz^4, (b - c)u^6 + (-b^3 - b^2 - bc - c)yzu^4$
$+(b - c)u^2xz^3 + (-b + c)z^4y^2 + (-b^4 + b^3c - 3b^3 + 3b^2c - 3b^2$
$+3bc - b + c)yz^5, u^4z + (-b + c)u^2yz^2 + xz^4 + y^3z^2, -u^4 + u^2xy$
$-xz^3 - y^3z, uxyz + (-b + c)z^2yu, -by^4 + xyz^2 - y^3z, xy^2 + yz^2,$

$by^2 + x^2 + yz\}$

Ex.11

$$\{u^7y, yzu^5, (b^2 - 2bc + c^2)u^5y + cu^3yz^2, uyz^3, (-b^3 + 3b^2c - 3bc^2 + c^3 + 1)u^6y + bcyz^6, (b - c)uy^2 + yzu, (-b^2 + 2bc - c^2)u^4y - cu^2yz^2 + (b^3 - 2b^2c + bc^2)y^2z^3 + (b^2 - 2bc + c^2)yz^4, by^5 + y^4z + yz^4, (b^3 - b^2c)u^4xy + (2b - c)yzu^4 + (-b^4 + b^3c)y^4z^2 + (b^4 - b^3c - b + c)yz^5, uxyz + (-b + c)z^2yu, -by^4 + xyz^2 - y^3z, xy^2 + yz^2, by^2 + x^2 + yz\}$$

Ex.12

$$\{z^2u^3y, uyz^3, yz^6, (b - c)uy^2 + zuyc, y^2z^5, (b^3 - b^2 - c^2)y^3z^3 + (3b^2c - 2bc)z^4y^2 + (2bc^2 - c^2)yz^5, by^2z^3 + cyz^4 + y^4z, c^3u^2z^2y + (b^4 - 2b^3c + b^2c^2 - b^3 + 2b^2c - bc^2)y^5 + (b^2c^2 - 2bc^3 + c^4)y^3z^2 + (-2b^4c + 4b^3c^2 - 2b^2c^3 + b^3c - 2b^2c^2 + bc^3)y^2z^3 + (-2b^3c^2 + 4b^2c^3 - 2bc^4 + b^2c^2 - 2bc^3 + c^4)yz^4, xz^2 + y^2z, -by^2z^2 - cyz^3 + xy^2z, by^2 + cyz + x^2\}$$

Ex.13

$$\{u^4yz^2, b^2cuyz^3 + u^2yz^2, (-b^9 - 3b^6 - 3b^3 - 1)z^2u^3y + (b^5c^3 + b^3c^3)yz^5, (b^4 + b)y^2u^2 + (b^3c + c)yu^2z + b^2c^2uyz^2, (b^3 + 1)y^2z + b^2cyz^2, b^7c^2u^3yz^2 + (b^8 + b^6 + 2b^5 + 2b^3 + b^2 + 1)y^6, bxz + yz, (b^9 + 3b^6 + 3b^3 + 1)xyu^3 + (-b^11 + b^10 - 3b^8 + 3b^7 - 3b^5 + 3b^4 - b^2 + b)y^5 + (b^7c^4 - b^6c^4 - b^4c^4)yz^4, -b^6c^2u^3yz^2 + (b^8 + b^6 + 2b^5 + 2b^3 + b^2 + 1)xy^5, by^2 + cyz + x^2\}$$

Ex.14

$$\{z^4u, uyz^2c + (-a + b - c)z^3u, (a - b)y^2 + yz - z^2, u^5xz, u^3xz^2, (abc - b^2c)u^3xz + (ad - bd + cd + d)uxz^3, xz^5, u^4xy, u^3xyz, (-a^2 + 2ab - b^2)xyu^3 + (a^2b^2 - 2ab^3 + b^4 + a^2b - ab^2 + ab)xyz^3 - abxz^4, -bxz + x^2\}$$

Ex.15

$$\{u^5, u^4z, (a^2bcd + 2abcd - bcd^2 + bcd)u^4 + (-a^2cd + a^2b - 2acd + cd^2 + 2ab - cd + b)u^2z^2, (ab^2 + b^2)u^4 + (-a^2bc^2d^2 + 2a^2b^2cd - 2abc^2d^2 + bc^2d^3 - a^2b^3 + 4ab^2cd - b^2cd^2 - bc^2d^2 - 2ab^3 + 2b^2cd - b^3)u^3z + (a^2c^2d^2 - 3a^2bcd + 2ac^2d^2 - c^2d^3 + 2a^2b^2 - 6abcd + 2bcd^2 + c^2d^2 + 4ab^2 - 3bcd + 2b^2)uz^3, (-a^2b^2d - 2ab^2d - b^2d)u^4 + z^4(-a^2cd + a^2b - 2acd + cd^2 + 2ab - cd + b), (-a^2cd + 2a^2b - 2acd + cd^2 + 4ab - bd - cd + 2b)u^4 + (-a^2bcd + a^2b^2 - 2abcd + bcd^2 + 2ab^2 - bcd + b^2)yu^3, (-ab^2cd^2 + ab^3d - b^2cd^2 + b^3d)u^4 + (a^2b^3cd^2 - a^2b^4d + 2ab^3cd^2 - b^3cd^3 - 2ab^4d + b^3cd^2 - b^4d)u^3z + (a^2c^2d^2 - 3a^2bcd + 2ac^2d^2 - c^2d^3 + 2a^2b^2 - 6abcd + 2bcd^2 + c^2d^2 + 4ab^2 - 3bcd + 2b^2)u^2yz, u^3 + yz^2, bdu^3 - duz^2$$

$$+uy^2, cy^3 + uz^2, (ab^2 + b^2)u^4 + (a^2c^2d^2 - 3a^2bcd + 2ac^2d^2 - c^2d^3 + 2a^2b^2$$
$$-6abcd + 2bcd^2 + c^2d^2 + 4ab^2 - 3bcd + 2b^2)u^3x + (-a^2b^2cd + a^2b^3$$
$$-2ab^2cd + b^2cd^2 + 2ab^3 - b^2cd + b^3)u^3z, (ab^2 + b^2)u^3 + u^2z + (a + 1)uxz$$
$$+buyz + (-ab - b)z^2u, byz + uz + xy, (a + 1)x^2 + ux\}$$

The results are shown in Table 1, with timings recorded on the aforementioned system configuration. The table columns represent:

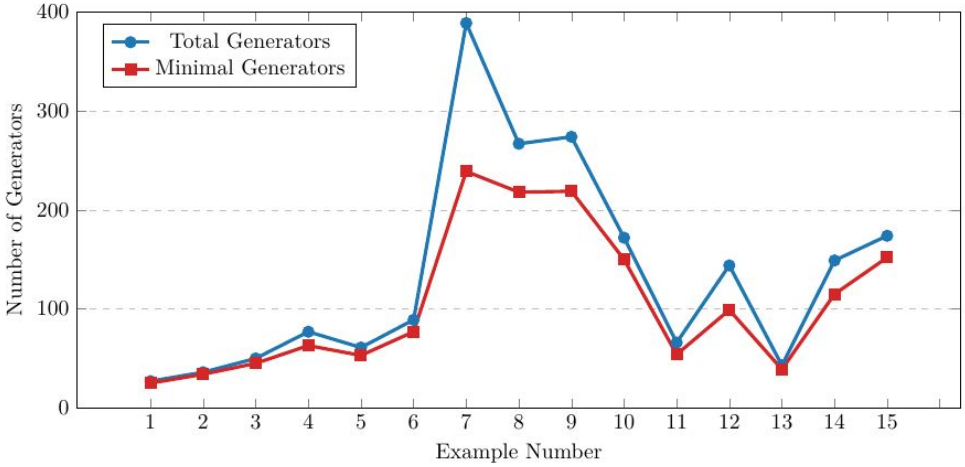
- **Example**: Identification number for each test case
- **Timing (s)**: CPU time in seconds for the complete computation
- **Memory (GB)**: Peak memory usage during computation
- **Branches**: Number of branches in the computed Gröbner system
- **Total Generators**: Sum of generators across all branches before minimization
- **Minimal Generators**: Sum of generators after minimization
- **Reduction**: Difference between total and minimal generators

TABLE 1. Performance results for the MGSYSTEM algorithm

Ex.	Time (s)	Mem (GB)	Branches	Total Gen	Min Gen	Reduction
1	0.082	0.015	7	27	25	2
2	0.202	0.040	6	36	34	2
3	0.681	0.147	8	50	45	5
4	2.110	0.550	11	77	63	14
5	0.348	0.038	12	61	53	8
6	0.896	0.195	14	89	77	12
7	158.000	22.550	29	389	239	150
8	47.180	7.240	23	267	218	49
9	10.320	1.830	27	274	219	55
10	62.360	16.290	11	172	150	22
11	2.610	0.510	6	66	54	12
12	23.890	4.360	15	144	99	45
13	23.090	4.110	6	43	39	4
14	22.840	3.910	19	149	115	34
15	12.470	3.200	15	174	152	22

The examples demonstrate several key aspects of the MGSYSTEM algorithm: The algorithm consistently reduces the number of generators across all examples, with reduction rates varying from 7.4% (Ex13) to 38.6% (Ex7) of the original generators. Memory usage scales with problem complexity, ranging from modest requirements (15 MB for Ex1) to substantial needs (22.55 GB for Ex7) for the largest computations. The algorithm handles particularly large reductions effectively, as seen in Ex7 where 150 redundant generators were eliminated from the original 389.

The following run chart provides a visual representation of the performance of the MGSystem algorithm across all 15 test examples. The blue bar denotes the total number of generators in the original Gröbner basis for each parametric ideal, while the red bar shows the minimized number of generators after applying our algorithm. The difference between the blue and red bars (the reduced portion) highlights the algorithm's effectiveness in eliminating redundant generators while preserving the ideal structure. The visualization demonstrates the ability of MGSystem across varying problem sizes and its efficiency maximization in high-redundancy cases. This makes it especially valuable for parametric ideals in fields like algebraic geometry and homological algebra, where minimal representations are essential for downstream computations.



Overall, the results confirm that the MGSystem algorithm provides an effective method for computing minimal generating systems across a wide range of parametric polynomial ideals, with performance characteristics that make it practical for many real-world applications. The significant reductions in generator counts achieved in several examples (particularly Ex7, Ex9, and Ex12) demonstrate the algorithm's ability to substantially simplify representations of parametric ideals.

Acknowledgments

This work was conducted as part of my project at the Institute for Research in Fundamental Sciences (IPM). I am grateful to the reviewers for their valuable comments and suggestions, which significantly improved the manuscript's quality.

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Appendix A. Appendix: Gröbner Basis Computations

This chapter presents the theoretical foundations of Gröbner bases computations, which are central to this paper. We provide an overview of how Gröbner basis computations work in practice.

A.1. **Polynomial Ring & Polynomial Ideal.**

Definition A.1. A **polynomial ring** over a field $\mathbb{K}$ in variables $x_1, \ldots, x_n$ is the set of all polynomials with coefficients in $\mathbb{K}$, denoted by $\mathbb{K}[x_1, \ldots, x_n]$. Each element has the form:

$$p = \sum_{\alpha} c_\alpha x^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $c_\alpha \in \mathbb{K}$.

Definition A.2. For polynomials $f_1, \ldots, f_s$, the **ideal generated by** them is:

$$\langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} g_i f_i \mid g_i \in \mathbb{K}[x_1, \ldots, x_n] \right\}$$

Theorem A.3 (Hilbert Basis Theorem)**.** *Every ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is finitely generated, meaning there exist polynomials $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_s \rangle$.*

A.2. **Monomial Orders.**

Definition A.4. A **monomial order** $\prec$ on $\mathbb{K}[x_1, \ldots, x_n]$ is a total order on monomials satisfying the following properties:

- Multiplicativity: $x^\alpha \prec x^\beta$ implies $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$ for any γ
- Well-ordering: Every non-empty set of monomials has a minimal element

Definition A.5. For monomials $x^\alpha, x^\beta \in \mathbb{K}[x_1, \ldots, x_n]$ with $x_1 \succ x_2 \succ \cdots \succ x_n$:

- **Lexicographic (lex)**:

$$x^\alpha \prec_{\text{lex}} x^\beta \Leftrightarrow \exists\, i \colon \alpha_i < \beta_i \text{ and } \alpha_j = \beta_j \ \forall j < i$$

- **Degree Lexicographic or Graded Lexicographic (grlex)**:

$$x^\alpha \prec_{\text{grlex}} x^\beta \Leftrightarrow \begin{cases} \deg(x^\alpha) < \deg(x^\beta), \text{ or} \\ \deg(x^\alpha) = \deg(x^\beta) \text{ and } x^\alpha \prec_{\text{lex}} x^\beta \end{cases}$$

- **Degree Reverse Lexicographic (drl)**:

$$x^\alpha \prec_{\text{drl}} x^\beta \Leftrightarrow \begin{cases} \deg(x^\alpha) < \deg(x^\beta), \text{ or} \\ \deg(x^\alpha) = \deg(x^\beta) \text{ and } \exists\, i \colon \alpha_i > \beta_i \text{ and } \alpha_j = \beta_j \ \forall j > i \end{cases}$$

Example A.6. Consider the monomials $x^\alpha = x_1^2 x_2^3 x_3$ and $x^\beta = x_1 x_2^4 x_3^2$ in $\mathbb{K}[x_1, x_2, x_3]$ with $x_1 \succ x_2 \succ x_3$.

Order	Relation	Decision Process
lex	$x^\alpha \succ_{\text{lex}} x^\beta$	$\exists i = 1 : 2 > 1$ and $\alpha_j = \beta_j$ for $j < i$ (vacuously true) $\Rightarrow$ First inequality determines order
grlex	$x^\alpha \succ_{\text{deglex}} x^\beta$	$\deg(x^\alpha) = 6 = \deg(x^\beta)$ Then lex comparison: $x^\alpha \succ_{\text{lex}} x^\beta$
drl	$x^\alpha \prec_{\text{drl}} x^\beta$	$\deg(x^\alpha) = 6 = \deg(x^\beta)$ $\exists i = 3 : 1 < 2$ and $\alpha_j = \beta_j$ for $j > i$ (vacuously true) $\Rightarrow$ Rightmost difference determines order

Key Observation: The orders disagree on this pair, showing how the same monomials can be ordered differently.

A.3. **Leading Terms.**

For a polynomial $f = \sum_\alpha c_\alpha x^\alpha$ with monomials ordered by $\prec$:

- Leading monomial: $\text{LM}(f)$ is the largest monomial in f with respect to $\prec$.
- Leading coefficient: $\text{LC}(f)$ is the coefficient of $\text{LM}(f)$
- Leading term: $\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f)$

A.4. **Division by a Single Polynomial.**

Definition A.7. For polynomials f, g, h with $g \neq 0$, we say f **reduces** to h modulo g, denoted $f \xrightarrow{g} h$, if $h = f - \frac{X}{\mathrm{LT}(g)} \cdot g$ for some term X of f divisible by $\mathrm{LM}(g)$.

A.5. **Division by Multiple Polynomials.**

Definition A.8. For polynomials f, h and a set $F = \{f_1, \ldots, f_s\}$, we say f reduces to h modulo F (denoted $f \xrightarrow{F} h$) if there exists a sequence of reductions:

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} \cdots \xrightarrow{f_{i_t}} h$$

Definition A.9. A polynomial r is in **normal form** (reduced) w.r.t. F if either $r = 0$ or no term of r is divisible by any $\mathrm{LM}(f_i)$ for $f_i \in F$.

Algorithm 2 Multivariate Division Algorithm

Require: Polynomials $f, f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$, where $f_i \neq 0$ for all i.
Ensure: $q_1, \ldots, q_s, r$ such that $f = \sum q_i f_i + r$ with r reduced
  Initialize: $q_i \leftarrow 0$ for all i, $r \leftarrow 0$, $h \leftarrow f$
  while $h \neq 0$ **do**
    if $\exists i$ with $\mathrm{LM}(f_i)$ divides $\mathrm{LM}(h)$ **then**
      Choose minimal such i
      $q_i \leftarrow q_i + \mathrm{LT}(h)/\mathrm{LT}(f_i)$
      $h \leftarrow h - (\mathrm{LT}(h)/\mathrm{LT}(f_i))f_i$
    else
      $r \leftarrow r + \mathrm{LT}(h)$
      $h \leftarrow h - \mathrm{LT}(h)$
    end if
  end while
  return $(q_1, \ldots, q_s, r)$

A.6. **Gröbner Bases.**

Definition A.10. A finite subset $G = \{g_1, \ldots, g_t\}$ of I is a **Gröbner basis** for I if:

$$\langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_t) \rangle = \langle \mathrm{LM}(I) \rangle$$

where $\mathrm{LM}(I) = \{\mathrm{LM}(f) \mid f \in I\}$.

A.7. **S-Polynomials.**

Definition A.11. For $f, g \neq 0$, let $L = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$. The **S-polynomial** is:

$$S(f, g) = \frac{L}{\mathrm{LT}(f)} f - \frac{L}{\mathrm{LT}(g)} g$$

A.8. **Buchberger's Algorithm.**

Theorem A.12 (Buchberger's Criterion)**.** *A generating set G of an ideal I is a Gröbner basis iff for all pairs $g_i, g_j \in G$, $S(g_i, g_j) \xrightarrow{G} 0$.*

Algorithm 3 BUCHBERGER'S ALGORITHM

Require: $F = \{f_1, \ldots, f_s\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ with $f_i \neq 0$
Ensure: Gröbner basis G for $\langle F \rangle$
  $G \leftarrow F$
  $\mathcal{P} \leftarrow \{\{f, g\} \mid f \neq g \in G\}$
  while $\mathcal{P} \neq \emptyset$ **do**
    Select $\{f, g\} \in \mathcal{P}$
    $\mathcal{P} \leftarrow \mathcal{P} \setminus \{\{f, g\}\}$
    Compute $S(f, g) \xrightarrow{G} h$ where h is reduced
    if $h \neq 0$ **then**
      $\mathcal{P} \leftarrow \mathcal{P} \cup \{\{h, g\} \mid g \in G\}$
      $G \leftarrow G \cup \{h\}$
    end if
  end while
  return (G)

Example A.13. Let $I = \langle f_1, f_2 \rangle \subseteq \mathbb{Q}[x, y]$ where:

$$f_1 = x^2y - y^2$$
$$f_2 = xy^2 - x^2$$

We compute a Gröbner basis for I with respect to the lexicographic order $y \prec x$.

Step 1: Initialization. Start with $G = \{f_1, f_2\}$.

Step 2: Compute S-polynomials. Compute $S(f_1, f_2)$:

(1) Compute LCM of leading terms:

$$\mathrm{LT}(f_1) = x^2y, \quad \mathrm{LT}(f_2) = xy^2$$

$$\mathrm{LCM}(x^2y, xy^2) = x^2y^2$$

(2) Compute S-polynomial:

$$S(f_1, f_2) = \frac{x^2y^2}{x^2y} f_1 - \frac{x^2y^2}{xy^2} f_2 = yf_1 - xf_2$$

$$= y(x^2y - y^2) - x(xy^2 - x^2)$$

$$= x^2y^2 - y^3 - x^2y^2 + x^3$$

$$= x^3 - y^3$$

Step 3: Reduce S-polynomial modulo G. Reduce $S(f_1, f_2) = x^3 - y^3$ with respect to G:

(1) Divide $x^3 - y^3$ by $\{f_1, f_2\}$:
  - No term is divisible by $\mathrm{LT}(f_1) = x^2y$ or $\mathrm{LT}(f_2) = xy^2$

(2) The remainder is $x^3 - y^3 \neq 0$

Step 4: Update Basis. Add the remainder to G:

$$G = \{f_1, f_2, f_3\} \quad \text{where} \quad f_3 = x^3 - y^3$$

Step 5: Compute New S-polynomials. Now compute all new S-polynomials:

(1) Compute $S(f_1, f_3)$:

$$\text{LCM}(x^2y, x^3) = x^3y$$

$$\begin{aligned} S(f_1, f_3) &= \frac{x^3y}{x^2y} f_1 - \frac{x^3y}{x^3} f_3 \\ &= xf_1 - yf_3 \\ &= x(x^2y - y^2) - y(x^3 - y^3) \\ &= x^3y - xy^2 - x^3y + y^4 \\ &= -xy^2 + y^4 \end{aligned}$$

Reduce modulo G:

$$-xy^2 + y^4 \xrightarrow{f_2} -xy^2 + y^4 + (xy^2 - x^2) = y^4 - x^2$$

Now reduce $y^4 - x^2$ by f_1 :

$y^4 - x^2 + (x^2y - y^2)$ (no reduction possible)

Remainder is $y^4 - x^2 \neq 0$

Add to basis:

$$G = \{f_1, f_2, f_3, f_4\} \quad \text{where} \quad f_4 = y^4 - x^2$$

(2) Compute $S(f_2, f_3)$:

$$\text{LCM}(xy^2, x^3) = x^3y^2$$

$$\begin{aligned} S(f_2, f_3) &= \frac{x^3y^2}{xy^2} f_2 - \frac{x^3y^2}{x^3} f_3 \\ &= x^2f_2 - y^2f_3 \\ &= x^2(xy^2 - x^2) - y^2(x^3 - y^3) \\ &= x^3y^2 - x^4 - x^3y^2 + y^5 \\ &= -x^4 + y^5 \end{aligned}$$

Reduce modulo G (several steps omitted for brevity)...

(3) Continue this process until all S-polynomials reduce to zero.

Final Step: Termination. After computing all remaining S-polynomials and finding they reduce to zero, we obtain the Gröbner basis:

$$G = \{x^2y - y^2, xy^2 - x^2, x^3 - y^3, y^4 - x^2\}$$

This is a Gröbner basis for the ideal I with respect to the lexicographic order $y \prec x$.

Journal of Algebraic Systems

MINIMAL GENERATING SETS FOR PARAMETRIC POLYNOMIAL IDEALS

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مجموعه‌های مولد مینیمال برای ایده‌آل‌های چندجمله‌ای پارامتری

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این مقاله قضیه سیزیجی شرایر را برای ایده‌آل‌های چندجمله‌ای با ضرایب پارامتری تعمیم می‌دهد و الگوریتمی را برای محاسبه مجموعه مولد مینیمال ایده‌آل‌های چندجمله‌ای پارامتری همگن که نسبت به متغیرها، یک پایه گربنر تشکیل می‌دهند، معرفی می‌کند. این الگوریتم، با به‌کارگیری دستگاه‌های گربنر فراگیر، فضای پارامتری را به زیرفضاهای متناهی تقسیم می‌کند و مجموعه مولد مینیمال را برای هر زیرفضا شناسایی و محاسبه می‌کند. الگوریتم‌های موجود در مقاله، در نرم افزارمیپل پیاده‌سازی شده‌اند و ساختار الگوریتم اصلی از طریق یک مثال ساده نشان داده شده است.

کلمات کلیدی: مجموعه مولد مینیمال، دستگاه‌های گربنر، چندجمله‌ای‌های پارامتری، دستگاه مولد مینیمال، قضیه سیزیجی شرایر.