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THE CONCEPT OF (I,J)-COHEN–MACAULAY MODULES

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Abstract. A generalization of the notion of depth of an ideal on a module is introduced by applying the concept of local cohomology modules with respect to a pair of ideals . The concept of (I,J)-Cohen–Macaulay modules is also introduced as a generalization of the concept of Cohen–Macaulay modules . This kind of modules is different from the Cohen–Macaulay modules, as shown in an example. Also an Artinian result is given for such modules.

1. Introduction

Let R be a commutative Noetherian ring, I, J be two ideals of R, and M be an R-module. Many generalizations of the notion of depth of an ideal on a module have been introduced for various purposes, including the depth, f-depth, g-depth, and k-depth $(k \geq -1)$ of an ideal, introduced in [10], [3], [8], and [9], respectively.
In this paper, a new depth related to a pair of ideals on a module is introduced, which concerns the local cohomology modules with respect to the pair of ideals introduced by Takahashi, Yoshino, and Yoshizawa [11].

In Section 2, the concept of depth of a pair of ideals (I, J) is introduced on an R-module M by depth(I, J, M). This invariant equals:

$$\inf \left\{ i \in \mathbb{N}_0 \mid \mathrm{H}^i_{I,J}(M) \neq 0 \right\}$$

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(see Proposition 2.3). We prove some formulas and inequalities for this invariant and examine how it behaves under various ideal theoretic operations and short exact sequences (see Corollary 2.5–Proposition 2.7). We also show that it is less than, or equal to, the ordinary depth of an ideal (see Corollary 2.5). The non-equality is shown in Example 3.4. For a finite R-module M, and any $\mathfrak{a} \in \tilde{W}(I,J)$ with $\mathfrak{a}M \neq M$, we present some conditions under which the $\operatorname{depth}(\mathfrak{a},M)$ equals the $\operatorname{depth}(I,J,M)$ (see Theorem 2.8). It is well-known that

$$\operatorname{depth}(I,M) = \inf\left\{i \in \mathbb{N}_0 \mid \mathrm{H}^i_I(M) \neq 0\right\}.$$

We show that if $t = \operatorname{depth}(I,J,M)$, and $\mathrm{H}^t_I(M) \neq 0$, then $t = \operatorname{depth}(I,M)$ (see Theorem 2.10).

In Section 3, by applying the concept of "depth of a pair of ideals", we introduce the concept of (I,J)-Cohen–Macaulay modules over Noetherian rings, as a generalization of Cohen–Macaulay modules over local rings, for $I = \mathfrak{m}$ and $J = 0$. These two concepts are not the same, as shown in Propositions 3.8 and 3.9, and Example 3.11. Indeed, for a local integral domain R, and a faithful finite R-module M, which is the (I,J)-Cohen–Macaulay module (with $J \neq 0$), if $t = \operatorname{depth}(I,J,M)$, and $\operatorname{Hom}_R\left(R/\mathfrak{m}, \mathrm{H}^t_{I,J}(M))\right) \neq 0$, then M is not Cohen–Macaulay. In Example 3.4, we show that the Grothendieck's non-vanishing theorem does not hold for local cohomology modules with respect to a pair of ideals, and in Example 3.11, we show that if the condition $\operatorname{Hom}_R\left(R/\mathfrak{m}, \mathrm{H}^t_{I,J}(M))\right) \neq 0$ does not hold in Proposition 3.8, the assertions do not necessarily hold. Finally, an Artinian result for M/JM is proved in Proposition 3.14 for a finite (I,J)-torsion R-module M, which is a (I,J)-Cohen–Macaulay module.

2. Depth of (I,J) on modules

In this section, we introduce the concept of depth of a pair of ideals (I,J) on a module M. By [5, Theorem 6.2.7], for an ideal $\mathfrak{a}$ of R and a finite R-module M with $\mathfrak{a}M \neq M$, the $\operatorname{depth}(\mathfrak{a},M)$ is the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M) \neq 0$. Having this in mind, we give the following definition, where $\tilde{W}(I,J)$ denotes the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n.

Definition 2.1. Assume that I, J are two ideals of R, and let M be an R-module. We define the depth of (I,J) on M by

$$\operatorname{depth}(I,J,M) = \inf\left\{\operatorname{depth}(\mathfrak{a},M) \mid \mathfrak{a} \in \tilde{W}(I,J)\right\},$$

if this infimum exists, and ∞ otherwise.

Remark 2.2. If M is a finite R-module such that $\mathfrak{a}M = M$ for any $\mathfrak{a} \in \tilde{W}(I,J)$, then by [5, Exercise 6.2.6] and [11, Theorem 3.2], we get $\mathrm{H}^i_{I,J}(M) = 0$ for all $i \in \mathbb{N}_0$. But if there exists $\mathfrak{a} \in \tilde{W}(I,J)$ such that $\mathfrak{a}M \neq M$, then $\operatorname{depth}(I,J,M) < \infty$. Also it is clear, by definition, that if $J = 0$, then $\operatorname{depth}(I,J,M)$ coincides with $\operatorname{depth}(I,M)$.

Proposition 2.3. *For a finite R-module M, we have the following equalities:*

$$\begin{aligned}\operatorname{depth}(I,J,M) &= \inf\left\{i \in \mathbb{N}_0 \mid \mathrm{H}^i_{I,J}(M) \neq 0\right\}\\ &= \inf\left\{\operatorname{depth}(\mathfrak{p},M) \mid \mathfrak{p} \in \mathrm{W}(I,J)\right\}\\ &= \inf\left\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{W}(I,J)\right\}.\end{aligned}$$

Proof. Assume,

$$s := \operatorname{depth}(I,J,M), \quad \text{and} \quad t := \inf\left\{i \in \mathbb{N}_0 \mid \mathrm{H}^i_{I,J}(M) \neq 0\right\}.$$

Thus there exists $\mathfrak{b} \in \tilde{W}(I,J)$ such that $s = \operatorname{depth}(\mathfrak{b},M)$. Since $\mathrm{V}(\mathfrak{b}) \subseteq \mathrm{W}(I,J)$, so, by [6, Proposition 1.2.10] and [11, Theorem 4.1], we have

$$\begin{aligned}s = \operatorname{depth}(\mathfrak{b},M) &= \inf\left\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{b})\right\}\\ &\geq \inf\left\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{W}(I,J)\right\} = t.\end{aligned}$$

If $t < s$, then $\mathrm{H}^t_{\mathfrak{a}}(M) = 0$ for all $\mathfrak{a} \in \tilde{W}(I,J)$. Thus by [11, Theorem 3.2], we get $\mathrm{H}^t_{I,J}(M) = 0$, which is a contradiction. Thus $t = s$. For the second equality, since $\operatorname{depth}(\mathfrak{a},M) = \inf\left\{\operatorname{depth}(\mathfrak{p},M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\right\}$, the equality follows from definition. Finally, the third equality follows from [11, Theorem 4.1]. □

Comparing the concept of depth of a pair of ideals with the ordinary depth of an ideal, we have Corollary 2.5. To achieve this, we need the following result in [2]. Recall that an R-module M is said to be ZD-module if for every submodule N of M, the set of zero-divisors of M/N is a union of finitely many prime ideals in $\operatorname{Ass}_R(M/N)$ (See [7]).

Proposition 2.4. *Let $\mathrm{H}^i_{I,J}(M) = 0$ for all $i < t$. Then the following statements hold for any $\mathfrak{a} \in \tilde{W}(I,J)$.*

(i) $$\begin{aligned}\operatorname{Ext}^t_R\left(R/\mathfrak{a},M\right) &\cong \operatorname{Hom}_R\left(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(M)\right)\\ &\cong \operatorname{Hom}_R\left(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a},J}(M)\right)\\ &\cong \operatorname{Hom}_R\left(R/\mathfrak{a},\mathrm{H}^t_{I,J}(M)\right).\end{aligned}$$

(ii) $\Gamma_{\mathfrak{a}}\big(\mathrm{H}^t_{\mathfrak{a}}(M)\big) \cong \mathrm{H}^t_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}\big(\mathrm{H}^t_{\mathfrak{a},J}(M)\big) \cong \Gamma_{\mathfrak{a}}\big(\mathrm{H}^t_{I,J}(M)\big)$.

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M) \subseteq \mathrm{H}^i_{\mathfrak{a},J}(M) \subseteq \mathrm{H}^i_{I,J}(M)$ *for all $i \leq t$.*

(iv) $\mathrm{H}^i_{\mathfrak{a},J}(M) = \mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for all $i < t$.*

(v) $\operatorname{Ass}\big(\mathrm{H}^i_{\mathfrak{a}}(M)\big) = \operatorname{Ass}\big(\mathrm{H}^i_{\mathfrak{a},J}(M)\big) \cap \mathrm{V}(\mathfrak{a})$

$= \operatorname{Ass}\big(\mathrm{H}^i_{I,J}(M)\big) \cap \mathrm{V}(\mathfrak{a})$ *for all* $i \leq t$.

(vi) *If* $\mathfrak{a} \neq 0$, *and* M *is* ZD-*module, then there exists a regular* M*-sequence of length* t *contained in* $\mathfrak{a}$.

Proof. See [2, Theorem 3.4]. □

Corollary 2.5. *For a finite* R*-module* M*, and all* $\mathfrak{a} \in \tilde{W}(I,J)$*, we have*

$$\operatorname{depth}(I,J,M) \leq \operatorname{depth}(\mathfrak{a},J,M) \leq \operatorname{depth}(\mathfrak{a},M).$$

In particular, $\operatorname{depth}(I,J,M) \leq \operatorname{depth}(I,M)$.

Proof. Apply Proposition 2.3 and Proposition 2.4 (iii). □

There is an example, which shows that this new invariant is different from the ordinary depth (see Example 3.4).
In the next two propositions, we prove some formulas and inequalities for this invariant. Also we examine how it behaves under various ideal theoretic operations and short exact sequences.

Proposition 2.6. *Let* I, J, $\mathfrak{b}$, $\mathfrak{c}$ *be ideals of* R*, and* M *be a finite* R*-module.*

(i) *If* $J^n \subseteq \mathfrak{b}^m$ *for some* $n,m \in \mathbb{N}$*, then*

$$\operatorname{depth}(I,\mathfrak{b},M) \leq \operatorname{depth}(I,J,M).$$

(ii) *If* $J^n \supseteq \mathfrak{c}^m$ *for some* $n,m \in \mathbb{N}$*, then*

$$\operatorname{depth}(I,J,M) = \operatorname{depth}(I+\mathfrak{c},J,M).$$

(iii) *If* $\sqrt{I} = \sqrt{\mathfrak{b}}$*, then* $\operatorname{depth}(I,J,M) = \operatorname{depth}(\mathfrak{b},J,M)$.

(iv) *If* $\sqrt{J} = \sqrt{\mathfrak{c}}$*, then* $\operatorname{depth}(I,J,M) = \operatorname{depth}(I,\mathfrak{c},M)$.

(v) $\operatorname{depth}(I,J,M) = \operatorname{depth}(\sqrt{I},J,M) =$
$\operatorname{depth}(I,\sqrt{J},M) = \operatorname{depth}(\sqrt{I},\sqrt{J},M)$.

(vi) $\operatorname{depth}(I,J\mathfrak{b},M) = \operatorname{depth}(I,J\cap\mathfrak{b},M)$.

Proof. All of these statements follow easily from Proposition 2.3, and [11, Propositions 1.4, 1.6]. As an illustration, we just prove statement (i). Since $J^n \subseteq \mathfrak{b}^m$, we have $\mathrm{W}(I,J) \subseteq \mathrm{W}(I,\mathfrak{b})$, by [11, Proposition 1.6]. Now, by Proposition 2.3, we get

$$\begin{aligned}\operatorname{depth}(I,\mathfrak{b},M) &= \inf\big\{\operatorname{depth}(\mathfrak{p},M) \mid \mathfrak{p} \in \mathrm{W}(I,\mathfrak{b})\big\}\\ &\leq \inf\big\{\operatorname{depth}(\mathfrak{q},M) \mid \mathfrak{q} \in \mathrm{W}(I,J)\big\} = \operatorname{depth}(I,J,M).\end{aligned}$$

□

Proposition 2.7. *Let* $0 \to U \to M \to N \to 0$ *be an exact sequence of finite* R*-modules. Let* $r := \operatorname{depth}(I,J,U)$*,* $t := \operatorname{depth}(I,J,M)$*, and* $s := \operatorname{depth}(I,J,N)$*. Then*

(i) $t \geq \min\{r, s\}$.
(ii) $r \geq \min\{t, s+1\}$.
(iii) $s \geq \min\{r-1, t\}$.
(iv) *One of the following equalities holds* :

$$t = r \ , \ t = s \ , \ t = r = s \ , \ s = r - 1.$$

Proof. For (i),(ii) and (iii), apply Corollary 2.5, and [6, Proposition 1.2.9].
To prove (iv), suppose that none of the equalities holds. Then only one of the following cases happens:

(a) $r < s < t$; (b) $s < r < t$; (c) $t < r < s$;
(d) $t < s < r$; (e) $s < t < r$; (f) $r < t < s$.

Let $r < s < t$. Then $s + 1 \leq t$ and by (ii), $s + 1 \leq r < s$, which is a contradiction.
If $s < r < t$, then $s \leq r - 1 < t$. Thus by (iii), $r - 1 \leq s$. Therefore, $s = r - 1$, which is a contradiction.
The same method can be applied to the other cases. □

Next, we give some conditions, for which the depth of a pair of ideals equals the ordinary depth.

Theorem 2.8. *Let M be a finite R-module. Let $t := depth(I, J, M)$. Then for any $\mathfrak{a} \in \tilde{W}(I, J)$ with $\mathfrak{a}M \neq M$, the following statements are equivalent:*

(i) $\operatorname{Hom}_R\big(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M)\big) \neq 0$;
(ii) $\operatorname{Hom}_R\big(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a},J}(M)\big) \neq 0$;
(iii) $\operatorname{Ext}^t_R\big(R/\mathfrak{a}, M\big) \neq 0$;
(iv) $\operatorname{Hom}_R\big(R/\mathfrak{a}, \operatorname{H}^t_{I,J}(M)\big) \neq 0$;
(v) $\operatorname{depth}(\mathfrak{a}, M) = t$.

Proof. All the equivalence parts (i)–(iv) are the consequences of Proposition 2.4 (i).
To prove (iii)⇔(v), apply Proposition 2.3, Proposition 2.4, and [6, Proposition 1.2.10 (e)]. □

Theorem 2.9. *Let M be a finite R-module, and $t := \operatorname{depth}(I, J, M)$. Then for any ideal $\mathfrak{b} \in \tilde{W}(I, 0)$ such that $\mathfrak{b}M \neq M$, the following statements are equivalent:*

(i) $\operatorname{Hom}_R\big(R/\mathfrak{b}, \operatorname{H}^t_I(M)\big) \neq 0$;
(ii) $\operatorname{Ext}^t_R\big(R/\mathfrak{b}, M\big) \neq 0$;
(iii) $\operatorname{depth}(\mathfrak{b}, M) = t$;

Proof. Apply [2, Corollary 3.8 (iii)], and Theorem 2.8. □

Theorem 2.10. *Let M be a finite R-module such that $IM \neq M$. Let $t := \operatorname{depth}(I, J, M)$. Then the following statements are fulfilled:*

(i) $\mathrm{H}^t_I(M) \neq 0$, *if and only if* $t = \operatorname{depth}(I, M)$;

(ii) *If* $\mathrm{H}^t_I(M) \neq 0$, *then* $\operatorname{depth}(I, M) \leq \operatorname{depth}(\mathfrak{a}, M)$ *for all* $\mathfrak{a} \in \tilde{W}(I, J)$;

(iii) $\operatorname{depth}(\mathfrak{p}, M) = t$, *for all* $\mathfrak{p} \in \operatorname{Ass}\left(\mathrm{H}^t_{I,J}(M)\right)$;

(iv) $\operatorname{depth}(\mathfrak{p}, M) \leq \operatorname{depth}(\mathfrak{a}, M)$, *for all* $\mathfrak{p} \in \operatorname{Ass}\left(\mathrm{H}^t_{I,J}(M)\right)$ *and all* $\mathfrak{a} \in \tilde{W}(I, J)$;

(v)
$$\begin{aligned}\operatorname{Hom}_R\left(R/I, \mathrm{H}^t_{I,J}(M)\right) &\cong \operatorname{Ext}^t_R\left(R/I, M\right);\\ &\cong \operatorname{Hom}_R\left(R/I, \mathrm{H}^t_I(M)\right);\\ &\cong \operatorname{Hom}_R\left(R/I, M/(x_1, x_2, \cdots, x_t)M\right),\end{aligned}$$
where $x_1, x_2, \cdots, x_t$ *is a poor regular M-sequence in I.*

Proof. (i) The assertion follows from Proposition 2.4 (iv), and [5, Theorem 6.2.7].

(ii) This is an immediate consequence of part (i), and Corollary 2.5.

(iii) Apply Proposition 2.3, and [12, Theorem 3.6].

(iv) Apply part (iii), and Corollary 2.5.

(v) Apply Proposition 2.4 (i), and [6, Lemma 1.2.4]. □

It should be remembered that Bijan-Zadeh, in [4] has introduced the concept of local cohomology with respect to a system of ideals. He showed that these local cohomology modules are a direct limit of certain Koszul cohomology. Moreover, using the Koszul complex, there is a natural generalization of the concept of grade of system of ideals (see [4, Definition 5.3]). On the other hand, by [11, Definition 3.1], and the proof of [11, Theorem 3.2], it is easy to see that the local cohomology modules with respect to a pair of ideals is a special case of local cohomology with respect to a system of ideals. To see this, we can consider $\tilde{W}(I, J)$ as a system of ideals. In the following theorem, we compare the concept of $\operatorname{depth}(I, J, M)$, and the concept of grade of system of ideals in Definition 5.3 of [4], for the system of ideals $\tilde{W}(I, J)$.

Theorem 2.11. *Let I,J be two ideals of R, and let M be a finitely generated R-module. Then in the sense of Definition 5.3 of* [4], $\tilde{W}(I, J)$-$\operatorname{grade}_M R$ *coincides with* $\operatorname{depth}(I, J, M)$ *in our sense.*

Proof. According to Definition 5.3 of [4], we have
$$\begin{aligned}\tilde{W}(I, J)\text{-}\operatorname{grade}_M R &= \mathit{inf}\left\{\operatorname{grade}_M R/\mathfrak{a} \mid \mathfrak{a} \in \tilde{W}(I, J)\right\}\\ &= \mathit{inf}\left\{\operatorname{grade}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\right\}\\ &= \mathit{inf}\left\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\right\}\\ &= \operatorname{depth}(I, J, M).\end{aligned}$$

Note that by a remark before Definition 5.3 of [4],

$$\operatorname{grade}_M R/\mathfrak{a} = \inf\{i \in \mathbb{N}_0 | \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\} = \operatorname{grade}(\mathfrak{a}, M).$$

□

3. (I,J)-Cohen–Macaulay modules

As a generalization of the concept of Cohen–Macaulay modules, we introduce the concept of (I,J)-Cohen–Macaulay modules using the concept of depth of a pair of ideals. The following lemma and proposition have a main role in some results on this section.

Lemma 3.1. *Let R be an integral domain, and $\mathfrak{a}$ be an ideal of R. Let M be a faithful finite R-module of finite Krull dimension d. Then* $\dim M/\mathfrak{a}M = d$*, if and only if* $\mathfrak{a} = 0$.

Proof. The assertion is obvious, since $(0) \in \operatorname{Supp}(M)$, and if $\mathfrak{a} \neq 0$, then $(0) \notin \operatorname{Supp}(M/\mathfrak{a}M)$. □

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a local integral domain, $I + J$ be an $\mathfrak{m}$-primary ideal, and M be a faithful finite R-module of Krull dimension d. Then the following statements are equivalent:*

(i) $\dim M/JM = d$*;*
(ii) $J = 0$*;*
(iii) $\operatorname{H}^d_{I,J}(M) \neq 0$.

Proof. Apply Lemma 3.1, and [11, Theorem 4.5]. □

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a local integral domain of dimension n, and J be a non-zero proper ideal of R. Then $\operatorname{H}^n_{I,J}(N) = 0$, for any R-module N.*

Proof. Since $J \neq 0$, then $\dim R/J < n$, by Lemma 3.1. Now, the assertion follows from [11, Corollary 4.4]. □

Example 3.4. By [5, Corollary 6.2.9], if $(R, \mathfrak{m})$ is a regular local ring of dimension n, then n is the unique integer i, for which $\operatorname{H}^i_{\mathfrak{m}}(R) \neq 0$. This result may not be true for local cohomology modules with respect to a pair of ideals. To see this, let k be a field, and $R := k[[x, y]]$. Let $\mathfrak{m} := (x, y)R$, and $J := (x)R$. Then R is a regular local ring, with $\dim R = 2 > \dim R/J = 1$. Thus $\operatorname{H}^2_{\mathfrak{m},J}(R) = 0$, by Corollary 3.3. Moreover, $\Gamma_{\mathfrak{m},J}(R) = 0$. Therefore, by Proposition 2.4 (vi), and Grothendieck's non-vanishing Theorem, we get $\operatorname{H}^1_{\mathfrak{m},J}(R) \neq 0$. Therefore, by Proposition 2.3, $\operatorname{depth}(\mathfrak{m}, J, R) = 1 = \dim R/J < \dim R = \operatorname{depth} R$. This motivates us to give the following definition:

Definition 3.5. Let R be a Noetherian ring, and I, J be two ideals of R. A finite R-module M is called an (I, J)-Cohen–Macaulay R-module, if $M \neq 0$, and $\operatorname{depth}(I, J, M) = \dim M/JM$ or if $M = 0$. If R itself is an (I, J)-Cohen–Macaulay module, we say that R is an (I, J)-Cohen–Macaulay ring.

By definition, it is obvious that if M is an (I, J)-Cohen–Macaulay module, then J is a proper ideal. Also if $(R, \mathfrak{m})$ is local, $I = \mathfrak{m}$, and $J = 0$, then the concept of (I, J)-Cohen–Macaulay R-modules coincide with Cohen–Macaulay R-modules (see [6, Definition 2.1.1]).

Remark 3.6. By Corollary 2.5, and [11, Theorem 4.3], if $(R, \mathfrak{m})$ is a local ring, then any (I, J)-Cohen–Macaulay R-module M is $(\mathfrak{m}, J)$-Cohen–Macaulay R-module. Moreover, by [11, Proposition 1.4 (6),(7)], if $I + J$ is $\mathfrak{m}$-primary, then M is (I, J)-Cohen–Macaulay, if and only if M is $(\mathfrak{m}, J)$-Cohen–Macaulay.

Proposition 3.7. *Let M be a finite R-module. If M is* (I, J)-Cohen–Macaulay, *then M is* $(\mathfrak{a}, J)$-Cohen–Macaulay *for any $\mathfrak{a} \in \tilde{W}(I, J)$.*

Proof. The assertion follows easily from Corollary 2.5. □

In the next two propositions, we show that the two concepts (I, J)-Cohen–Macaulay and Cohen–Macaulay modules are not the same.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a local integral domain, and $J \neq 0$. Let M be a faithful finite R-module, and let $t = \operatorname{depth}(I, J, M)$ be such that $\operatorname{Hom}_R\left(R/\mathfrak{m}, \mathrm{H}^t_{I,J}(M)\right) \neq 0$. If M is (I, J)-Cohen–Macaulay, then M is not Cohen–Macaulay.*

Proof. By Remark 3.6, M is $(\mathfrak{m}, J)$-Cohen–Macaulay. Now, the assertion follows from the definition, Theorem 2.8 for $\mathfrak{a} = \mathfrak{m}$, and Lemma 3.1. □

Proposition 3.9. *Let $(R, \mathfrak{m})$ be a local integral domain, and $J \neq 0$. Let M be a faithful finite R-module, and let $t = \operatorname{depth}(\mathfrak{m}, J, M)$ be such that $\mathrm{H}^t_{\mathfrak{m}}(M) \neq 0$. If M is $(\mathfrak{m}, J)$-Cohen–Macaulay, then M is not Cohen–Macaulay.*

Proof. Apply Theorem 2.10, and Lemma 3.1. □

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a local ring, and I, J be the proper ideals of R. Let $M \neq 0$ be an* (I, J)-Cohen–Macaulay *R-module. If $t = \operatorname{grade}(I, J, M) \geq 1$, then $\mathrm{H}^t_{I,J}(M)$ is not finite.*

Proof. Since M is (I, J)-Cohen–Macaulay, so $\dim M/JM = t$. Now, the assertion follows from [1, Corollary 4.12 (ii)]. □

Remark 3.11. It is considerable that, if $\operatorname{Hom}_R\left(R/\mathfrak{m}, \mathrm{H}^t_{I,J}(M)\right)=0$ in Proposition 3.8, and $\mathrm{H}^t_{\mathfrak{m}}(M)=0$ in Proposition 3.9, then the assertions do not necessarily hold. For example, let k be a field, $R:=k[[x,y]]$, $\mathfrak{m}:=(x,y)R$, and $J:=(x)R$.

(i) R is a Cohen–Macaulay ring, and, as we saw in Example 3.4, $\operatorname{depth}(\mathfrak{m},J,R)=1$. thus R is an $(\mathfrak{m},J)$-Cohen–Macaulay ring.
(ii) By Proposition 3.10, and Example 3.4, $\mathrm{H}^1_{\mathfrak{m},J}(R)$ is not finite.
(iii) If $I:=(x^2)R$, then $\Gamma_{I,J}(R)=R$, and so R is not an (I,J)-Cohen–Macaulay.

The next result can be a generalization of [5, Corollary 6.2.8], which gives a bound for non-vanishing of local cohomology modules in the local case.

Proposition 3.12. *Let M be a non-zero finite module over the local ring $(R,\mathfrak{m})$, and $J\neq R$. Then any integer i, for which $\mathrm{H}^i_{\mathfrak{m},J}(M)\neq 0$, must satisfy*

$$\operatorname{depth}(\mathfrak{m},J,M)\leq i\leq \dim M/JM,$$

while, for i at either extremity of this range, we do have $\mathrm{H}^i_{\mathfrak{m},J}(M)\neq 0$.

Proof. Apply Proposition 2.3, and [11, Theorem 4.5]. □

An immediate consequence of Proposition 3.12 is the following result.

Corollary 3.13. *Let $(R,\mathfrak{m})$ be a local ring, and $M\neq 0$ be a finite R-module. Suppose that $I+J$ is an $\mathfrak{m}$-primary ideal. Then there is exactly one integer i, for which, $\mathrm{H}^i_{I,J}(M)\neq 0$ if and only if $\operatorname{depth}(I,J,M)=\dim M/JM$, i.e. if and only if M is an (I,J)-Cohen–Macaulay R-module.*

Proposition 3.14. *Let M be a finite (I,J)-torsion R-module. If M is (I,J)-Cohen–Macaulay, then M/JM is Artinian.*

Proof. It is enough to show that $\operatorname{depth}(I,J,M)=0$. Since $\Gamma_{I,J}(M)=M$, $\mathrm{H}^i_{I,J}(M)=0$, for all $i\geq 1$, by [11, Corollary 1.13]. Thus $\operatorname{depth}(I,J,M)=0$, as required. □

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References

1. M. Aghapournahr, Kh. Ahmadi-Amoli and M. Y. Sadeghi, Cofiniteness and Artinianness of certain local cohomology modules, To appear in *Ricerche di Matematica.*
2. M. Aghapournahr, Kh. Ahmadi-Amoli and M. Y. Sadeghi, Some results on the local cohomology modules with respect to a pair of ideals, *Advances and Applications in Mathematical Sciences* **13** (2014), 203-214.
3. Kh. Ahmadi-Amoli, Filter regular sequences, local cohomology modules and singular sets, Ph. D. Thesis, University for Teacher Education, Iran (1996).
4. M. H. Bijan-Zadeh, A commen generalization of local cohomology theories, *Glasgow Math. J.* **21**(1980), 173-181.
5. M. P. Brodmann and R.Y. Sharp, Local cohomology : An Algebraic Introduction with Geometric Applications, Cambridge University Press, 1998.
6. W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, revised ed., 1998.
7. K. Divaani-Aazar and M. A. Esmkhani, Artinianness of local cohomology modules of ZD-modules, *Comm. Algebra* **33** (2005), 2857–2863.
8. L. Melkersson, Some applications of a criterion for artinianness of a module, *J. Pure. Appl. Algebra.* **101**(3) (1995), 291–303.
9. L. T. Nhan, On generalized regular sequences and the finiteness for associated primes of local cohomology modules, *Comm. Algebra* **33**(3) (2005), 793–806.
10. D. Rees, The grade of an ideal or module, *Proc. Camb. Philos. Soc.* **53** (1957), 28–42.
11. R. Takahashi, Y. Yoshino, and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, *J. Pure. Appl. Algebra* **213** (2009), 582–600.
12. A. Tehranian and A. P. Talemi, Cofiniteness of local cohomology based on a nonclosed support defined by a pair of ideals, *Bull. Iranian Math. Soc.* **36** (2010), 145–155.

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THE CONCEPT OF (I, J)-COHEN–MACAULAY MODULES

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مفهوم مدولهای (I, J)-کوهن مکولی

محرم آقاپورنهر[1]، خدیجه احمدی آملی [2]و میر یوسف صادقی [3]
اراک-خیابان شهید بهشتی-دانشگاه اراک-دانشکده علوم-گروه ریاضی-کدپستی ۳۸۱۵۶۸۳۸۴۹ [1]
تهران-حکیمیه-خیابان بامتری شیرازی-دانشگاه پیام نور مرکز تهران شرق [2,3]

در این مقاله تعمیمی از تعریف نمرهء یک ایدآل روی یک مدول، با بکار بردن مفهوم مدولهای کوهمولوژی موضعی تعریف شده روی یک جفت ایدآل معرفی میگردد. همچنین بعنوان تعمیمی از تعریف مدولهای کوهن مکولی، مفهوم مدولهای (I, J)-کوهن مکولی معرفی میگردد و با مثالی نشان داده میشود این مدولها بطور اکید شامل مدولهای کوهن مکولی است. در پایان یک نتیجه آرتینی بودن برای این نوع مدولها ثابت میشود.

کلمات کلیدی: مدولهای کوهمولوژی موضعی، نمره یک جفت ایدآل، مدولهای (I, J)-کوهن مکولی، مدولهای کوهن مکولی.