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GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

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Abstract. The rank-k numerical range has a close connection to the construction of quantum error correction code for a noisy quantum channel. For a noisy quantum channel, a quantum error correcting code of dimension k exists, if and only if the associated joint rank-k numerical range is non-empty. In this paper, the notion of joint rank-k numerical range is generalized, and some statements of [2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.] are extended.

1. Introduction

Let M_n be the set of $n \times n$ complex matrices, and $A \in M_n$. Furthermore, assume that $k \in \{1, ..., n\}, \alpha \subset \{1, \cdots, n\}$. Throughout this paper, the following notations are fixed:

$$\omega_k = \exp\left(\frac{2\pi i}{k}\right)$$
$$\Omega_k = \left\{\omega_k^0, \omega_k^1, \cdots, \omega_k^{k-1}\right\}.$$

Besides, the symbol $\sigma(A)$ stands for the spectrum of the matrix A, and $A\alpha$ refers to the principal submatrix of A that lies in the rows and columns of A indexed by α.

Recently, the joint higher rank numerical range [5] has played a key role in finding quantum error correcting codes[4], and some researchers

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have taken this into consideration. The present article mainly concentrates on extending the notion of joint rank-k numerical range of $A = (A_1, ..., A_m) \in M_n^m$, i.e. the set of all $(a_1, ..., a_m) \in \mathbb{C}^m$, such that there exists the orthogonal projector P of rank k that satisfies

$$PA_jP = a_jP \ \forall j.$$

Definition 1.1. Let $A = (A_1, ..., A_m) \in M_n^m, B \in M_k$, and $k \leq n$. Then the set

$$\begin{aligned}&{}_B\Lambda_k(A) = \\ &\{(a_1, ..., a_m) \in \mathbb{C}^m : \exists U \in M_{n,k}, s.t., U^*U = I_k, U^*A_jU = a_jB \ \forall j\}\end{aligned}$$

is called the joint matrix higher rank numerical range.

When $B = diag(b_1, ..., b_k)$, we abbreviate ${}_B\Lambda_k(A)$ as ${}_{b_1,...,b_k}\Lambda_k(A)$, and in the case $b_1 = k = 1$, the joint numerical range of A is defined as $W(A) =_{b_1,...,b_k} \Lambda_k(A)$.

Definition 1.2. Let $A = (A_1, \cdots, A_m) \in M_n^m, k \leq n$. The kth joint matrix numerical range of A is the set

$$W_k(A) = \{(U^*A_1U, U^*A_2U, \cdots, U^*A_mU) : U \in M_{n,k}, U^*U = I_k\}$$

In [3], The authors have introduced "$k-$generalized projector". They have said that $A \in M_n$ is the $k-$generalized projector, if $A^k = A^*$ and $k > 1$.

Theorem 1.3. [3] *Let $A \in M_n$, and $k \in \mathbb{N}$, $k > 1$. Then the following statements are equivalent:*

(a) *A is a $k-$generalized projector.*
(b) *A is a normal matrix, and $\sigma(A) \subset \{0\} \cup \Omega_{k+1}$.*

Now, it is natural to extend "joint higher rank numerical range" as follows:

Definition 1.4. Let $A = (A_1, ..., A_m) \in M_n^m$, and k and k' are positive integers. Then the set

$$\left\{ (a_1, ..., a_m) \in \mathbb{C}^m : \begin{array}{l} \exists k' - generalized\ projector\ of\ rank\ k\ \ (P), \\ s.t., PA_jP = a_jP \ \forall j \end{array} \right\}$$

is called the $k'-$generalized joint rank-k numerical range, and is abbreviated as $G\Lambda_{k',k}(A)$.

Notice that the recent definition is an obvious extension of "generalized higher rank numerical range," which has been defined in [1].

2. Main results

The proof of the following results is elementary and hence, we leave it to the interested reader.

Proposition 2.1. *Let* $A = (A_1, \cdots, A_m) \in M_n^m, k \leq n,$ *and* $1 < k'$. *The following statements are equivalent:*

(i) $a = (a_1, a_2, \cdots, a_m) \in G\Lambda_{k',k}(A_1, A_2, \cdots, A_m)$.

(ii) *There exist* $b_1, \cdots, b_k \in \Omega_{k'+1}$ *and unitary matrix* $U \in M_n$, *such that for any* $j \in \{1, \cdots, m\}$,

$$(U^* A_j U)\{1, 2, \cdots, k\} = a_j \operatorname{diag}\left(\{b_i\}_{i=1}^{k}\right).$$

(iii) *There exist* $b_1, \cdots, b_k \in \Omega_{k'+1}$, *and* $X = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \in M_{n,k}$, *such that* $X^* X = I_k$, *and*

$$\forall j \in \{1, \cdots, m\}, X^* A_j X = a_j \operatorname{diag}\left(\{b_i\}_{i=1}^{k}\right).$$

(iv) *There exists* $b_1, \cdots, b_k \in \Omega_{k'+1}$, *and orthonormal vectors* $u_1, \cdots, u_k \in \mathbb{C}^n$, *such that*

$$\forall r \in \{1, \cdots, m\} \forall i, j \in \{1, \cdots, k\}, \langle A_r u_i, u_j \rangle = a_r b_i \delta_{ij}.$$

Proof. One can deduce, from Theorem 1.3, the equivalency of (i) and (ii). Notice that P is a $k'-$generalized projector of rank k, if and only if there exists a unitary matrix U, and numbers $b_1, \cdots, b_k \in \Omega_{k'+1}$ such that

$$P = U^* \operatorname{diag}\left(b_1, \cdots, b_k, \underbrace{0, \cdots, 0}_{n-k,\, 0's}\right) U.$$

Equivalence of parts (ii), (iii), and (iv) is obvious. □

Corollary 2.2. *Let* $A = (A_1, \cdots, A_m) \in M_n^m$. *Then:*

(i) $\Lambda_k(A) \subset G\Lambda_{k',k}(A)$;

(ii) $G\Lambda_{k',k}(A) \subset G\Lambda_{k',k}((A_1, \cdots, A_{m-1})) \times \mathbb{C}$;

(iii) $G\Lambda_{k',k+1}(A) \subset G\Lambda_{k',k}(A)$;

Proof. (i) is trivial, since every orthogonal projector of rank k is a $k'-$generalized projector of rank k.
(ii) is obvious.
(iii) Assume that $(\lambda_1, \cdots, \lambda_m) \in G\Lambda_{k',k+1}(A)$. Then there exist orthonormal vectors $u_1, \cdots, u_{k+1} \in \mathbb{C}^n$, and $b_1, \cdots, b_{k+1} \in \Omega_{k'+1}$, such that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$, for $r \in \{1, \cdots, m\}$, $1 \leq i, j \leq k+1$. Therefore, by considering the orthonormal vectors $u_1, \cdots, u_k \in \mathbb{C}^n$ and $b_1, \cdots, b_k \in \Omega_{k'+1}$, we see that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$ for $r \in \{1, \cdots, m\}$, $1 \leq i, j \leq k$, and therefore, $(\lambda_1, \cdots, \lambda_m) \in G\Lambda_{k',k}(A)$. □

Corollary 2.3. *Let* $A = (A_1, \cdots, A_m) \in M_n^m$*, and* $k', k > 1$*. If* $n \geq (k-1)(m+1)^2$*, then* $G\Lambda_{k',k}(A) \neq \emptyset$*.*

Proof. It suffices to consider [5, Proposition 2.4], and Corollary 2.2(i). □

Proposition 2.4. *Let* $k' > 1$*, and* $A \in M_n^m$*. Then:*

(i) $G\Lambda_{k',k}(A) = \bigcup_{b_1,\cdots,b_k \in \Omega_{k'+1}} {}_{b_1,\cdots,b_k}\Lambda_k(A)$;

(ii) $G\Lambda_{k',1}(A) = \bigcup_{b \in \Omega_{k'+1}} bW(A)$;

Proof. By definition, (i), and (ii) can readily be verified. □

Corollary 2.5. *Consider the Pauli matrices:*

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and let $A_1, \cdots, A_m \in \{I, X, Y, Z\}$*, and* $k' > 1$*. Then:*

$$G\Lambda_{k',1}(I_{2^n}, A_1 \otimes \cdots \otimes A_n) = \begin{cases} \bigcup_{b \in \Omega_{k'+1}} \{(b, b)\} & : \{A_1, \cdots, A_m\} = \{I\} \\ \bigcup_{b \in \Omega_{k'+1}} b\{(1, a) : a \in [-1, 1]\} & : elsewhere \end{cases}$$

Proof. It suffices to note that Pauli matrices are normal, and:

$$\sigma(X) = \sigma(Y) = \sigma(Z) = \{-1, 1\}.$$

□

The proofs of the next two results are straightforward, and thus are omitted.

Corollary 2.6.

$$G\Lambda_{k',1}(\{diag(a_{i1}, \cdots, a_{in})\}_{i=1}^m) = \bigcup_{b \in \Omega_{k'+1}} b\text{conv}\left(\{(a_{1j}, \cdots, a_{mj})\}_{j=1}^n\right)$$

Lemma 2.7. *Let* $A_j = diag(a_{1j}, \cdots, a_{nj}), j = 1, \cdots, m$*. Then:*

$$\begin{aligned} G\Lambda_{k',n}(A_1, \cdots, A_m) \subset & \left(\bigcup_{b_1,\cdots,b_n \in \Omega_{k'+1}} \left\{ c_1 : c_1 = \tfrac{a_{11}}{b_1} = \cdots = \tfrac{a_{n1}}{b_n} \right\} \right) \\ & \times \left(\bigcup_{b_1,\cdots,b_n \in \Omega_{k'+1}} \left\{ c_2 : c_2 = \tfrac{a_{12}}{b_1} \cdots = \tfrac{a_{n2}}{b_n} \right\} \right) \\ & \times \cdots \\ & \times \left(\bigcup_{b_1,\cdots,b_n \in \Omega_{k'+1}} \left\{ c_m : c_m = \tfrac{a_{1m}}{b_1} \cdots = \tfrac{a_{nm}}{b_n} \right\} \right) \end{aligned}$$

Definition 2.8. [2] Let S be a convex set, and $R := S^{\frac{1}{k}} = \left\{z \in \mathbb{C} : z^k \in S\right\}$. Then R is called the **convex kth root set**.

Corollary 2.9. *Let $A_j = diag\,(a_{1j}, \cdots, a_{nj}), j \in \{1, \cdots, m\}, k' > 1$, and there exists $i_1, i_2 \in 1, \cdots, n$, such that $\frac{a_{i_1,j}}{a_{i_2,j}} \in \mathbb{C} \backslash \left(\mathbb{R}^{\frac{1}{k'+1}}\right)$,. Then:*

$$G\Lambda_{k',n}(A_1, \cdots, A_m) = \emptyset.$$

Now, we extend [5, Proposition 2.5]:

Proposition 2.10. *Let $A = (A_1, \cdots, A_m) \in M_n^m, B \in M_k$, and $1 \le r < k \le n$. Then:*

$$_B\Lambda_k(A) \subset \bigcap_{\substack{X \in M_{n,n-r},\\ X^*X = I_{n-r}}} \left\{(a_1, \cdots, a_m) \in \mathbb{C}^m : \begin{pmatrix} W_{k-r}(a_1B, \cdots, a_mB) \cap \\ W_{k-r}(X^*A_1X, \cdots, X^*A_mX) \end{pmatrix} \neq \emptyset\right\}$$

Proof. Let $a = (a_1, \cdots, a_m) \in {_B\Lambda_k}(A)$, and $X \in M_{n,n-r}$ be such that $X^*X = I_{n-r}$. Then there exists $U \in M_{n,k}$, such that:

$$U^*U = I_k, U^*A_jU = a_jB \quad \forall j.$$

We can choose the orthonormal vectors $x_1, \cdots, x_{k-r} \in (X\mathbb{C}^{n-r}) \cap (U\mathbb{C}^k)$, and therefore, there exist $Y = [y_1, \cdots, y_{k-r}] \in M_{k,k-r}, Z = [z_1, \cdots, z_{k-r}] \in M_{n-r,k-r}$, such that $XZ = [x_1, \cdots, x_{k-r}] = UY$, and $Y^*Y = I_{k-r} = Z^*Z$. Therefore,

$$Z^*X^*A_jXZ = a_jY^*BY \quad \forall j,$$

and the proof is completed. □

The following lemma can directly follow from the definition.

Lemma 2.11. *Let $A = (A_1, \cdots, A_m) \in M_{n_1}^m, C = (C_1, \cdots, C_m) \in M_{n_2}^m, B \in M_k, k \le \min\{n_2, n_1\}$, and there exists the matrix $V \in M_{n_1,n_2}$, such that $V^*V = I_{n_2}$, and for any $1 \le j \le m$, $C_j = V^*A_jV$. Then:*

$$_B\Lambda_k(C) \subset {_B\Lambda_k}(A)$$

Lemma 2.12. *Let $A = (A_1, \cdots, A_m) \in M_{n_1}^m, C = (C_1, \cdots, C_m) \in M_{n_2}^m, B \in M_k, k \le n_1 \le n_2$, and for any j, $A_j = C_j\{1, \cdots, n_1\}$. Then:*

$$_B\Lambda_k(A) \subset {_B\Lambda_k}(C).$$

Proof. Let $(a_1, \cdots, a_m) \in {_B\Lambda_k}(A)$. Therefore, there exist $X \in M_{n_1,k}$, such that $X^*X = I_k$, and $X^*A_jX = a_jB$, for all $j \in \{1, \cdots, m\}$. Now, define $Y = \begin{bmatrix} X \\ 0 \end{bmatrix}_{n_2 \times k}$. Then, one can see that $Y^*Y = I_k$, and $Y^*C_jY = a_jB$, for all $j \in \{1, \cdots, m\}$.

□

Corollary 2.13. *Let* $A = (A_1, \cdots, A_m) \in M_{n_1}^m, C = (C_1, \cdots, C_m) \in M_{n_2}^m, k \le n_1 \le n_2,$ *and for any* $j,\ A_j = C_j \{1, \cdots, n_1\}, 1 < k'$; *and there exists the matrix* $V \in M_{n_1,n_2}$, *such that* $V^*V = I_{n_2}$, *and for any* $1 \le j \le m,\ D_j = V^* A_j V$. *Then:*

$$G\Lambda_{k',k}(D) \subset G\Lambda_{k',k}(A) \subset G\Lambda_{k',k}(C).$$

The following theorem is an extension of [5, Theorem 3.1].

Theorem 2.14. *Let* $A = (A_1, \cdots, A_m) \in M_n^m, \hat{k} \ge (m+2)k,\ B \in M_k, (0, \cdots, 0) \in {}_B\Lambda_{\hat{k}}(A),$ *and* $(a_1, \cdots, a_m) \in {}_B\Lambda_k(A)$. *Then for any* $t \in [0,1],$

$$t(a_1, \cdots, a_k) \in {}_B\Lambda_k(A).$$

Proof. Assume that there exist $X \in M_{n,k}$, and $V \in M_{n,(m+2)k}$, such that:

$$\begin{array}{l} X^*X = I_k, \forall j, X^*A_jX = a_jB, \\ V^*V = I_{(m+2)k}, \forall j, V^*A_jV = 0_{(m+2)k}. \end{array}$$

By the Lemma 2.11 and Lemma 2.12, it suffices to show that there is the non-singular matrix $Z \in M_{n,(m+2)k}$, such that:

$$\begin{cases} Z^*Z = I_{(m+2)k}, \\ \forall j, Z^*A_jZ = \begin{bmatrix} \begin{bmatrix} a_jB & 0_k \\ 0_k & 0_k \end{bmatrix} & * \\ * & * \end{bmatrix}_{(m+2)k\times(m+2)k} \end{cases}. \tag{2.1}$$

Because, in this case, we have:

$$\begin{array}{l} {}_B\Lambda_k\left(\begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \end{bmatrix}, \cdots, \begin{bmatrix} a_mB & 0_k \\ 0_k & 0_k \end{bmatrix}\right) \\ \subset {}_B\Lambda_k(Z^*A_1Z, \cdots, Z^*A_mZ) \\ \subset {}_B\Lambda_k(A_1, \cdots, A_m) \end{array}$$

and

$$\forall t \in [0,1], t(a_1, \cdots, a_k) \in {}_B\Lambda_k\left(\begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \end{bmatrix}, \cdots, \begin{bmatrix} a_mB & 0_k \\ 0_k & 0_k \end{bmatrix}\right)$$

(Note that for any $t \in [0,1]$ there exists $U = \begin{bmatrix} \sqrt{t}I_k \\ \sqrt{1-t}I_k \end{bmatrix}$ such that for all $j, U^* \begin{bmatrix} a_jB & 0_k \\ 0_k & 0_k \end{bmatrix} U = ta_jB$.)

Now, we want to select $Y \in M_{n,k}$, and $W \in M_{n,mk}$, such that their columns selected from columns of V and $Z = \begin{bmatrix} X & Y & W \end{bmatrix}$ satisfy in 2.1. But, 2.1 is equivalent to:

$$\begin{cases} Y^*Y = I_k, W^*W = I_{mk}, X^*Y = 0_k, X^*W = 0_{k,mk}, Y^*W = 0_{mk}, \\ \forall j, X^*A_jY = Y^*A_jX = Y^*A_jY = 0_k. \end{cases}$$

Thus, in order to find the Y, it is sufficient to find the k columns of columns of V, such that they lie in the space $H^{\perp}$, such that:

$$H = \text{span}\begin{pmatrix} \{\text{columns of } X\} \cup \{\text{columns of } A_1X\} \\ \cup \cdots \cup \{\text{columns of } A_mX\} \end{pmatrix}.$$

But $\dim(H) \leq (m+1)k$, while V has $(m+2)k$ columns. Therefore, we can construct Y. Now, $\text{span}(\{\text{columns of } X\} \cup \{\text{columns of } Y\})$ is a space with dimension $2k$ and so we can find the mk columns of V, such that they lie not in this space, and assume W, such that their columns are constructed by those columns. □

Also, we can extend [5, proposition 2.1], as follows:

Proposition 2.15. *Suppose* $A = (A_1, \cdots, A_m) \in M_n^m, k \leq n, 1 < k'$ *and* $S = (s_{ij})$ *is an* $m \times n$ *matrix. If* $B_j = \sum_{i=1}^{m} s_{ij}A_i$, *for* $j = 1, \cdots, n$, *then:*

$$\{aT : a \in G\Lambda_{k',k}(A)\} \subset G\Lambda_{k',k}(B).$$

Equality holds, if $\{A_1, \cdots, A_m\}$ *is linearly independent and:*

$$\text{span}\{A_1, \cdots, A_m\} = \text{span}\{B_1, \cdots, B_n\}.$$

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GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

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برد عددی رتبه-بالاتر توأم تعمیم یافته

حمیدرضا افشین، صدیقه باقری، و محمدعلی مهرجوفرد
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برد عددی k رتبه رابطه ی نزدیکی با ساختن کد تصحیح خطای کوانتومی برای کانال کوانتومی پر پارازیت دارد. در کانال کوانتومی پر پارازیت، کد تصحیح خطای کوانتومی از بعد k وجود دارد اگر و تنها اگر برد عددی k رتبه ی توأم مرتبط با آن، $\mathbb{A}$ ناتهی باشد. در این مقاله مفهوم برد عددی k رتبه ی توأم تعمیم داده شده و برخی گزاره های مقاله
[2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.]
توسعه داده شده است.

کلمات کلیدی: تصویرگر تعمیم یافته، برد عددی رتبه بالاتر توأم، برد عددی ماتریسی توأم، برد عددی رتبه بالاتر ماتریسی توأم، برد عددی رتبه بالاتر توأم تعمیم یافته.